# Flag Algebras and Some Applications

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Iowa State University

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Boží Dar
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(Joint results with many friends...)

# Outline

- Introduction to the use of Flag Algebras
- Example of Flag Algebras application
- Applications of Flag Algebras

# Flag algebras

Seminal paper:
Razborov, Flag Algebras, *Journal of Symbolic Logic* **72** (2007), 1239–1282.
David P. Robbins Prize by AMS for Razborov in 2013

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## Example (Goodman, Razborov)

If density of edges is at least $\rho > 0$, what is the minimum density of triangles?

- designed to attack extremal problems.
- works well if constraints as well as desired value can be computed by checking small subgraphs (or average over small subgraphs)
- the results are in limit (very large graphs)

# Applications (incomplete list)

| Author | Year | Application/Result |
|---|---|---|
| Razborov | 2008 | edge density vs. triangle density |
| Hladký, Kráľ, Norin | 2009 | Bounds for the Caccetta-Haggvist conjectu |
| Razborov | 2010 | On 3-hypergraphs with forbidden 4-vertex c |
| Hatami, Hladký,Kráľ,Norine,Razborov / Grzesik | 2011 | Erdős Pentagon problem |
| Hatami, Hladký, Kráľ, Norin, Razborov | 2012 | Non-Three-Colourable Common Graphs Exist |
| Balogh, Hu, L., Liu / Baber | 2012 | 4-cycles in hypercubes |
| Reiher | 2012 | edge density vs. clique density |
| Das, Huang, Ma, Naves, Sudakov | 2013 | minimum number of $k$-cliques |
| Baber, Talbot | 2013 | A Solution to the 2/3 Conjecture |
| Falgas-Ravry, Vaughan | 2013 | Turán density of many 3-graphs |
| Cummings, Kráľ, Pfender, Sperfeld, Treglown, Young | 2013 | Monochromatic triangles in 3-edge colored |
| Kramer, Martin, Young | 2013 | Boolean lattice |
| Balogh, Hu, L., Pikhurko, Udvari, Volec | 2013 | Monotone permutations |
| Norin, Zwols | 2013 | New bound on Zarankiewicz's conjecture |
| Huang, Linial, Naves, Peled, Sudakov | 2014 | 3-local profiles of graphs |
| Balogh, Hu, L., Pfender, Volec, Young | 2014 | Rainbow triangles in 3-edge colored graphs |
| Balogh, Hu, L., Pfender | 2014 | Induced density of $C_5$ |
| Goaoc, Hubard, de Verclos, Séréni, Volec | 2014 | Order type and density of convex subsets |
| Coregliano, Razborov | 2015 | Tournaments |
| Alon, Naves, Sudakov | 2015 | Phylogenetic trees |
| ... | ... | ... |

Applications to graphs, oriented graphs, hypergraphs, hypercubes, permutations, crossing number of graphs, order types, geometry, . . . Razborov: Flag Algebra: an Interim Report

# Example extremal problem


## Theorem (Mantel 1907)

*A triangle-free graph contains at most $\frac{1}{4}n^2$ edges.*

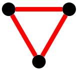
## Problem

*Maximize a graph parameter (# of edges) over a class of graphs (triangle-free).*

- local condition and global parameter
- threshold
- bound and extremal example

# Flag algebras definitions

Let $G$ be a 2-edge-colored complete graph on $n$ vertices.



The probability that three random vertices in $G$ span a red triangle.

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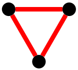

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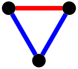

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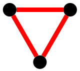

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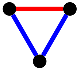

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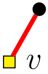


The probability that a random vertex other than $v$ is connected to $v \in V(G)$ by a red edge, i.e., the red degree of $v$ divided by $n-1$.

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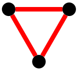

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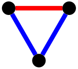

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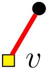


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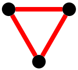

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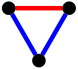

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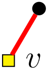


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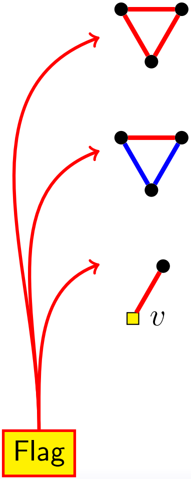


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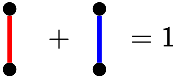
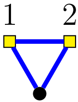
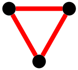
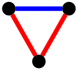
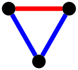
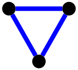



*Type* is a flag induced by labeled vertices

# Flag algebras identities

Let $G$ be a 2-edge-colored complete graph on $n$ vertices. Then

 +  +  +  = 1.

Same kind as

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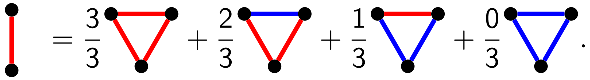
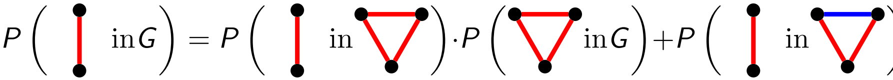



Expanded version where pictures mean graphs:



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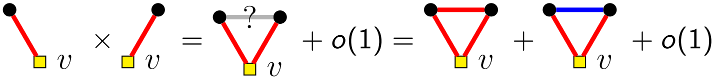


$o(1)$ as $|V(G)| \to \infty$ (will be omitted on next slides)

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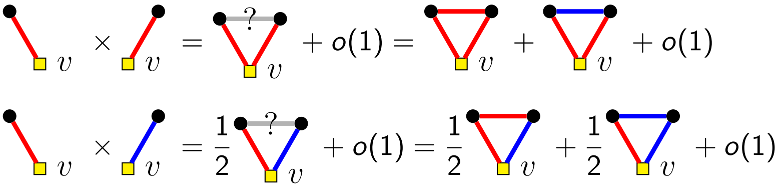


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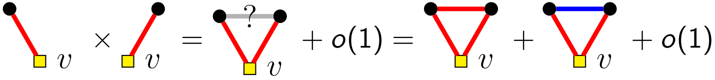
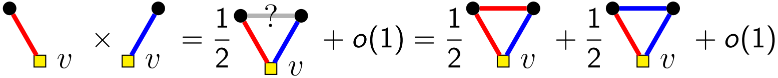
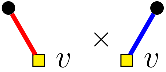
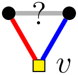



: The probability that choosing two vertices $u_1, u_2$ other than $v$ gives red $vu_1$ and blue $vu_2$.




: The probability that choosing two different vertices $u_1, u_2$ other than $v$ gives one of $vu_1$ and $vu_2$ is red and the other is blue.

