Unique Maximum Facial Colorings

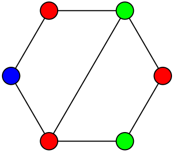
Vesna Andova Bernard Lidický Borut Lužar
Kacy Messerschmidt Riste Škrekovski

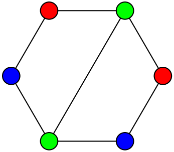
AMS Sectional meeting #1132
Buffalo, NY
Sep 16, 2017

Graph Coloring

Definition

A *(proper) coloring* of a graph G is a mapping $\varphi : V(G) \to C$ such that for every $uv \in E(G) : \varphi(u) \neq \varphi(v)$.





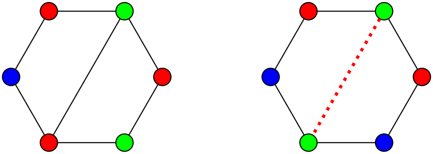
G is *k-colorable* if there is a (proper) coloring of G with $|C| = k$.

Minimum k such that G is k-colorable is denote by $\chi(G)$.

Here we color with $\{1, 2, \ldots, k\}$ instead of arbitrary C.

DEFINITION

A *(proper) coloring* of a graph G is a mapping $\varphi : V(G) \to C$ such that for every $uv \in E(G) : \varphi(u) \neq \varphi(v)$.



G is *k-colorable* if there is a (proper) coloring of G with $|C| = k$.

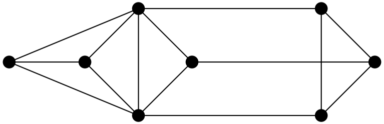
Minimum k such that G is k-colorable is denote by $\chi(G)$.

Here we color with $\{1, 2, \ldots, k\}$ instead of arbitrary C.

Plane Graphs

Definition

A graph *G* is *planar* if it can be embeded in the plane, where vertices are points and edges are non-crossing curves.



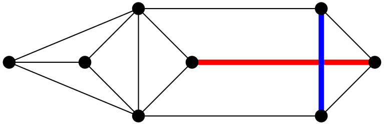
G is *plane* if it is embedded in the plane.

Connected regions of the *plane* − *G* are *faces*.

Plane Graphs

Definition

A graph *G* is *planar* if it can be embeded in the plane, where vertices are points and edges are non-crossing curves.

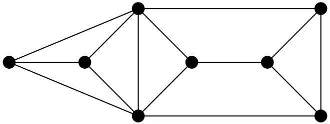


G is *plane* if it is embedded in the plane.

Connected regions of the *plane* − *G* are *faces*.

Definition

A graph *G* is *planar* if it can be embeded in the plane, where vertices are points and edges are non-crossing curves.

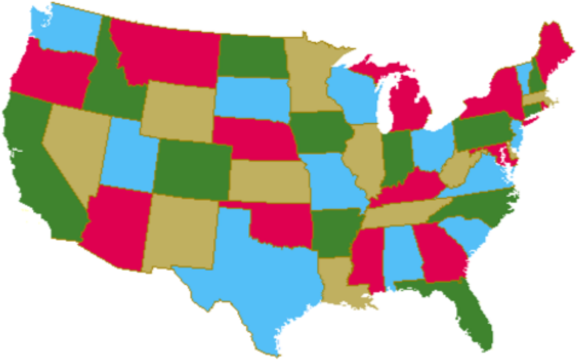


G is *plane* if it is embedded in the plane.

Connected regions of the *plane* − *G* are *faces*.

Theorem (Appel and Haken 1977)

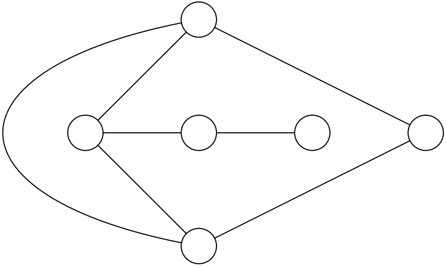
Every planar graph is 4-colorable.



Conjecture

Conjecture (Fabrici and Göring)

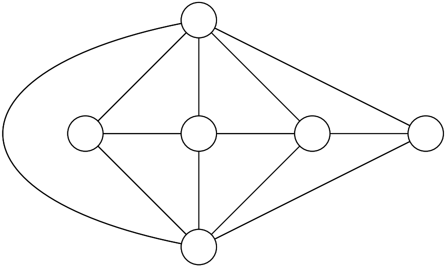
If G is a plane graph, then there is a proper coloring of the vertices of G by colors in $\{1, 2, 3, 4\}$ such that every face contains a unique vertex colored with the maximal color appearing on that face.



Conjecture

Conjecture (Fabrici and Göring)

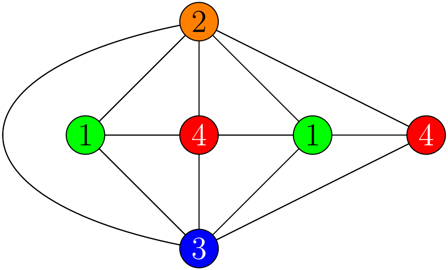
If G is a plane graph, then there is a proper coloring of the vertices of G by colors in $\{1, 2, 3, 4\}$ such that every face contains a unique vertex colored with the maximal color appearing on that face.



