Decomposing graphs into edges and triangles

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Motivation: Designing experiments.
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Theorem (Erdős, Goodman, Pósa (1966))

The edges of any graph $G$ of order $n$ can be decomposed into at most $\left\lfloor \frac{n^2}{4} \right\rfloor$ cliques.
Some history

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Quick check:
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Quick check:
- No triangles $\rightarrow K_{\frac{n}{2}, \frac{n}{2}}$ can be decomposed into $\frac{n^2}{4}$ edges;
Theorem (Erdős, Goodman, Pósa (1966))

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Quick check:

- No triangles $\rightarrow K_{\frac{n}{2}, \frac{n}{2}}$ can be decomposed into $\frac{n^2}{4}$ edges;
- "All triangles" $\rightarrow K_n$ can be decomposed into $\frac{1}{3} \binom{n}{2} \approx \frac{n^2}{6}$ triangles.
The edges of any graph $G$ of order $n$ can be decomposed into cliques $C_1, \ldots, C_\ell$ with $\sum_i |C_i| \leq \frac{n^2}{2}$.
Some history

Theorem (Chung (1981); Győri, Kostochka (1980))

The edges of any graph $G$ of order $n$ can be decomposed into cliques $C_1, \ldots, C_\ell$ with $\sum_i |C_i| \leq \frac{n^2}{2}$.

\[ 3 \leq \frac{7^2}{2} \]
Theorem (Chung (1981); Győri, Kostochka (1980))

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3 + 3 \leq \frac{7^2}{2}$
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$$2 + 4 + 3 + 3 \leq \frac{7^2}{2}$$
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**Theorem (Győri and Tuza (1987))**

The edges of any graph $G$ of order $n$ can be decomposed into edges and triangles $C_1, \ldots, C_\ell$ with $\sum_i |C_i| \leq \frac{9}{16} n^2$.

**Conjecture (Győri and Tuza (1987))**

The edges of any graph $G$ of order $n$ can be decomposed into edges and triangles $C_1, \ldots, C_\ell$ with $\sum_i |C_i| \leq n^2 + o(n^2)$.

**Cost**

$$\text{cost} = 2\#K_2 + 3\#K_3.$$
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In general:

- Assign cost $c_r$ to a clique $K_r$.
- Minimize $\sum_r c_r \cdot \#K_r$. 

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Cost = $2\#K_2 + 3\#K_3$
Main result 1

**Theorem (Král, L., Martins, Pehova 2019)**

The edges of any graph $G$ of order $n$ can be decomposed into edges and triangles $C_1, \ldots, C_\ell$ with $\sum_i |C_i| \leq \frac{n^2}{2} + o(n^2)$. 

Proof outline

1. Obtain a fractional decomposition into edges and triangles. (flag algebras method)
2. Fractional to full decomposition. (regularity method)
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**Fractional decomposition**

**Definition**

A *decomposition* of a graph $G$ into triangles $\mathcal{T}$ and edges $\mathcal{E}$ is an assignment $w : \mathcal{T} \cup \mathcal{E} \rightarrow \{ 0, 1 \}$ such that for each $e \in E(G)$:

$$\sum_{T \supseteq e} w(T) + \sum_{e \in \mathcal{E}} w(e) = 1.$$ 

$K_4$ doesn't have a triangle decomposition but it has a fractional triangle decomposition.

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$\frac{1}{2} (3 + 3)$

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$\begin{align*}
3 + 2 + 2 + 2 &= 9 \\
\frac{1}{2} (3 + 3 + 3 + 3) &= 6
\end{align*}$

$K_4$ doesn’t have a triangle decomposition but it has a fractional triangle decomposition.
**Finding a fractional decomposition**

**Definition**

For a graph $G$, let

$$\pi_3(G) = \min \text{ cost of a triangle-edge decomposition of } G$$

Our Theorem first step: $\pi_3(G), f(G) \leq \frac{1}{2}n^2 + o(n^2)$
**Finding a fractional decomposition**

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$\pi_3(K_4) = 9$, $\pi_{3,f}(K_4) = 6$
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Our Theorem first step: $\pi_{3,f}(G) \leq \frac{n^2}{2} + o(n^2)$
Finding a fractional decomposition