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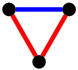
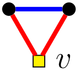
$\frac{1}{3}$  $= \frac{1}{|V(G)|} \sum_{v \in V(G)}$  $v$

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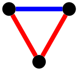
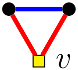
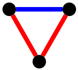
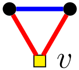
$\frac{1}{3}$  $= \frac{1}{|V(G)|} \sum_{v \in V(G)}$  $v$

 $\binom{n}{3} = \sum_{v \in V(G)}$  $v \binom{n-1}{2}$

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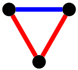
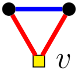
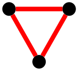
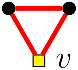
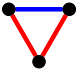
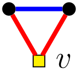
$\frac{1}{3}$  $= \frac{1}{|V(G)|} \sum_{v \in V(G)}$  $v$

 $= \frac{1}{|V(G)|} \sum_{v \in V(G)}$  $v$

 $\binom{n}{3} = \sum_{v \in V(G)}$  $v \binom{n-1}{2}$

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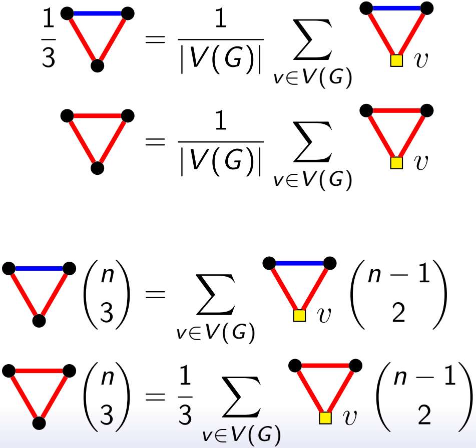

# Identities Summary

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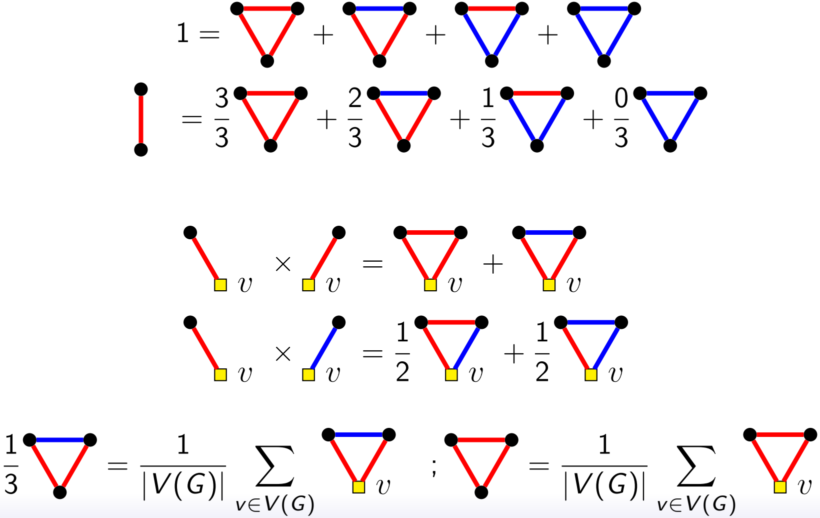

# First try for Mantel's theorem

- How to use the equations to prove something
- Gives bounds as well as helps with extremal examples

# Example - Mantel's theorem, 1st try

## Theorem (Mantel 1907)

*A triangle-free graph contains at most $\frac{1}{4}n^2$ edges.*

Assume edges are red and non-edges are blue.

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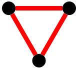
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

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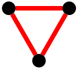
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$0 \leq \Big(1 - 2$  $v\Big)^2$

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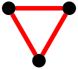
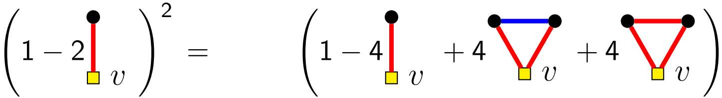
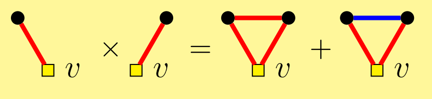
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$0 \leq$ 

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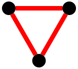
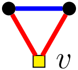
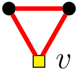
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$0 \leq \frac{1}{n}\sum_{v}\Big(1 - 2$  $v\Big)^2 = \frac{1}{n}\sum_{v}\Big(1 - 4$  $v + 4$  $v + 4$  $v\Big)$

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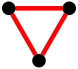
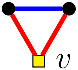
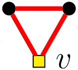
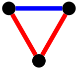
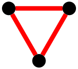
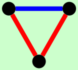
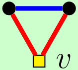
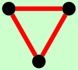
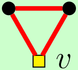
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$0 \leq \frac{1}{n}\sum_v \Big(1 - 2$  $v \Big)^2 = \frac{1}{n}\sum_v \Big(1 - 4$  $v + 4$  $v + 4$  $v\Big)$

$= 1 - 4$  $+ \frac{4}{3}$  $+ 4$ 

$\frac{1}{3}$  $= \frac{1}{|V(G)|}\sum_{v \in V(G)}$  $v$

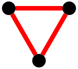
 $= \frac{1}{|V(G)|}\sum_{v \in V(G)}$  $v$

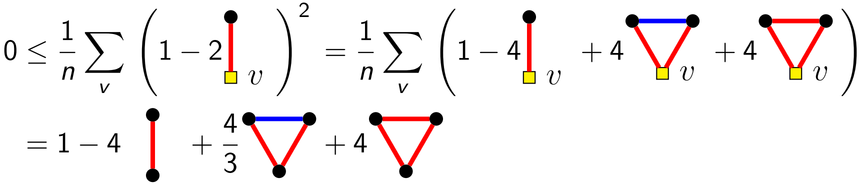
# Example - Mantel's theorem, 1st try

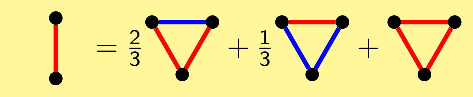
## Theorem (Mantel 1907)

*A triangle-free graph contains at most $\frac{1}{4}n^2$ edges.*

Assume edges are red and non-edges are blue.

Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)



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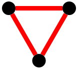
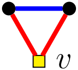
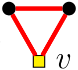
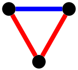
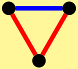
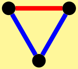
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$0 \leq \frac{1}{n}\sum_v \Big(1 - 2$  $\Big)^2 = \frac{1}{n}\sum_v \Big(1 - 4$  $+ 4$  $+ 4$  $\Big)$

$= 1 - 4$  $+ \frac{4}{3}$ 

 $= \frac{2}{3}$  $+ \frac{1}{3}$ 

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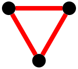
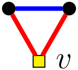
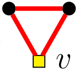
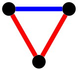
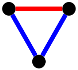
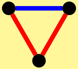
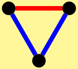
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$0 \leq \frac{1}{n}\sum_v \Big(1 - 2$  $v\Big)^2 = \frac{1}{n}\sum_v \Big(1 - 4$  $v + 4$  $v + 4$  $v\Big)$

$= 1 - 4$  $+ \frac{4}{3}$ 

$= 1 - 2$  $- \frac{2}{3}$ 

$2$  $= \frac{4}{3}$  $+ \frac{2}{3}$ 

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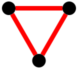
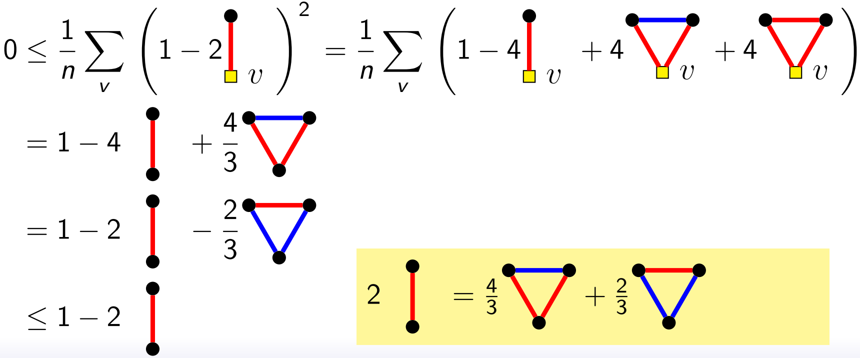
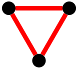
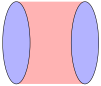
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)



# Example - stability for Mantel

Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$ .