Conjecture

Conjecture (Fabrici and Göring)

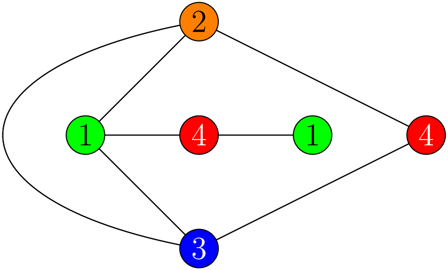
If G is a plane graph, then there is a proper coloring of the vertices of G by colors in $\{1, 2, 3, 4\}$ such that every face contains a unique vertex colored with the maximal color appearing on that face.



Conjecture

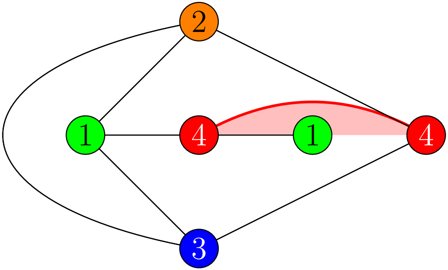
Conjecture (Fabrici and Göring)

If G is a plane graph, then there is a proper coloring of the vertices of G by colors in $\{1, 2, 3, 4\}$ such that every face contains a unique vertex colored with the maximal color appearing on that face.



Conjecture (Fabrici and Göring)

If G is a plane graph, then there is a proper coloring of the vertices of G by colors in $\{1, 2, 3, 4\}$ such that every face contains a unique vertex colored with the maximal color appearing on that face.



Note: Add or delete edges carefully!

Conjecture

Conjecture (Fabrici and Göring)

If G is a plane graph, then $\chi_{\text{fum}}(G) \leq 4$.

A proper coloring of a graph G embedded on some surface, where

(1) colors are natural numbers, and

(2) every face has a unique vertex colored with its maximal color,

is called a *facial unique-maximum coloring* or *FUM-coloring*.

The minimum number k such that G admits a FUM-coloring with colors $\{1, 2, \ldots, k\}$ is called the FUM chromatic number of G, denoted by $\chi_{\text{fum}}(G)$.

Conjecture

Conjecture (Fabrici and Göring)

If G is a plane graph, then $\chi_{\text{fum}}(G) \leq 4$.

Conjecture

Conjecture (Fabrici and Göring)

If G is a plane graph, then $\chi_{\mathrm{fum}}(G) \leq 4$.

Theorem (Fabrici and Göring 2015)

If G is a plane graph, then $\chi_{\mathrm{fum}}(G) \leq 6$.

Conjecture

Conjecture (Fabrici and Göring)

If G is a plane graph, then $\chi_{\text{fum}}(G) \leq 4$.

Theorem (Fabrici and Göring 2015)

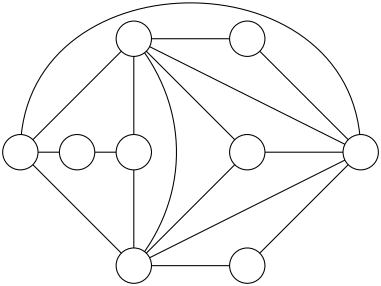
If G is a plane graph, then $\chi_{\text{fum}}(G) \leq 6$.

Theorem (Wendland 2016)

If G is a plane graph, then $\chi_{\text{fum}}(G) \leq 5$.

Idea

Theorem (Fabrici and Göring 2015)
If G is a plane graph, then $\chi_{\text{fum}}(G) \leq 6$.

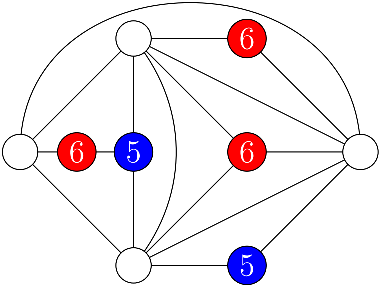


Idea

Theorem (Fabrici and Göring 2015)

If G is a plane graph, then $\chi_{\text{fum}}(G) \leq 6$.

Color *some* vertices of G by colors 5 and 6 such that each face contains unique 6 or (no 6 and unique 5).

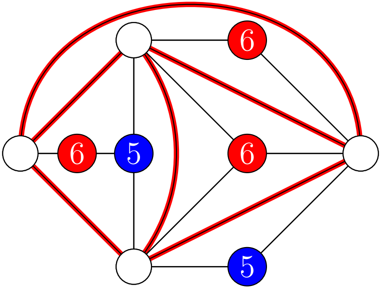


Idea

Theorem (Fabrici and Göring 2015)

If G is a plane graph, then $\chi_{\text{fum}}(G) \leq 6$.

Color *some* vertices of G by colors 5 and 6 such that each face contains unique 6 or (no 6 and unique 5).