**Key lemma (using flag algebras)**

Let $G$ be a (large) graph and $W$ be a uniformly chosen random subset of 7 vertices of $G$. Then

$$E[\pi_3, f(G[W])] \leq 21 + o(1).$$
Finding a fractional decomposition

Key lemma (using flag algebras)

Let $G$ be a (large) graph and $W$ be a uniformly chosen random subset of 7 vertices of $G$. Then

$$E[\pi_3,f(G[W])] \leq 21 + o(1).$$

Fractional decomposition of $G$:

- sum optimal decompositions for all $W \in \binom{V}{7}$
- divide by $\binom{n-2}{5}$

Each edge is in $\binom{n-2}{5}$ $W$s so it is a fractional decomposition of $G$. 
Finding a fractional decomposition

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Each edge is in $\binom{n-2}{5}$ $W$s so it is a fractional decomposition of $G$.

$$\pi_{3,f}(G) \leq \frac{1}{\binom{n-2}{5}} \sum_{W} \pi_{3,f}(G[W])$$

$$\leq \frac{1}{\binom{n-2}{5}} \binom{n}{7} (21 + o(1)) = \frac{n^2}{2} + o(n^2).$$
Main result 1

**Theorem (Král, L., Martins, Pehova 2019)**

For any sufficiently large graph $G$, $\pi_3(G) \leq \frac{n^2}{2} + o(n^2)$.

**Proof**

✓ Obtain a fractional decomposition into edges and triangles $\pi_{3,f}(G) \leq \frac{n^2}{2} + o(n^2)$ (flag algebra methods).

\[ \text{cost} = 2\#K_2 + 3\#K_3 \]
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• Fractional to full decomposition (regularity method).
Fractional to full decomposition

Definition

$H$ fixed small graph, $G$ large graph.
**Fractional to full decomposition**

**Definition**

\( H \) fixed small graph, \( G \) large graph.

\[ \nu_H(G) = \text{max size of } H\text{-packing of } G, \]

\[ \nu_{fH}(G) = \text{max size of a fractional } H\text{-packing of } G, \]

\[ \nu_{K_3}(K_4) = 1 \]
**Fractional to full decomposition**

**Definition**

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$v_H(G) = \text{max size of } H\text{-packing of } G,$

$v^f_H(G) = \text{max size of a fractional } H\text{-packing of } G.$

$v_{K_3}(K_4) = 1$ \hspace{1cm} $v^f_{K_3}(K_4) = 2$
**Fractional to Full Decomposition**

**Definition**

$H$ fixed small graph, $G$ large graph.

$\nu_H(G) = \text{max size of } H\text{-packing of } G$, 

$\nu^f_H(G) = \text{max size of a fractional } H\text{-packing of } G$.

$\nu_H(G) \leq \nu^f_H(G)$.

$\nu_{K_3}(K_4) = 1$  $\nu^f_{K_3}(K_4) = 2$
**Definition**

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\[ \nu_H(G) \leq \nu^f_H(G). \]

\[ \nu_{K_3}(K_4) = 1 \]

\[ \nu^f_{K_3}(K_4) = 2 \]

**Theorem (Haxell, Rödl 2001)**

For any graph $H$ and an $n$-vertex graph $G$ we have

\[ \nu^f_H(G) \leq \nu_H(G) + o(n^2). \]
Fractional to full decomposition

Theorem (Haxell, Rödl 2001)

For any graph $H$ and an $n$-vertex graph $G$ we have

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Notice $\pi_3(G) = 2e(G) - 3\nu_{K_3}(G)$
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$\text{cost} = 2\#K_2 + 3\#K_3$
**Fractional to Full Decomposition**