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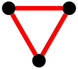
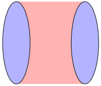
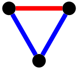
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$ .

$0 \leq 1 - 2$  $- \frac{2}{3}$ 

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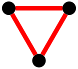
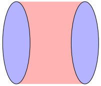
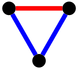
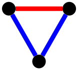
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$  .

$0 \leq 1 - 2$  $- \frac{2}{3}$ 

$0 \leq -\frac{2}{3}$ 

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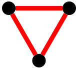
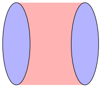
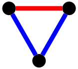
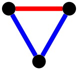
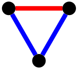
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$ .

$0 \leq 1 - 2$  $- \frac{2}{3}$ 

$0 \leq -\frac{2}{3}$ 

$0 \geq$ 

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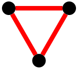
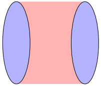
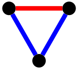
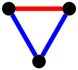
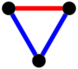
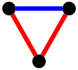
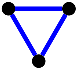
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$  .

$0 \leq 1 - 2$  $- \frac{2}{3}$ 

$0 \leq -\frac{2}{3}$ 

$0 \geq$ 

Only  and  appear in $G$.

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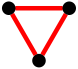
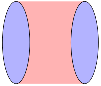
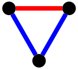
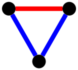
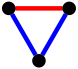
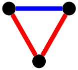
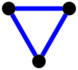
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$  .

$0 \leq 1 - 2$  $- \frac{2}{3}$ 

$0 \leq -\frac{2}{3}$ 

$0 \geq$ 

Only  and  appear in $G$.

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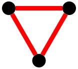
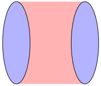
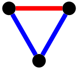
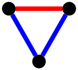
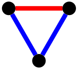
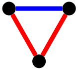
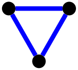
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$ .

$0 \leq 1 - 2$  $- \frac{2}{3}$ 

$0 \leq -\frac{2}{3}$ 

$0 \geq$ 

Only  and  appear in $G$.

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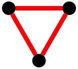
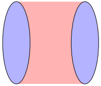
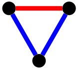
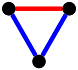
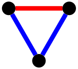
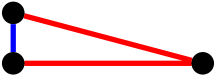
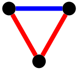
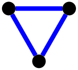
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$ .

$0 \leq 1 - 2$  $- \dfrac{2}{3}$ 

$0 \leq -\dfrac{2}{3}$ 

$0 \geq$ 



Only  and  appear in $G$.

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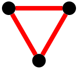
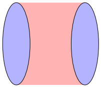
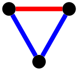
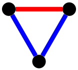
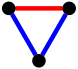
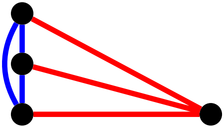
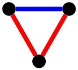
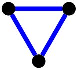
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$ .

$0 \leq 1 - 2$  $- \frac{2}{3}$ 

$0 \leq -\frac{2}{3}$ 

$0 \geq$ 



Only  and  appear in $G$.

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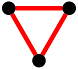
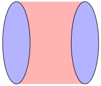
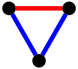
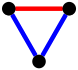
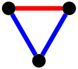
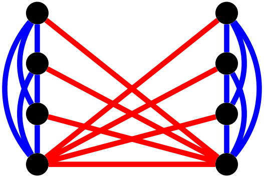
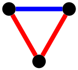
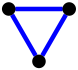
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$ .

$0 \leq 1 - 2$  $- \dfrac{2}{3}$ 

$0 \leq -\dfrac{2}{3}$ 

$0 \geq$ 



Only  and  appear in $G$.

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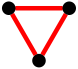
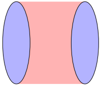
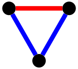
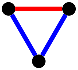
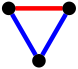
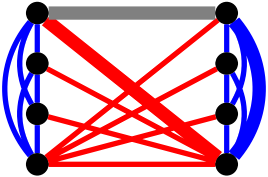
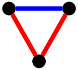
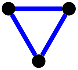
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$  .

$0 \leq 1 - 2$  $- \frac{2}{3}$ 

$0 \leq -\frac{2}{3}$ 

$0 \geq$ 



Only  and  appear in $G$.

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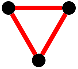
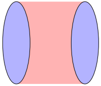
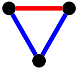
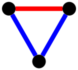
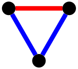
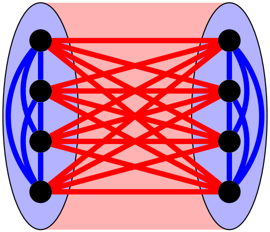
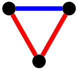
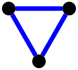
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$  .

$0 \leq 1 - 2$  $- \dfrac{2}{3}$ 

$0 \leq -\dfrac{2}{3}$ 

$0 \geq$ 



Only  and  appear in $G$.

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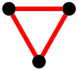
- we optimize on $\mathrm{LIM}^{\mathrm{T}} = \Big\{ q \in \mathrm{LIM} : q\Big($  $\Big) = 0 \Big\}$

$\frac{1}{2} \geq \max_{q \in \mathrm{LIM}^{\mathrm{T}}} q\Big($  $\Big)$

# More automatic approach

- How to use computer to guess the right equation for you.

$0 \leq \Big(1 - 2$  $\Big)^2$

# Example - Mantel's theorem, 2nd try

## Theorem (Mantel 1907)

*A triangle-free graph contains at most $\frac{1}{4}n^2$ edges.*

Assume edges are red and non-edges are blue.

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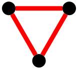
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

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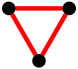
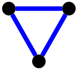
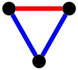
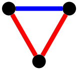
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

 $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

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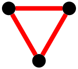
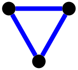
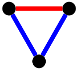
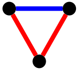
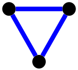
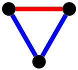
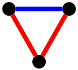
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

 $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

 $\leq \frac{2}{3}\Big($  $+$  $+$  $\Big)$

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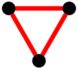
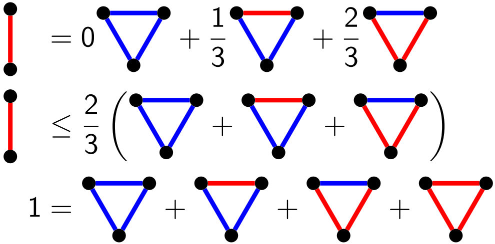
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)



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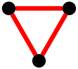
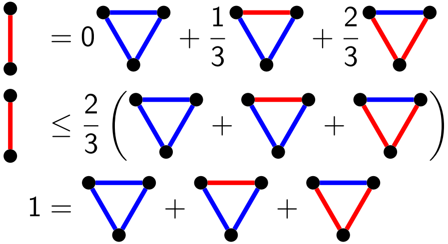
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)



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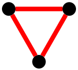
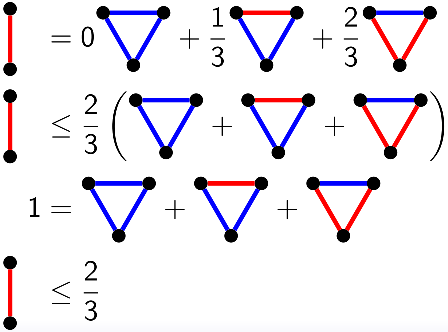
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)



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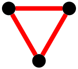
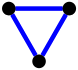
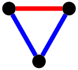
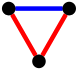
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

 $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

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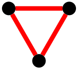
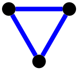
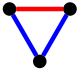
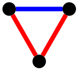
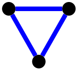
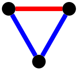
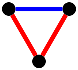
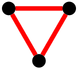
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

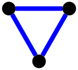
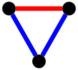
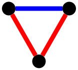
 $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

Idea: find $c_1, c_2, c_3 \in \mathbb{R}$ such that for every graph $G$

$0 \leq c_1$  $+ c_2$  $+ c_3$  .