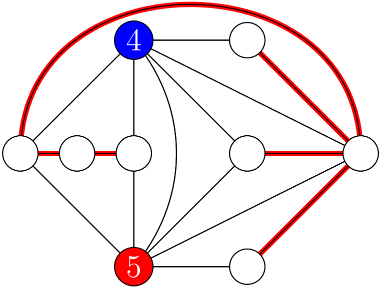
Color rest by 4-color theorem with $\{1, 2, 3, 4\}$.

Idea

Theorem (Fabrici and Göring 2015)

If G is a plane graph, then $\chi_{\mathrm{fum}}(G) \leq 6$.

Color *some* vertices of G by colors 5 and 6 such that each face contains unique 6 or (no 6 and unique 5).



Color rest by 4-color theorem with $\{1, 2, 3, 4\}$.
Wendland: Make *the rest* triangle-free and use Grötzsch's theorem.
Just $\{4, 5\} \cup \{1, 2, 3\}$ colors needed in total.

Our Results

Conjecture (Fabrici and Göring)

If G is a plane graph, then $\chi_{\mathrm{fum}}(G) \leq 4$.

Our Results

Conjecture (Fabrici and Göring)

If G is a plane graph, then $\chi_{\text{fum}}(G) \leq 4$.

Theorem (Andova, L., Lužar, Škrekovski)

If G is a plane subcubic graph, then $\chi_{\text{fum}}(G) \leq 4$.

Our Results

Conjecture (Fabrici and Göring)

If G is a plane graph, then $\chi_{\mathrm{fum}}(G) \leq 4$.

Theorem (Andova, L., Lužar, Škrekovski)

If G is a plane subcubic graph, then $\chi_{\mathrm{fum}}(G) \leq 4$.

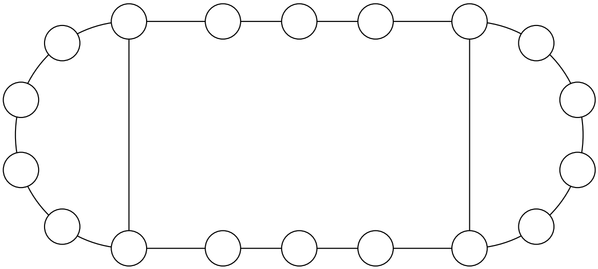
Theorem (Andova, L., Lužar, Škrekovski)

If G is an outerplane graph, then $\chi_{\mathrm{fum}}(G) \leq 4$.

Both results are tight.

Tight Example

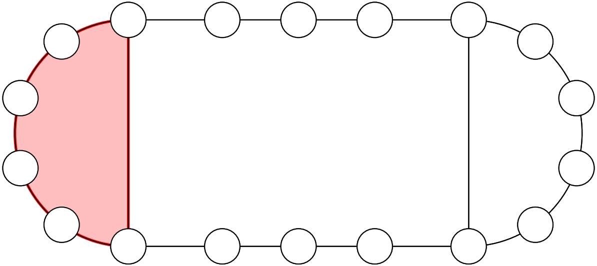
For the following graph G, $\chi_{\text{fum}}(G) > 3$.
Suppose for contradiction $\chi_{\text{fum}}(G) = 3$:



Notice G is subcubic, bipartite, 2-connected, and outerplane.

Tight Example

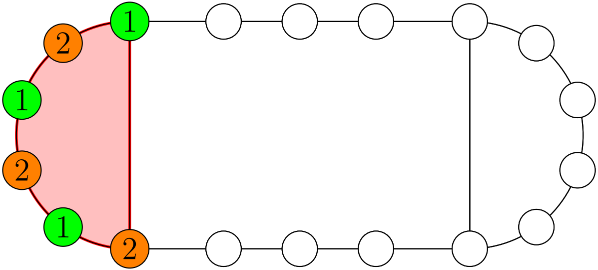
For the following graph G, $\chi_{\text{fum}}(G) > 3$.
Suppose for contradiction $\chi_{\text{fum}}(G) = 3$:



Notice G is subcubic, bipartite, 2-connected, and outerplane.

Tight Example

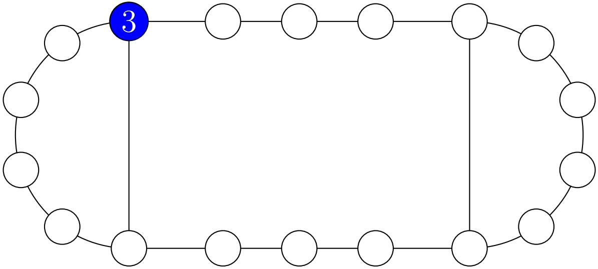
For the following graph G, $\chi_{\text{fum}}(G) > 3$.
Suppose for contradiction $\chi_{\text{fum}}(G) = 3$:



Notice G is subcubic, bipartite, 2-connected, and outerplane.