**Theorem (Haxell, Rödl 2001)**

For any graph $H$ and an $n$-vertex graph $G$ we have

$$\nu_H^f(G) \leq \nu_H(G) + o(n^2).$$

Notice $\pi_3(G) = 2e(G) - 3\nu_{K_3}(G) \leq 2e(G) - 3\nu_{K_3}^f(G) + o(n^2)$

**Corollary**

$$\pi_3(G) \leq \pi_{3,f}(G) + o(n^2) \leq \frac{n^2}{2} + o(n^2)$$

$cost = 2\#K_2 + 3\#K_3$
**Fractional to Full Decomposition**

**Theorem (Haxell, Rödl 2001)**

For any graph $H$ and an $n$-vertex graph $G$ we have

$$\nu^f_H(G) \leq \nu_H(G) + o(n^2).$$

Notice $\pi_3(G) = 2e(G) - 3\nu_{K_3}(G) \leq 2e(G) - 3\nu^f_{K_3}(G) + o(n^2)$

**Corollary**

$$\pi_3(G) \leq \pi_{3,f}(G) + o(n^2) \leq \frac{n^2}{2} + o(n^2)$$

**Theorem (Yuster 2004)**

For a fixed family $\mathcal{F}$ of graphs and an $n$ vertex graph $G$ we have

$$\nu^f_{\mathcal{F}}(G) \leq \nu_{\mathcal{F}}(G) + o(n^2).$$

\[\text{cost} = 2\#K_2 + 3\#K_3\]
Recap

Theorem (Král, L., Martins, Pehova 2019)

For any sufficiently large graph $G$

$$\pi_3(G) \leq \frac{n^2}{2} + o(n^2).$$
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What is needed for $o(n^2)$?
**Theorem (Král, L., Martins, Pehova 2019)**

For any sufficiently large graph $G$

$$\pi_3(G) \leq \frac{n^2}{2} + o(n^2).$$

What is needed for $o(n^2)$?

Examples:

$$\pi_3\left( K_{\frac{n}{2}, \frac{n}{2}} \right) = \frac{n^2}{2}$$

$$\pi_3\left( K_{\frac{n-1}{2}, \frac{n+1}{2}} \right) = \frac{n^2-1}{2}$$

cost $= 2\#K_2 + 3\#K_3$
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Theorem (Král, L., Martins, Pehova 2019)

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$$\pi_3\left(K\frac{n-1}{2}, \frac{n+1}{2}\right) = \frac{n^2-1}{2}$$

$$\pi_3(K_n) = ?$$
Decomposing $K_n$

Conditions for triangle decomposition
- every vertex has even degree
- number of edges is divisible by 3
Decomposing $K_n$

Conditions for triangle decomposition
- every vertex has even degree
- number of edges is divisible by 3

$e = 15, d = 5$
$n = 6 \equiv 0$

$e = 21, d = 6$
$n = 7 \equiv 1$

$e = 28, d = 7$
$n = 8 \equiv 2$

$e = 36, d = 8$
$n = 9 \equiv 3$

$e = 45, d = 9$
$n = 10 \equiv 4$

$e = 55, d = 10$
$n = 11 \equiv 5$
Decomposing $K_n$

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\[ e = 51, \ d = 8, 10 \]
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Decomposing $K_n$

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\begin{align*}
e &= 51, \quad d = 8, 10 \\
n &= 11 \equiv 5
\end{align*}
Decomposing $K_n$

Conditions for triangle decomposition
- every vertex has even degree
- number of edges is divisible by 3

\[ e = 12, \quad d = 4 \]
\[ n = 6 \equiv 0 \]

\[ e = 21, \quad d = 6 \]
\[ n = 7 \equiv 1 \]

\[ e = 24, \quad d = 6 \]
\[ n = 8 \equiv 2 \]

\[ e = 36, \quad d = 8 \]
\[ n = 9 \equiv 3 \]

\[ e = 40, \quad d = 8 \]
\[ n = 10 \equiv 4 \]

\[ e = 51, \quad d = 8, 10 \]
\[ n = 11 \equiv 5 \]
Decomposing $K_n$

Conditions for triangle decomposition
- every vertex has even degree
- number of edges is divisible by 3

$e = 12, d = 4$ \hspace{1cm} $e = 21, d = 6$ \hspace{1cm} $e = 24, d = 6$

$n = 6 \equiv 0$ \hspace{1cm} $n = 7 \equiv 1$ \hspace{1cm} $n = 8 \equiv 2$

$e = 36, d = 8$ \hspace{1cm} $e = 40, d = 8$ \hspace{1cm} $e = 51, d = 8, 10$

$n = 9 \equiv 3$ \hspace{1cm} $n = 10 \equiv 4$ \hspace{1cm} $n = 11 \equiv 5$
Decomposing $K_n$