# Example - Mantel's theorem, 2nd try

Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

 $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

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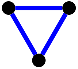
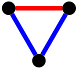
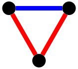
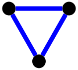
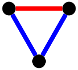
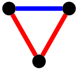
$0 \leq c_1$  $+ c_2$  $+ c_3$  .

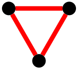
After summing together

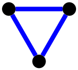
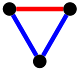
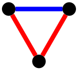
 $\leq c_1$  $+ \left(\frac{1}{3} + c_2\right)$  $+ \left(\frac{2}{3} + c_3\right)$ 

and

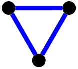
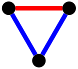
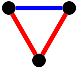
 $\leq \max\left\{(0 + c_1), \frac{1}{3} + c_2, \frac{2}{3} + c_3\right\}.$

# Example - Mantel's theorem, 2nd try

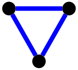
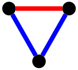
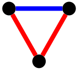
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

 $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

Idea: find $c_1, c_2, c_3 \in \mathbb{R}$ such that for every graph $G$

$0 \leq c_1$  $+ c_2$  $+ c_3$  .

After summing together

 $\leq c_1$  $+ \left(\frac{1}{3} + c_2\right)$  $+ \left(\frac{2}{3} + c_3\right)$ 

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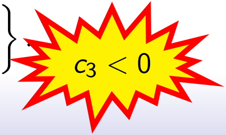
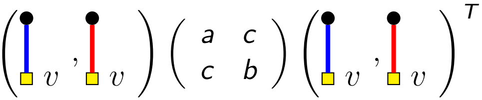


# Candidates for $c_1, c_2, c_3$

$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$ (matrix is positive semidefinite)

# Candidates for $c_1, c_2, c_3$

$0 \leq$




$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$ (matrix is positive semidefinite)

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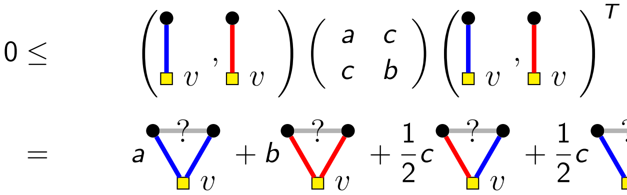

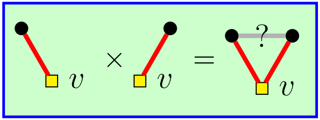

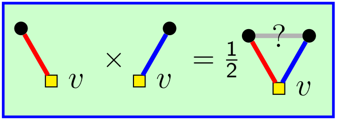

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

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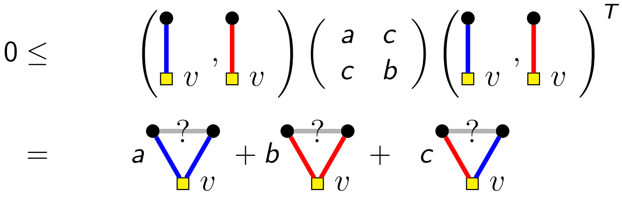


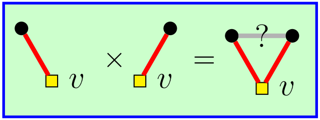


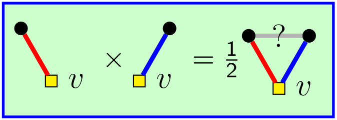


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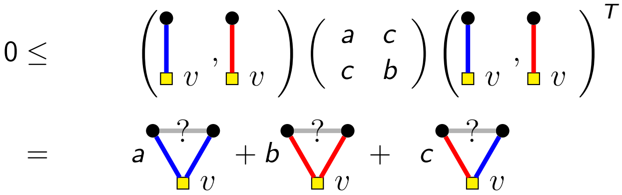


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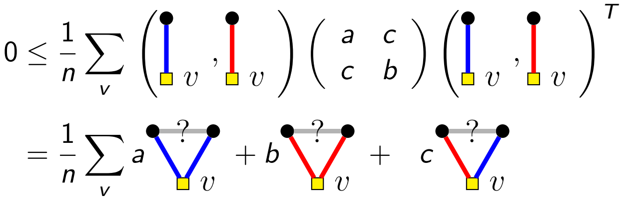


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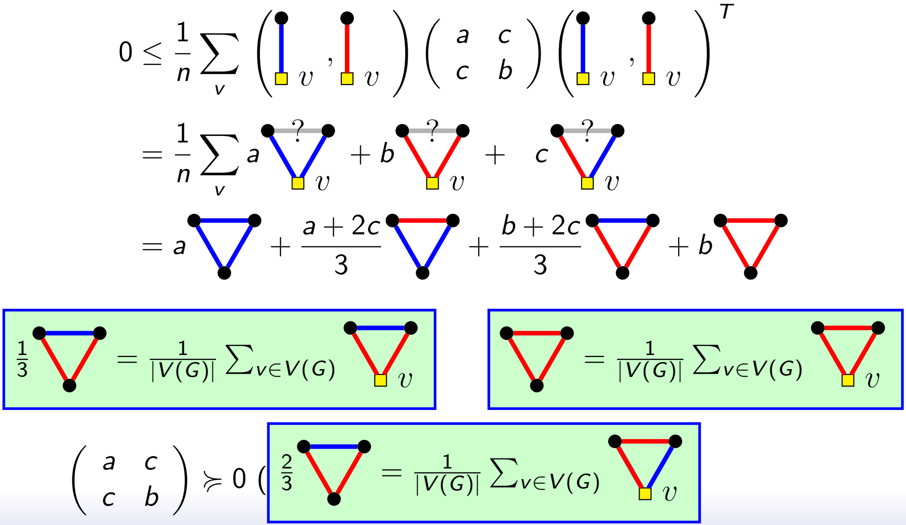

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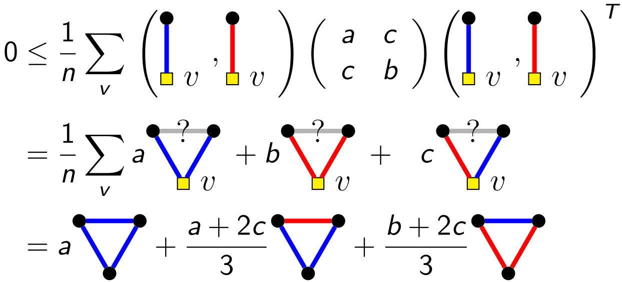
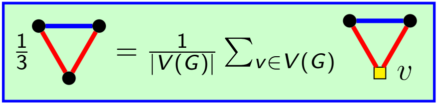
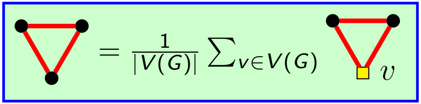
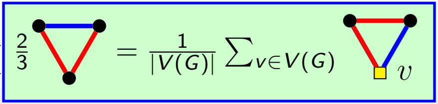