Tight Example

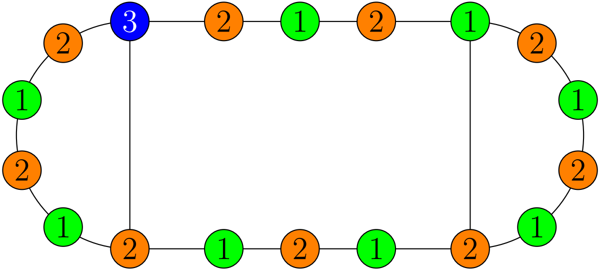
For the following graph G, $\chi_{\text{fum}}(G) > 3$.
Suppose for contradiction $\chi_{\text{fum}}(G) = 3$:



Notice G is subcubic, bipartite, 2-connected, and outerplane.

Tight Example

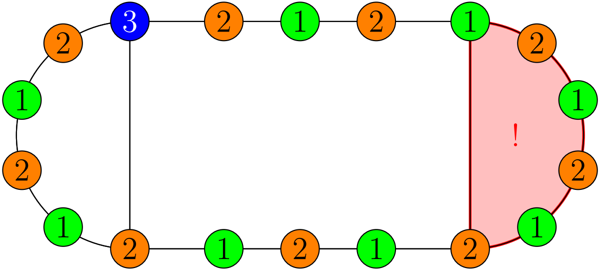
For the following graph G, $\chi_{\text{fum}}(G) > 3$.
Suppose for contradiction $\chi_{\text{fum}}(G) = 3$:



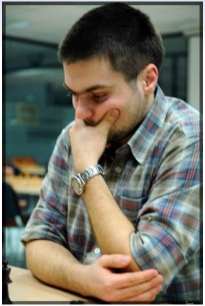
Notice G is subcubic, bipartite, 2-connected, and outerplane.

Tight Example

For the following graph G, $\chi_{\text{fum}}(G) > 3$.
Suppose for contradiction $\chi_{\text{fum}}(G) = 3$:



Notice G is subcubic, bipartite, 2-connected, and outerplane.
Also, G can have arbitrarily large girth.

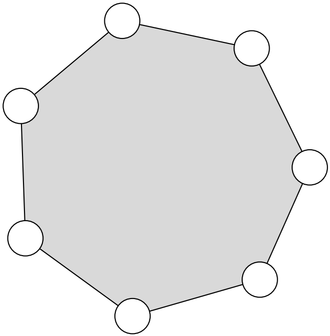


CCCP

Proof Idea

If G is a plane subcubic graph, then $\chi_{\mathrm{fum}}(G) \leq 4$.

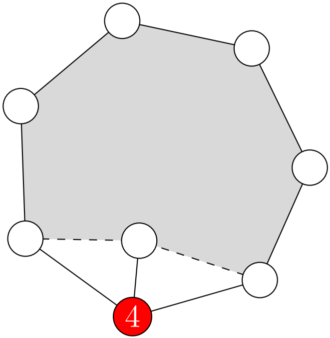
Use precoloring extension method (Thomassen's 5-list coloring)
(Nice induction)



Proof Idea

If G is a plane subcubic graph, then $\chi_{\mathrm{fum}}(G) \leq 4$.

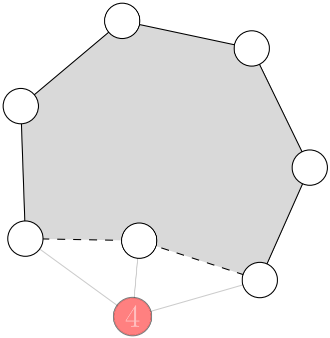
Use precoloring extension method (Thomassen's 5-list coloring)
(Nice induction)



Proof Idea

If G is a plane subcubic graph, then $\chi_{\text{fum}}(G) \leq 4$.

Use precoloring extension method (Thomassen's 5-list coloring)
(Nice induction)

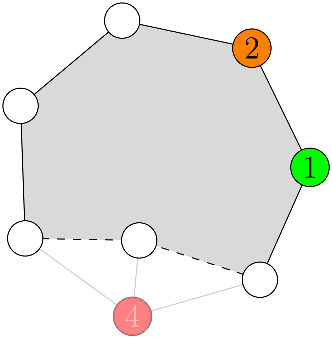


outer face colored by $\{1, 2, 3\}$

Proof Idea

If G is a plane subcubic graph, then $\chi_{\mathrm{fum}}(G) \leq 4$.

Use precoloring extension method (Thomassen's 5-list coloring)
(Nice induction)

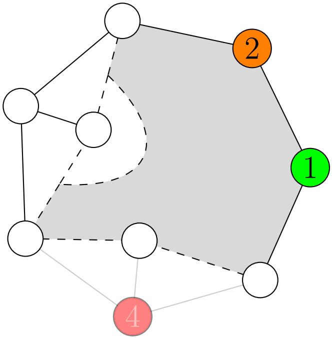


outer face colored by $\{1, 2, 3\}$
up to two precolored vertices (eliminate cut vertices and chords)

Proof Idea

If G is a plane subcubic graph, then $\chi_{\mathrm{fum}}(G) \leq 4$.