Conditions for triangle decomposition
- every vertex has even degree
- number of edges is divisible by 3

\[
e = \begin{cases} 
12, & d = 4 \\
21, & d = 6 \\
24, & d = 6 \\
36, & d = 8 \\
39, & d = 6, 8 \\
51, & d = 8, 10 
\end{cases}
\]

\[
n = \begin{cases} 
6 \equiv 0 \\
7 \equiv 1 \\
8 \equiv 2 \\
9 \equiv 3 \\
10 \equiv 4 \\
11 \equiv 5 
\end{cases}
\]
Decomposing $K_n$

Conditions for triangle decomposition
- every vertex has even degree
- number of edges is divisible by 3

\[
\begin{align*}
&\text{Conditions: } e = 12, d = 4, \quad n = 6 \equiv 0 \\
&\text{Conditions: } e = 21, d = 6, \quad n = 7 \equiv 1 \\
&\text{Conditions: } e = 24, d = 6, \quad n = 8 \equiv 2 \\
&\text{Conditions: } e = 36, d = 8, \quad n = 9 \equiv 3 \\
&\text{Conditions: } e = 39, d = 6, 8, \quad n = 10 \equiv 4 \\
&\text{Conditions: } e = 51, d = 8, 10, \quad n = 11 \equiv 5
\end{align*}
\]
Decomposing $K_n$

Conditions for triangle decomposition
- every vertex has even degree
- number of edges is divisible by 3

\[ e = 12, \ d = 4 \]
\[ n = 6 \equiv 0 \]
\[ e = 21, \ d = 6 \]
\[ n = 7 \equiv 1 \]
\[ e = 24, \ d = 6 \]
\[ n = 8 \equiv 2 \]
\[ e = 36, \ d = 8 \]
\[ n = 9 \equiv 3 \]
\[ e = 39, \ d = 6, 8 \]
\[ n = 10 \equiv 4 \]
\[ e = 51, \ d = 8, 10 \]
\[ n = 11 \equiv 5 \]
Decomposing $K_n$

Conditions for triangle decomposition
- every vertex has even degree
- number of edges is divisible by 3

\[ e = 12, d = 4 \]
\[ n = 6 \equiv 0 \]
\[ \frac{n^2}{2} \]
\[ n \text{ even} \]

\[ e = 21, d = 6 \]
\[ n = 7 \equiv 1 \]
\[ \binom{n}{2} \]
\[ n \text{ odd} \]

\[ e = 24, d = 6 \]
\[ n = 8 \equiv 2 \]
\[ \frac{n^2}{2} + 1 \]

\[ e = 36, d = 8 \]
\[ n = 9 \equiv 3 \]
\[ \binom{n}{2} + 4 \]

\[ e = 39, d = 6, 8 \]
\[ n = 10 \equiv 4 \]

\[ e = 51, d = 8, 10 \]
\[ n = 11 \equiv 5 \]
Decomposing $K_n$

Conditions for triangle decomposition
- every vertex has even degree
- number of edges is divisible by 3

\[ e = \frac{n^2}{2}, \quad d = 4 \]
\[ n = 6 \equiv 0 \]
\[ e = 12, \quad d = 4 \]
\[ n = 6 \equiv 0 \]

\[ e = 21, \quad d = 6 \]
\[ n = 7 \equiv 1 \]

\[ e = 24, \quad d = 6 \]
\[ n = 8 \equiv 2 \]

\[ e = \frac{n^2}{2} + 1 \]
\[ n = 9 \equiv 3 \]
\[ e = 36, \quad d = 8 \]

\[ e = 39, \quad d = 6, 8 \]
\[ n = 10 \equiv 4 \]
\[ n = 10 \equiv 4 \]
\[ n = 11 \equiv 5 \]

\[ e = 51, \quad d = 8, 10 \]
Theorem (Blumenthal, L., Pikhurko, Pehova, Pfender, Volec)

For sufficiently large \( n \),

\[
\pi_3(G) \leq \begin{cases} 
\frac{n^2}{2} & \text{if } n \equiv 0, 2 \pmod{6} \quad \ldots \quad K_{\frac{n}{2}, \frac{n}{2}} \text{ and } K_n, \\
\frac{n^2-1}{2} & \text{if } n \equiv 1, 3, 5 \pmod{6} \quad \ldots \quad K_{\frac{n-1}{2}, \frac{n+1}{2}}, \\
\frac{n^2}{2} + 1 & \text{if } n \equiv 4 \pmod{6} \quad \ldots \quad K_n.
\end{cases}
\]

Note \( \pi_3(K_5) = 14 > \frac{n^2}{2} + 1. \)

The theorem cannot be extended to all \( n \) without adding exception(s).
If $\pi_3, f(G) \leq (\frac{1}{2} - \varepsilon) n^2$ then $\pi_3(G) < \frac{1}{2} n^2$ by Yuster/Haxell, Rödl.