$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$ (



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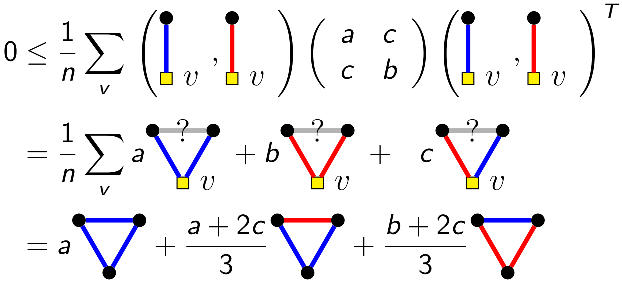
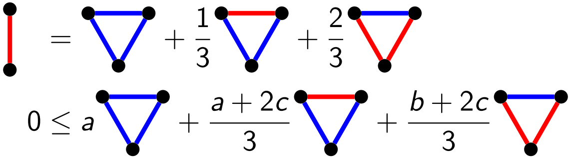


$$c_1 = a,\ c_2 = \frac{a+2c}{3},\ c_3 = \frac{b+2c}{3}$$

$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$ (matrix is positive semidefinite)

# Using $c_1, c_2, c_3$

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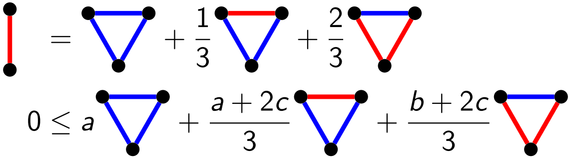


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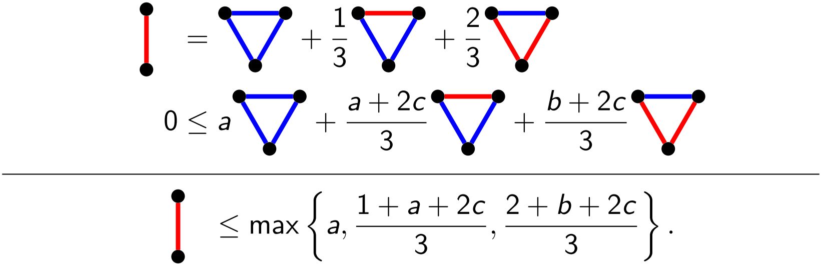


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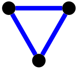
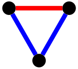
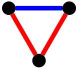
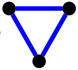
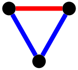
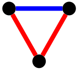
# Using $c_1, c_2, c_3$

 $=$  $+\frac{1}{3}$  $+\frac{2}{3}$ 

$0 \leq a$  $+\frac{a+2c}{3}$  $+\frac{b+2c}{3}$ 

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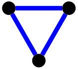
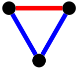
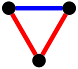
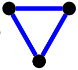
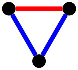
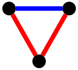
 $\leq \max\left\{a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3}\right\}.$

Try

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

# Using $c_1, c_2, c_3$

 $=$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

$0 \leq a$  $+ \frac{a+2c}{3}$  $+ \frac{b+2c}{3}$ 

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 $\leq \max\left\{a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3}\right\}.$

Try

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

It gives

 $\leq \max\left\{\frac{1}{2}, \frac{1}{6}, \frac{1}{2}\right\} = \frac{1}{2}.$

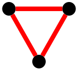
# Optimizing $a, b, c$

$$\leq \max\left\{a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3}\right\}$$

$$(SDP)\begin{cases} \text{Minimize} & d \\ \text{subject to} & a \leq d \\ & \frac{1+a+2c}{3} \leq d \\ & \frac{2+b+2c}{3} \leq d \\ & \begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \end{cases}$$

($SDP$) can be solved on computers using CSDP or SDPA.
Rounding may be needed for exact results.

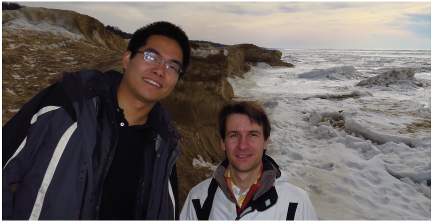
# Applications

Recall  is the probability that 3 randomly chosen vertices form a red triangle.

J. Balogh

P. Hu

## Hypercubes and posets

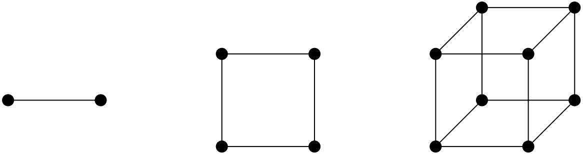


H. Liu | B. L.

Application to sparse structure.

# Hypercube

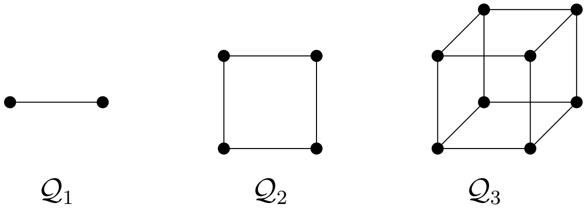
$\mathcal{Q}_n$ is $n$-dimensional hypercube ($n$-cube)



$\mathcal{Q}_1$ $\mathcal{Q}_2$ $\mathcal{Q}_3$

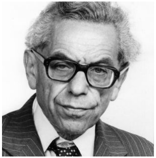
# Hypercube

$\mathcal{Q}_n$ is $n$-dimensional hypercube ($n$-cube)



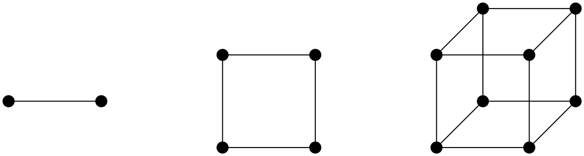

## Problem (Erdős 1984)

*What is the maximum number of edges in a subgraph of $\mathcal{Q}_n$ with no $\mathcal{Q}_2$?*

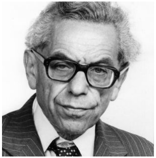


# Hypercube

$\mathcal{Q}_n$ is $n$-dimensional hypercube ($n$-cube)

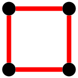


$\mathcal{Q}_1$ $\mathcal{Q}_2$ $\mathcal{Q}_3$



## Problem (Erdős 1984)

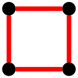
*What is the maximum number of edges in a subgraph of $\mathcal{Q}_n$ with no $\mathcal{Q}_2$?*

maximize  subject to  $= 0$

# Lower bound

## Conjecture (Erdős 1984)

*In $\mathcal{Q}_n$ where $n \to \infty$:*

 $= 0 \Rightarrow$  $\leq \frac{1}{2}$.

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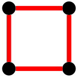
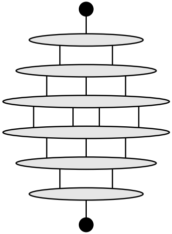
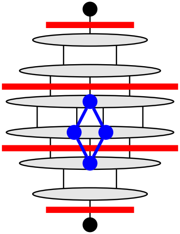
 $= 0 \Rightarrow$  $\leq \frac{1}{2}.$

As posets



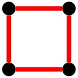
$\mathcal{Q}_7$



$\mathcal{Q}_7$

By removing every second layer,  $\geq 1/2.$

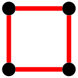
# Results about hypercubes

If  $= 0$ then

### Theorem (Chung 1992)

 $\leq 0.62284.$

# Results about hypercubes

If  = 0 then

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### Theorem (Thomason and Wagner 2009)

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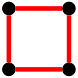
# Results about hypercubes

If  $= 0$ then

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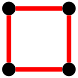
If  = 0 then

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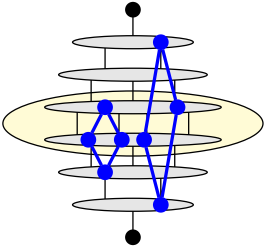
## Theorem (Balogh, Hu, **L.**, Liu 2014; Baber 2014+)

 $\leq 0.6068$. *(Uses $\mathcal{Q}_3$ instead of $\mathcal{Q}_2$.)*

# Related results - boolean lattice

Let $\mathcal{B}_n$ denote $n$-dimensional boolean lattice.
Let $F$ be a subposet of $\mathcal{B}_n$ not containing $\Diamond$.