Use precoloring extension method (Thomassen's 5-list coloring)
(Nice induction)

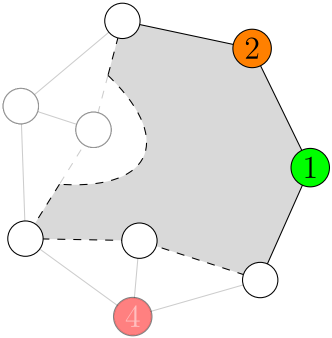


outer face colored by $\{1, 2, 3\}$
up to two precolored vertices (eliminate cut vertices and chords)

Proof Idea

If G is a plane subcubic graph, then $\chi_{\text{fum}}(G) \leq 4$.

Use precoloring extension method (Thomassen's 5-list coloring)
(Nice induction)

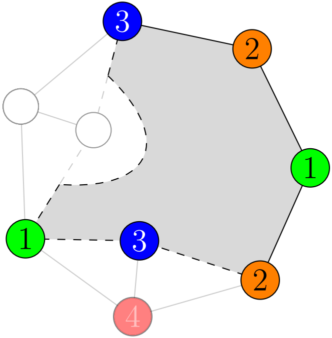


outer face colored by $\{1, 2, 3\}$
up to two precolored vertices (eliminate cut vertices and chords)

Proof Idea

If G is a plane subcubic graph, then $\chi_{\text{fum}}(G) \leq 4$.

Use precoloring extension method (Thomassen's 5-list coloring)
(Nice induction)

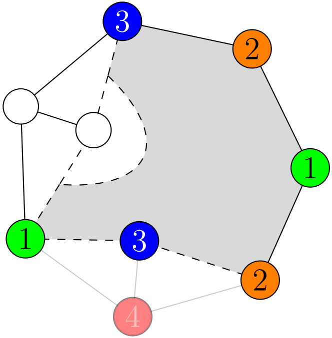


outer face colored by $\{1, 2, 3\}$
up to two precolored vertices (eliminate cut vertices and chords)

Proof Idea

If G is a plane subcubic graph, then $\chi_{\text{fum}}(G) \leq 4$.

Use precoloring extension method (Thomassen's 5-list coloring)
(Nice induction)

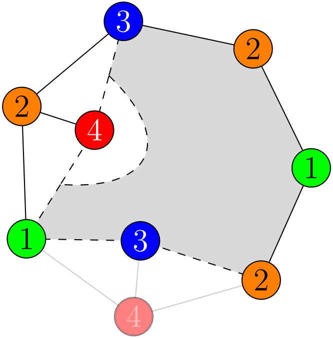


outer face colored by $\{1, 2, 3\}$
up to two precolored vertices (eliminate cut vertices and chords)

Proof Idea

If G is a plane subcubic graph, then $\chi_{\mathrm{fum}}(G) \leq 4$.

Use precoloring extension method (Thomassen's 5-list coloring)
(Nice induction)

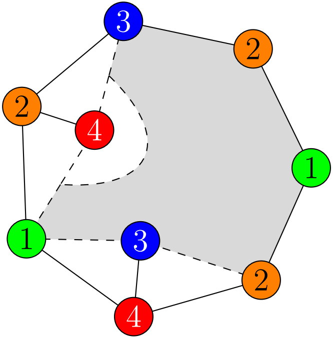


outer face colored by $\{1, 2, 3\}$
up to two precolored vertices (eliminate cut vertices and chords)

Proof Idea

If G is a plane subcubic graph, then $\chi_{\mathrm{fum}}(G) \leq 4$.

Use precoloring extension method (Thomassen's 5-list coloring)
(Nice induction)



outer face colored by $\{1, 2, 3\}$
up to two precolored vertices (eliminate cut vertices and chords)

More Results

Conjecture (Fabrici and Göring)

If G is a plane graph, then $\chi_{\mathrm{fum}}(G) \leq 4$.

Theorem (Andova, L., Lužar, Škrekovski)

If G is a plane subcubic graph, then $\chi_{\mathrm{fum}}(G) \leq 4$.

More Results

Conjecture (Fabrici and Göring)

If G is a plane graph, then $\chi_{\mathrm{fum}}(G) \leq 4$.

Theorem (Andova, L., Lužar, Škrekovski)

If G is a plane subcubic graph, then $\chi_{\mathrm{fum}}(G) \leq 4$.

Theorem (L., Messerschmidt, Škrekovski)

If G is a plane graph, where vertices of degree at least 4 induce a matching, then $\chi_{\mathrm{fum}}(G) \leq 4$.

More Results

Conjecture (Fabrici and Göring)

If G is a plane graph, then $\chi_{\text{fum}}(G) \leq 4$.

Theorem (Andova, L., Lužar, Škrekovski)

If G is a plane subcubic graph, then $\chi_{\text{fum}}(G) \leq 4$.