If $\pi_3, f(G) \geq (\frac{1}{2} - \varepsilon) n^2$, by flag algebra methods the following graphs

$$\mathcal{F} = \left\{ \begin{array}{c}
\begin{array}{c}
\emptyset \\
\end{array}
, \begin{array}{c}
\{0,0,0\}
\end{array}
, \begin{array}{c}
\{0,0,0\}
\end{array}
\end{array} \right\}$$

have density at most $\delta$, where $\delta \to 0$ as $\varepsilon \to 0$. By Induced removal lemma, $G$ is $\mathcal{F}$-free up to $\delta' n^2$ edges.

Hence $G$ is $\begin{array}{c}
\begin{array}{c}
\emptyset \\
\end{array}
\end{array}$ or $\begin{array}{c}
\begin{array}{c}
\{0,0,0\}
\end{array}
\end{array}$ up to $\delta' n^2$ edges.
**Exact Result** $K_{\left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil}$

$G$ is close to $K_{\left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil}$, show $\pi_3(G) \leq \pi_3(K_{\left\lfloor \frac{n}{2} \right\rfloor, \left\lceil \frac{n}{2} \right\rceil})$.

- take maxcut ($|E(G)| \geq \frac{n^2}{4}$)
**Exact Result** $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$

$G$ is close to $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, show $\pi_3(G) \leq \pi_3(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil})$.

- take maxcut ($|E(G)| \geq \frac{n^2}{4}$)
- extra edges, missing edges
Exact result $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$

$G$ is close to $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$, show $\pi_3(G) \leq \pi_3(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil})$.

- take maxcut ($|E(G)| \geq \frac{n^2}{4}$)
- extra edges, missing edges
- high extra degree vertices $U$

Theorem (Gy˝ori 1988)

If $G$ is a graph with $n$ vertices and $\frac{n^2}{4} + k$ edges, where $n \to \infty$ and $k = o(\frac{n^2}{4})$, then it has at least $k - O(k^2/n^2)$ edge-disjoint triangles.
**Exact result** $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$

$G$ is close to $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, show $\pi_3(G) \leq \pi_3(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil})$.

- take maxcut ($|E(G)| \geq \frac{n^2}{4}$)
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- high extra degree vertices $U$
Exact result $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$

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- take maxcut ($|E(G)| \geq n^2/4$)
- extra edges, missing edges
- high extra degree vertices $U$
- triangles with 1 extra edge with $U$
**Exact result** $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$

$G$ is close to $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$, show $\pi_3(G) \leq \pi_3(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil})$.

- take maxcut ($|E(G)| \geq \frac{n^2}{4}$)
- extra edges, missing edges
- high extra degree vertices $U$
- triangles with 1 extra edge with $U$
Exact result $K_{\left\lfloor n/2 \right\rfloor, \left\lceil n/2 \right\rceil}$

$G$ is close to $K_{\left\lfloor n/2 \right\rfloor, \left\lceil n/2 \right\rceil}$, show $\pi_3(G) \leq \pi_3(K_{\left\lfloor n/2 \right\rfloor, \left\lceil n/2 \right\rceil})$.

- take maxcut ($|E(G)| \geq \frac{n^2}{4}$)
- extra edges, missing edges
- high extra degree vertices $U$
- triangles with 1 extra edge with $U$
- other triangles with extra edges
Exact result $K_{\left\lfloor \frac{n}{2} \right\rfloor,\left\lceil \frac{n}{2} \right\rceil}$

$G$ is close to $K_{\left\lfloor \frac{n}{2} \right\rfloor,\left\lceil \frac{n}{2} \right\rceil}$, show $\pi_3(G) \leq \pi_3(K_{\left\lfloor \frac{n}{2} \right\rfloor,\left\lceil \frac{n}{2} \right\rceil})$.