$\mathcal{B}_7$

# Related results - boolean lattice

Let $\mathcal{B}_n$ denote $n$-dimensional boolean lattice.
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**Theorem**

$|F| \leq (c + o(1))\binom{n}{\lfloor n/2 \rfloor}$, *where*
$c \leq 2.3$ *[Griggs, Lu 2009]*
$c \leq 2.284$ *[Axenovich, Manske, Martin 2012]*
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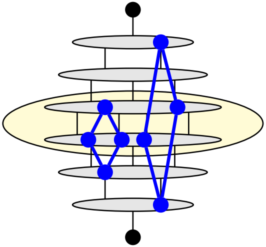

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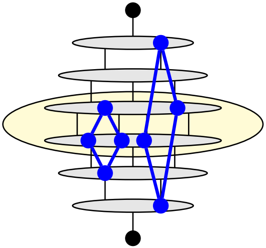

$\mathcal{B}_7$

### Theorem

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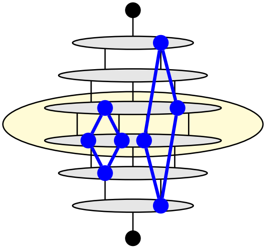

$\mathcal{B}_7$

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$c = 2$ [Kramer, Martin 2015, announced]

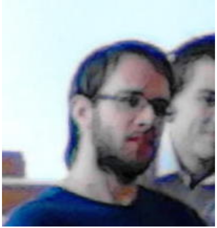
J. Balogh

P. Hu

B. L.

# Permutations


O. Pikhurko



B. Udvari

J. Volec

Application with exact result.

# Permutations and extremal problems

## Problem

*What is the minimum number of monotone subsequences of size $k$ in a permutation of $[n]$?*

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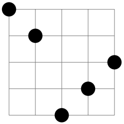
n = 5



(5,4,1,2,3)

(5,4,1),(5,4,2),(5,4,3)
(1,2,3)

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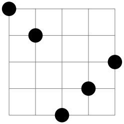

$(5{,}4{,}1{,}2{,}3)$

(5,4,1),(5,4,2),(5,4,3)
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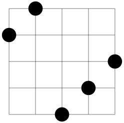

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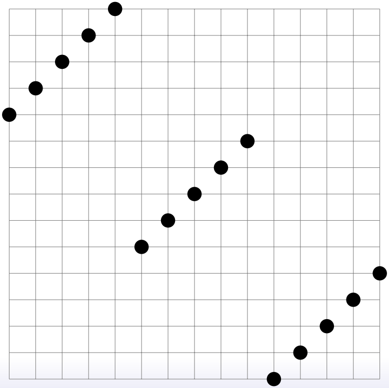
# Conjecture

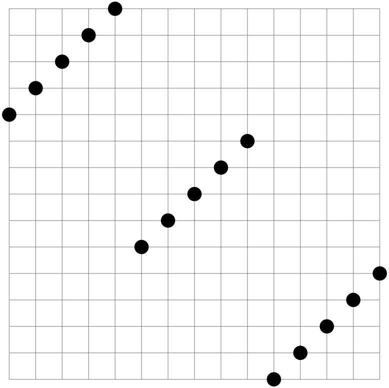
## Conjecture (Myers 2002)

*The number of monotone subsequences of length $k$ is minimized by a permutation on $[n]$ with $k-1$ increasing runs of as equal lengths as possible.*



$k = 4, n = 15$

# Extremal case is not unique



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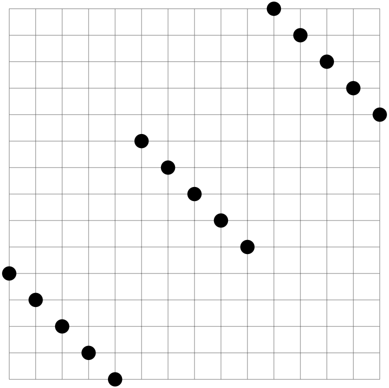

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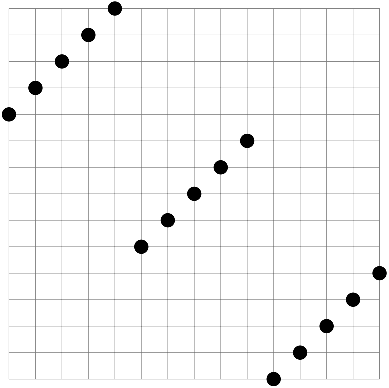

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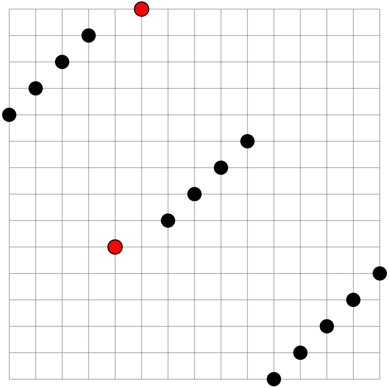

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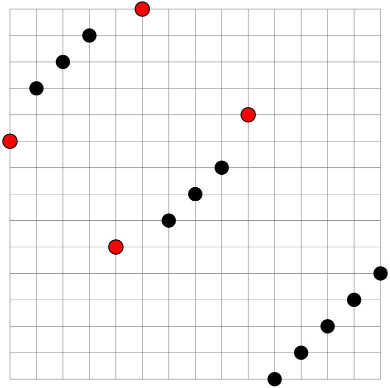

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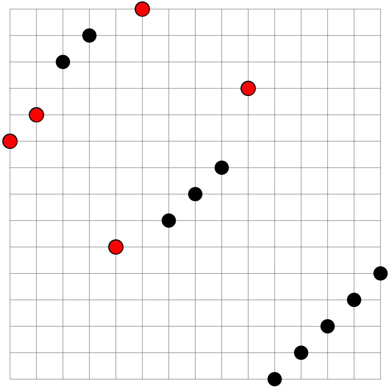

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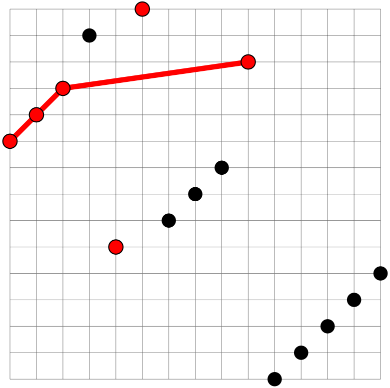

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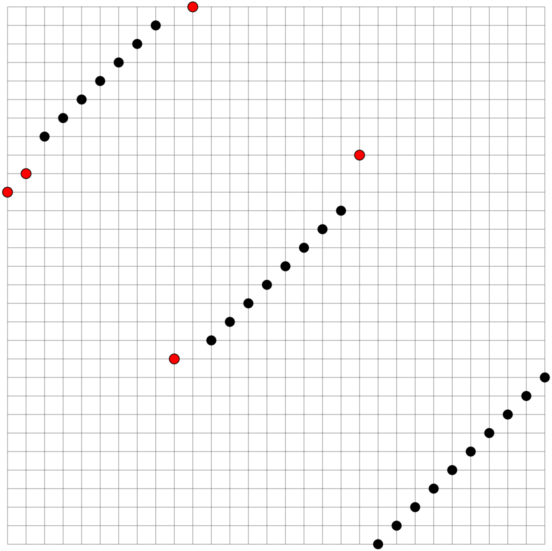