Theorem (L., Messerschmidt, Škrekovski)

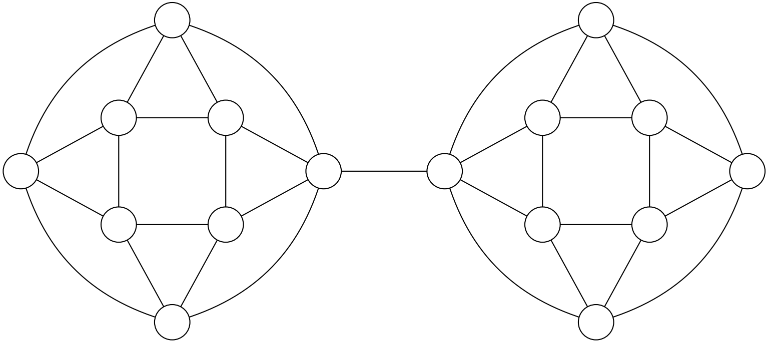
If G is a plane graph, where vertices of degree at least 4 induce a matching, then $\chi_{\text{fum}}(G) \leq 4$.

Theorem (L., Messerschmidt, Škrekovski)

There exists a plane graph G with $\chi_{\text{fum}}(G) = 5$.

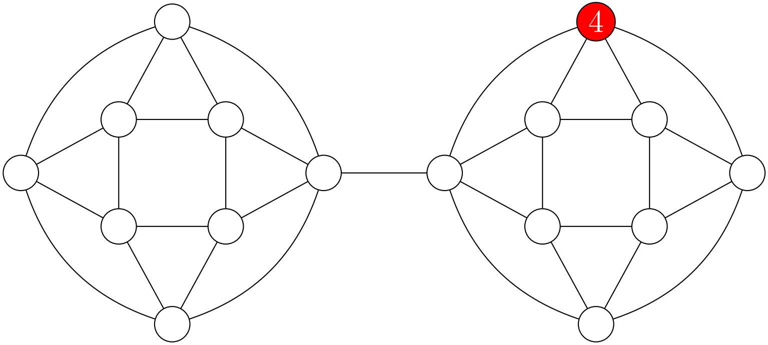
Counterexample

Plane graph with $\chi_{\text{fum}}(G) = 5$.
Suppose for contradiction that $\chi_{\text{fum}}(G) = 4$:



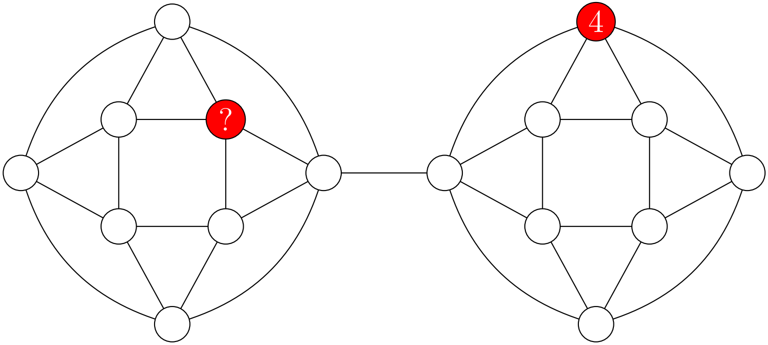
Counterexample

Plane graph with $\chi_{\text{fum}}(G) = 5$.
Suppose for contradiction that $\chi_{\text{fum}}(G) = 4$:



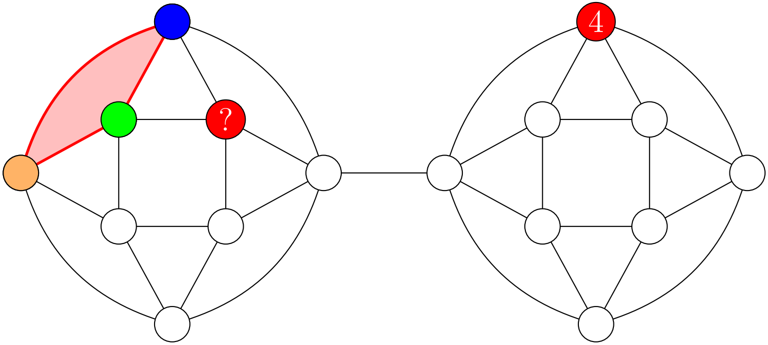
Counterexample

Plane graph with $\chi_{\text{fum}}(G) = 5$.
Suppose for contradiction that $\chi_{\text{fum}}(G) = 4$:



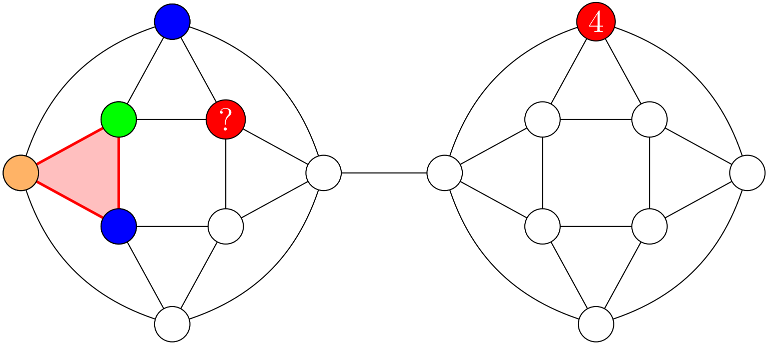
Counterexample

Plane graph with $\chi_{\text{fum}}(G) = 5$.
Suppose for contradiction that $\chi_{\text{fum}}(G) = 4$:



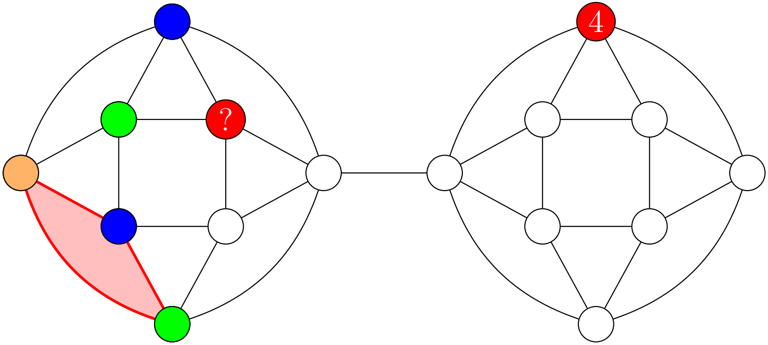
Counterexample

Plane graph with $\chi_{\text{fum}}(G) = 5$.
Suppose for contradiction that $\chi_{\text{fum}}(G) = 4$:



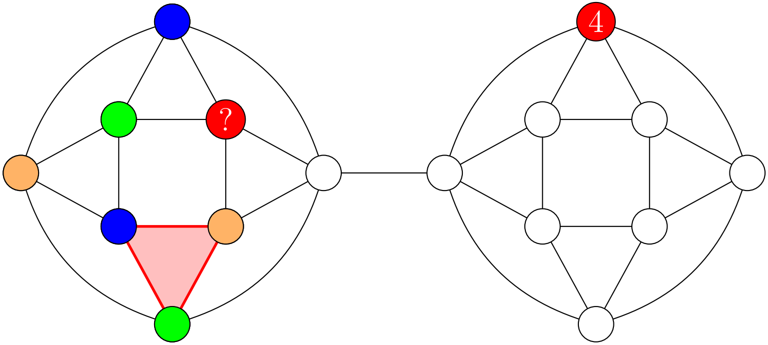
Counterexample

Plane graph with $\chi_{\text{fum}}(G) = 5$.
Suppose for contradiction that $\chi_{\text{fum}}(G) = 4$:



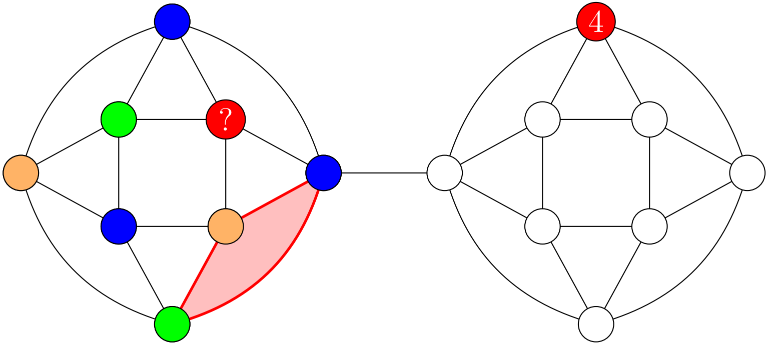
Counterexample

Plane graph with $\chi_{\text{fum}}(G) = 5$.
Suppose for contradiction that $\chi_{\text{fum}}(G) = 4$:



Counterexample

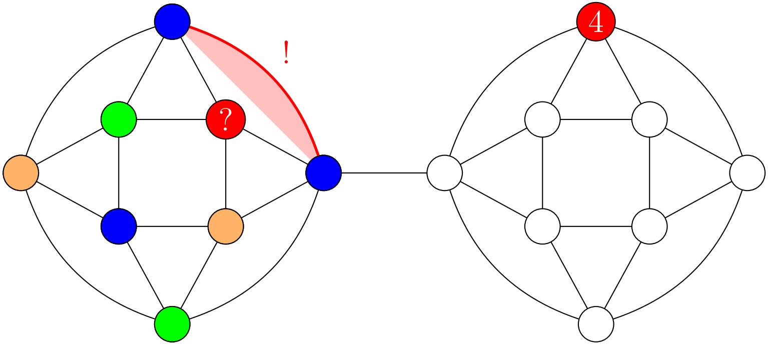
Plane graph with $\chi_{\text{fum}}(G) = 5$.
Suppose for contradiction that $\chi_{\text{fum}}(G) = 4$:



Counterexample

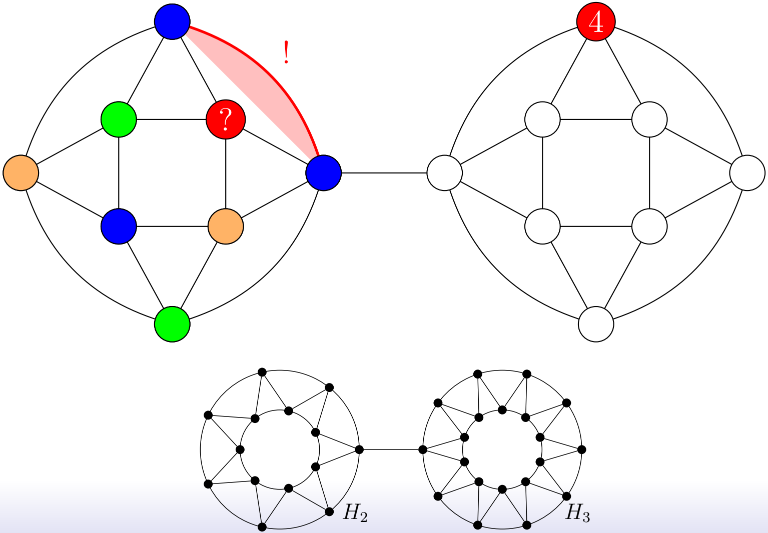
Plane graph with $\chi_{\text{fum}}(G) = 5$.

Suppose for contradiction that $\chi_{\text{fum}}(G) = 4$:



Counterexample

Plane graph with $\chi_{\text{fum}}(G) = 5$.
Suppose for contradiction that $\chi_{\text{fum}}(G) = 4$:



Edge Version

Conjecture (Fabrici, Jendrol', Vrbjarová 2015)

If G is a 2-edge-connected plane graph, then $\chi'_{\mathrm{fum}}(G) \leq 4$.

Warning: $\chi'_{\mathrm{fum}}(G)$ is not usual edge coloring. Only edges that are consecutive in some facial walk need different colors.

Edge Version

Conjecture (Fabrici, Jendrol', Vrbjarová 2015)
If G is a 2-edge-connected plane graph, then $\chi'_{\mathrm{fum}}(G) \leq 4$.

They proved that $\chi'_{\mathrm{fum}}(G) \leq 6$.

Warning: $\chi'_{\mathrm{fum}}(G)$ is not usual edge coloring. Only edges that are consecutive in some facial walk need different colors.

Edge Version

Conjecture (Fabrici, Jendrol', Vrbjarová 2015)
If G is a 2-edge-connected plane graph, then $\chi'_{\mathrm{fum}}(G) \leq 4$.

They proved that $\chi'_{\mathrm{fum}}(G) \leq 6$.

Wendland implies $\chi'_{\mathrm{fum}}(G) \leq 5$.

Warning: $\chi'_{\mathrm{fum}}(G)$ is not usual edge coloring. Only edges that are consecutive in some facial walk need different colors.

Edge Version

Conjecture (Fabrici, Jendrol', Vrbjarová 2015)
If G is a 2-edge-connected plane graph, then $\chi'_{\mathrm{fum}}(G) \leq 4$.

They proved that $\chi'_{\mathrm{fum}}(G) \leq 6$.

Wendland implies $\chi'_{\mathrm{fum}}(G) \leq 5$.

Theorem (Andova, L., Lužar, Škrekovski)
If G is a 2-vertex-connected plane graph, then $\chi'_{\mathrm{fum}}(G) \leq 4$.

Warning: $\chi'_{\mathrm{fum}}(G)$ is not usual edge coloring. Only edges that are consecutive in some facial walk need different colors.

List Version

Theorem (Wendland 2016)

If each vertex of a plane graph is assigned a list of 7 integers, then there exists a FUM-coloring assigning each vertex a color from its list.

List Version

Theorem (Wendland 2016)

If each vertex of a plane graph is assigned a list of 7 *integers, then there exists a FUM-coloring assigning each vertex a color from its list.*

Conjecture (Wendland 2016)

If each vertex of a plane graph is assigned a list of 5 *integers, then there exists a FUM-coloring assigning each vertex a color from its list.*

List Version

Theorem (Wendland 2016)

If each vertex of a plane graph is assigned a list of 7 integers, then there exists a FUM-coloring assigning each vertex a color from its list.

Conjecture (Wendland 2016)

If each vertex of a plane graph is assigned a list of 5 integers, then there exists a FUM-coloring assigning each vertex a color from its list.

Question

If each edge of a plane graph is assigned a list of 4 integers, then there exists a FUM-edge-coloring assigning each edge a color from its list.

More Questions

Problem

Determine k such that $\chi_{\text{fum}}(G) \leq k$ for all G embedded on a surface Σ.

We know $k = 5$ for the sphere.

Problem

If G is a connected plane graph with maximum degree 4, then $\chi_{\text{fum}}(G) \leq 4$.

(our example has degree 5, connected needed)

Problem

How about not requiring uniqueness but at most k occurrences of the largest color?

More Questions

Problem

Determine k such that $\chi_{\text{fum}}(G) \leq k$ for all G embedded on a surface Σ.

We know $k = 5$ for the sphere.

Problem

If G is a connected plane graph with maximum degree 4, then $\chi_{\text{fum}}(G) \leq 4$.

(our example has degree 5, connected needed)

Problem

How about not requiring uniqueness but at most k occurrences of the largest color?

Thank you for your attention!