- take maxcut ($|E(G)| \geq \frac{n^2}{4}$)
- extra edges, missing edges
- high extra degree vertices $U$
- triangles with 1 extra edge with $U$
- other triangles with extra edges
- rest taken as $K_2$s
**Exact Result**  $K\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil$

$G$ is close to $K\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil$, show $\pi_3(G) \leq \pi_3(K\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)$.

- take maxcut ($|E(G)| \geq \frac{n^2}{4}$)
- extra edges, missing edges
- high extra degree vertices $U$
- triangles with 1 extra edge with $U$
- other triangles with extra edges
- rest taken as $K_2$s

**Theorem (Győri 1988)**

If $G$ is a graph with $n$ vertices and $\frac{n^2}{4} + k$ edges, where $n \to \infty$ and $k = o(n^2)$, then it has at least $k - O(k^2/n^2)$ edge-disjoint triangles.
Theorem (Barber, Kuhn, Lo, Osthus; Dross; Gustavsson)

Every large graph $G$ on $n$ vertices, where $|E(G)|$ is a multiple of 3 and all vertices have even degree at least $(9/10 + o(1))n$ has a triangle decomposition.
**Exact result** $K_n$

**Theorem (Barber, Kuhn, Lo, Osthus; Dross; Gustavsson)**

Every large graph $G$ on $n$ vertices, where $|E(G)|$ is a multiple of 3 and all vertices have even degree at least $(9/10 + o(1))n$ has a triangle decomposition.

$K_n$ proof overview

- special treatment for low degree vertices
- make all degrees even and $|E(G)|$ divisible by 3
- apply Theorem

Desired conclusion:
If $G$ is close to $K_n$ then $\pi_3(G) \leq \pi_3(K_n)$. 
Exact result $K_n$

$G$ is close to $K_n$ and maximize try to show $\pi_3(G) \leq \pi_3(K_n)$.

- $\delta(G) \geq n/8$
**Exact result** \( K_n \)

\( G \) is close to \( K_n \) and maximize try to show \( \pi_3(G) \leq \pi_3(K_n) \).

- \( \delta(G) \geq n/8 \)
- degree < 0.99n go to \( U \), rest in \( W \)
**Exact Result** $K_n$

$G$ is close to $K_n$ and maximize try to show $\pi_3(G) \leq \pi_3(K_n)$.

- $\delta(G) \geq n/8$
- degree $< 0.99n$ go to $U$, rest in $W$
- $S \subseteq W$ with odd degree in $G$
Exact result \( K_n \)

\( G \) is close to \( K_n \) and maximize try to show \( \pi_3(G) \leq \pi_3(K_n) \).

- \( \delta(G) \geq n/8 \)
- degree < 0.99\( n \) go to \( U \), rest in \( W \)
- \( S \subseteq W \) with odd degree in \( G \)
- make degrees in \( S \) even
**Exact Result** $K_n$

$G$ is close to $K_n$ and maximize try to show $\pi_3(G) \leq \pi_3(K_n)$.

- $\delta(G) \geq n/8$
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Exact result $K_n$

$G$ is close to $K_n$ and maximize try to show $\pi_3(G) \leq \pi_3(K_n)$.

- $\delta(G) \geq n/8$
- degree $< 0.99n$ go to $U$, rest in $W$
- $S \subseteq W$ with odd degree in $G$
- make degrees in $S$ even
- for $u$ in $U$ cover $N_u[W]$ by triangles
Exact result $K_n$

$G$ is close to $K_n$ and maximize try to show $\pi_3(G) \leq \pi_3(K_n)$.

- $\delta(G) \geq n/8$
- degree $< 0.99n$ go to $U$, rest in $W$
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Exact result $K_n$

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- $\delta(G) \geq n/8$
- degree $< 0.99n$ go to $U$, rest in $W$
- $S \subseteq W$ with odd degree in $G$
- make degrees in $S$ even
- for $u$ in $U$ cover $N_u[W]$ by triangles
- cover $U$ edges using $K_2$s.
**Exact result** $K_n$

$G$ is close to $K_n$ and maximize try to show $\pi_3(G) \leq \pi_3(K_n)$.