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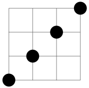
## Theorem (Samotij, Sudakov '14+)

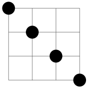
*Myers' conjecture is true for sufficiently large $k$ and $n \leq k^2 + ck^{3/2}\log k$, where $c$ is an absolute positive constant.*

## Theorem (Balogh, Hu, **L.**, Pikhurko, Udvari, Volec '14+)

*Myers' conjecture is true for $k = 4$ and $n$ sufficiently large.*



(1,2,3,4)



(4,3,2,1)

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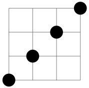

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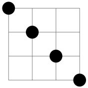


(4,3,2,1)

Use of flag algebras, $k = 5, 6$ also doable, 7 not.

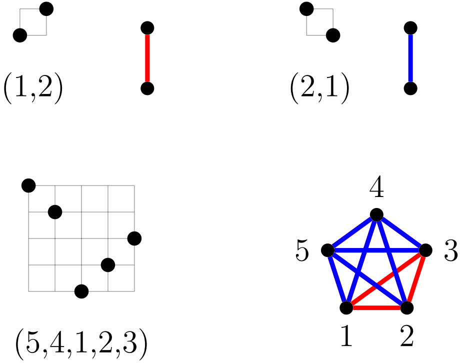
# From permutations to permutation graphs

(1,2)

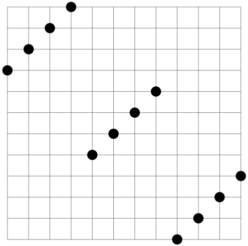
(2,1)

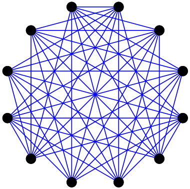
# From permutations to permutation graphs

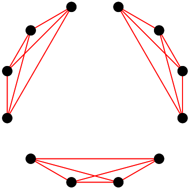
k = 3
n = 5

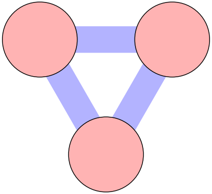


# Extremal example ($k = 4$)

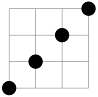




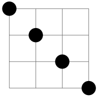




# As flag algebra question ($k = 4$)



$(1,2,3,4)$



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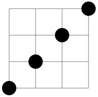

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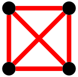

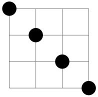

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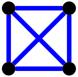

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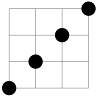

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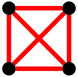
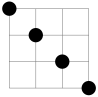
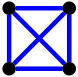
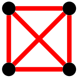
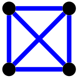




(4,3,2,1)



minimize  $+$ 

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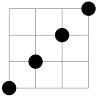
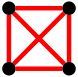
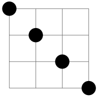
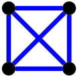


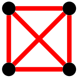
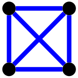
(1,2,3,4)



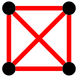
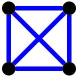


(4,3,2,1)



minimize  $+$ 

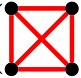
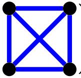
**Theorem (Balogh, Hu, L., Pikhurko, Udvari, Volec '14+)**

 $+$  $\geq \frac{1}{27}$

*for every permutation graph.*

# Only for permutation graphs

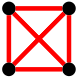
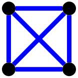
Theorem (Balogh, Hu, **L.**, Pikhurko, Udvari, Volec '14+)

$$\min\Big($$  $$+$$  $$\Big) = \frac{1}{27}$$

*over permutation graphs (and extremal permutations described using Myers' results - stability arguments).*

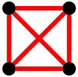
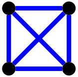
# ONLY FOR PERMUTATION GRAPHS

THEOREM (BALOGH, HU, **L.**, PIKHURKO, UDVARI, VOLEC '14+)

min (  +  ) $= \frac{1}{27}$

*over permutation graphs (and extremal permutations described using Myers' results - stability arguments).*

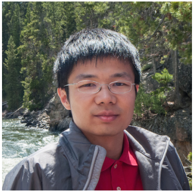
THEOREM (SPERFELD '12; THOMASON '89)

$\frac{1}{35} <$ min (  +  ) $< \frac{1}{33}$

*over* **all** *sufficiently large* 2*-edge-colored complete graphs.*

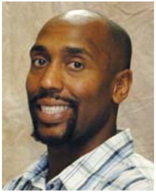
F. Pfender B. L. J. Balogh

## Rainbow Triangles



P. Hu

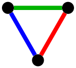

J. Volec



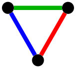
M. Young

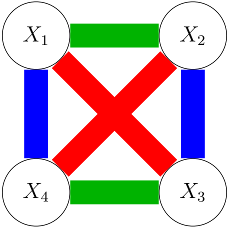
Application with exact result and iterated extremal construction.

# The problem

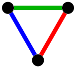
$F(n) :=$ max  over all 3-edge-colorings of $K_n$

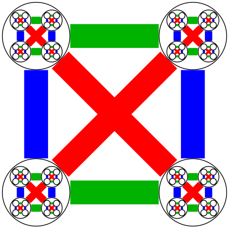
# The problem

$F(n) :=$ max  over all 3-edge-colorings of $K_n$

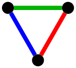


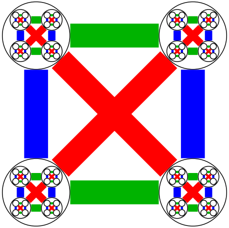
# The problem

$F(n) :=$ max  over all 3-edge-colorings of $K_n$



# The problem

$F(n) :=$ max  over all 3-edge-colorings of $K_n$



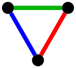
## Conjecture (Erdős, Sós 1972-)

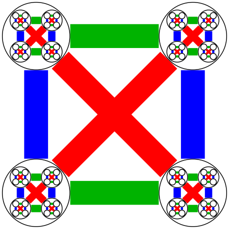
*This construction is the best possible. In other words,*

$$F(n) = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 + \sum_i F(x_i),$$

*where* $x_1 + x_2 + x_3 + x_4 = n$*, and* $|x_i - x_j| \leq 1$*.*

# The problem

$F(n) :=$ max  over all 3-edge-colorings of $K_n$



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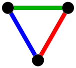
*where* $x_1 + x_2 + x_3 + x_4 = n$, *and* $|x_i - x_j| \leq 1$.

Our result: The conjecture is true for $n$ large and for any $n = 4^k$.

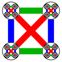
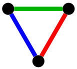
# Direct application of Flag Algebras

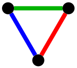
$F(n) :=$ max  over all 3-edge-colorings of $K_n$

**Theorem (Balogh, Hu, L., Pfender, Volec, Young)**

$$F(n) = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 + \sum_i F(x_i),$$

*where $x_1 + x_2 + x_3 + x_4 = n$, and $|x_i - x_j| \leq 1$ and $n$ is large or $n = 4^k$.*

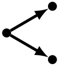
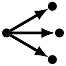
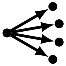
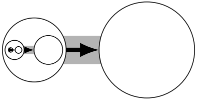
Construction  :  $\geq 0.4$

FA:  $\leq 0.4006$

Usual stability approach with excluded subgraphs does not work (nothing is excluded). Not tight result from FA is typical if the extremal construction is iterated.

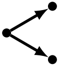
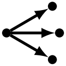
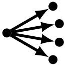
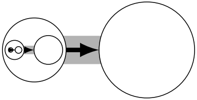
# Results with iterated constructions

## Theorem (Falgas-Ravry, Vaughan 2012)

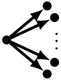
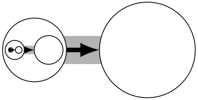
*Density of* ,  *and*  *is maximized by* .

# Results with iterated constructions

## Theorem (Falgas-Ravry, Vaughan 2012)

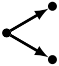
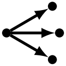
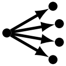
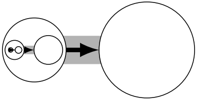
*Density of*  ,  *and*  *is maximized by*  .

## Theorem (Huang 2014)

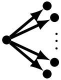
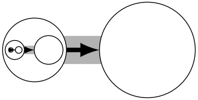
*Density of*  *is maximized by*  .

# Results with iterated constructions

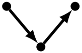
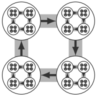
**Theorem (Falgas-Ravry, Vaughan 2012)**

*Density of*  ,  *and*  *is maximized by*  .

**Theorem (Huang 2014)**

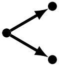
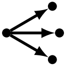
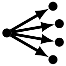
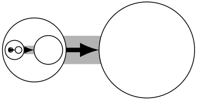
*Density of*  *is maximized by*  .

**Theorem (Hladký, Kráľ, Norin)**

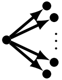
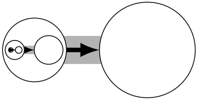
*Density of*  *is maximized by*  .

# Results with iterated constructions

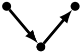
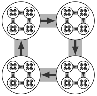
## Theorem (Falgas-Ravry, Vaughan 2012)

*Density of*  *,*  *and*  *is maximized by*  *.*

## Theorem (Huang 2014)

*Density of*  *is maximized by*  *.*

## Theorem (Hladký, Kráľ, Norin)

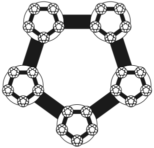
*Density of*  *is maximized by*  *.*

## Theorem (Pikhurko 2014)

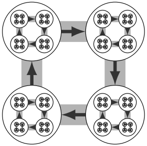
*Iterated blow-up of $r$-graph is extremal for $\pi(\mathcal{F})$ for some family $\mathcal{F}$.*

# Related results

**Theorem (Balogh, Hu, L., Pfender, 2014+)**

*# of induced $C_5$s is maximized by* 

**Theorem (Hu, L., Pfender, Volec)**

*# of induced oriented $C_4$s is maximized by* 

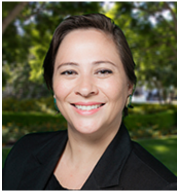
E. Gethner

L. Hogben

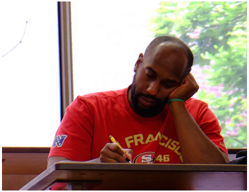
B. L.

# Crossing numbers

F. Pfender



A. Ruiz



M. Young

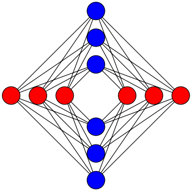
Application to graph drawing.

For a graph $G$, $cr(G)$ is *crossing number*.

## Conjecture (Zarankiewicz 1954)

$$cr(K_{m,n}) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{(n-1)}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{(m-1)}{2} \right\rfloor.$$

## Theorem (Norin, Zwols 2013+)



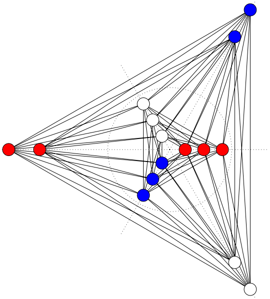
$K_{6,6}$

$$cr(K_{m,n}) \geq 0.9 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{(n-1)}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{(m-1)}{2} \right\rfloor$$

*for large m and n. (Zarankiewicz's conjecture is 90% true)*

For a graph $G$, $\overline{cr}(G)$ is the *rectilinear crossing number*.

## Conjecture

*$\overline{cr}(K_{n_1,n_2,n_3})$ is minimized by*  $K_{5,5,5}$

## Theorem (Gethner, Hogben, **L.**, Pfender, Ruiz, Young)

*$\overline{cr}(K_{n_1,n_2,n_3})$ conjecture is 97.3% true.*

## Problem

*What about more partite graphs?*

## Ramsey numbers

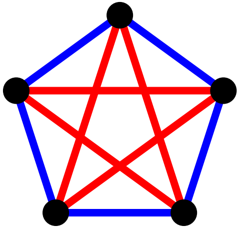
F. Pfender B.L.

Application to something seemingly unrelated.

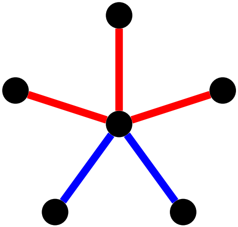
## Definition

$R(G_1, G_2, \ldots, G_k)$ is the smallest integer $n$ such that any $k$-edge coloring of $K_n$ contains a copy of $G_i$ in color $i$ for some $1 \leq i \leq k$.

$R(K_3, K_3) > 5$



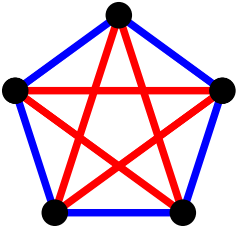
$R(K_3, K_3) \leq 6$



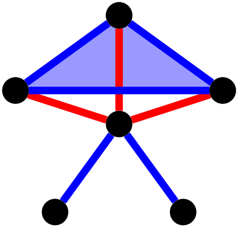
## Definition

$R(G_1, G_2, \ldots, G_k)$ is the smallest integer $n$ such that any $k$-edge coloring of $K_n$ contains a copy of $G_i$ in color $i$ for some $1 \leq i \leq k$.

$R(K_3, K_3) > 5$



$R(K_3, K_3) \leq 6$



Theorem (Ramsey 1930)

*$R(K_m, K_n)$ is finite.*

$R(G_1, \ldots, G_k)$ is finite

Questions:

- study how $R(G_1, \ldots, G_k)$ grows if $G_1, \ldots, G_k$ grow (large)
- study $R(G_1, \ldots, G_k)$ for fixed $G_1, \ldots, G_k$ (small)

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Radziszowski - *Small Ramsey Numbers*
Electronic Journal of Combinatorics - Survey

*[Erdős] Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack.*

# New upper bounds (so far)

| Problem | Lower | New upper | Old upper |
|---|---|---|---|
| $R(K_4^-, K_4^-, K_4^-)$ | 28 | 28 | 30 |
| $R(K_3, K_4^-, K_4^-)$ | 21 | 23 | 27 |
| $R(K_4, K_4^-, K_4^-)$ | 33 | 47 | 59 |
| $R(K_4, K_4, K_4^-)$ | 55 | 104 | 113 |
| $R(C_3, C_5, C_5)$ | 17 | 18 | 21? |
| $R(K_4, K_7^-)$ | 37 | 52 | 59 |
| $R(K_{2,2,2}, K_{2,2,2})$ | 30 | 32 | 60? |
| $R(K_5^-, K_6^-)$ | 31 | 38 | 39 |
| $R(K_5, K_6^-)$ | 43 | 62 | 67 |

Thank you for your attention!