- $\delta(G) \geq n/8$
- degree $< 0.99n$ go to $U$, rest in $W$
- $S \subseteq W$ with odd degree in $G$
- make degrees in $S$ even
- for $u$ in $U$ cover $N_u[W]$ by triangles
- cover $U$ edges using $K_2$s.
- make rest triangle divisible
**Exact result** $K_n$

$G$ is close to $K_n$ and maximize try to show $\pi_3(G) \leq \pi_3(K_n)$.

- $\delta(G) \geq n/8$
- degree $< 0.99n$ go to $U$, rest in $W$
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- $\delta(G) \geq n/8$
- degree $< 0.99n$ go to $U$, rest in $W$
- $S \subseteq W$ with odd degree in $G$
- make degrees in $S$ even
- for $u$ in $U$ cover $N_u[W]$ by triangles
- cover $U$ edges using $K_2$s.
- make rest triangle divisible

- $G$ is $K_n$ or
- $G$ is $K_n$ without a matching of size 2 (mod 3) and $n \equiv 1, 3 \pmod{6}$. 
$K_n$ WITHOUT A MATCHING

$n \equiv 1, 3 \pmod{6}$ means $K_n$ is triangle divisible

\[ e = 105, \, d = 14 \quad n = 15 \equiv 3 \]
\[ e = 104, \, d = 13, 14 \quad n = 15 \equiv 3 \]
\[ e = 103, \, d = 13, 14 \quad n = 15 \equiv 3 \]

$\pi_3(K_n) =$
$\pi_3(K_n^-) =$
$\pi_3(K_n^-) =$

\[ \text{cost} = 2\#K_2 + 3\#K_3 \]
$K_n$ WITHOUT A MATCHING

$n \equiv 1, 3 \pmod{6}$ means $K_n$ is triangle divisible

\[
e = 105, \quad d = 14 \quad n = 15 \equiv 3
\]

\[
e = 104, \quad d = 13, 14 \quad n = 15 \equiv 3
\]

\[
e = 103, \quad d = 13, 14 \quad n = 15 \equiv 3
\]

\[
\pi_3(K_n) = \binom{n}{2}
\]

\[
\pi_3(K_n^-) =
\]

\[
\pi_3(K_n^\sim) =
\]

\[
\text{cost} = 2\#K_2 + 3\#K_3
\]
$K_n$ WITHOUT A MATCHING

\[ n \equiv 1, 3 \pmod{6} \text{ means } K_n \text{ is triangle divisible} \]

\[ e = 105, d = 14 \quad n = 15 \equiv 3 \]

\[ e = 102, d = 12, 14 \quad n = 15 \equiv 3 \]

\[ e = 103, d = 13, 14 \quad n = 15 \equiv 3 \]

\[ \pi_3(K_n) = \binom{n}{2} \]

\[ \pi_3(K_n^-) = \]

\[ \pi_3(K_n^-) = \]

\[ \text{cost} = 2\#K_2 + 3\#K_3 \]
$K_n$ WITHOUT A MATCHING

$n \equiv 1, 3 \pmod{6}$ means $K_n$ is triangle divisible

$\pi_3(K_n) = \binom{n}{2}$

$\pi_3(K_n^-) = \pi_3(K_n^-) = \pi_3(K_n^-) =$

$\text{cost} = 2\#K_2 + 3\#K_3$
$K_n$ WITHOUT A MATCHING

$n \equiv 1, 3 \pmod{6}$ means $K_n$ is triangle divisible

\[
e = 105, \ d = 14 \quad n = 15 \equiv 3
\]

\[
e = 102, \ d = 12, 14 \quad n = 15 \equiv 3
\]

\[
e = 103, \ d = 13, 14 \quad n = 15 \equiv 3
\]

$\pi_3(K_n) = \binom{n}{2}$

$\pi_3(K_n^-) = \binom{n}{2} + 1$

$\pi_3(K_n^-) =$

\[\text{cost} = 2\#K_2 + 3\#K_3\]
\( K_n \) WITHOUT A MATCHING

\( n \equiv 1, 3 \pmod{6} \) means \( K_n \) is triangle divisible

\[
e = 105, \ d = 14 \\
n = 15 \equiv 3
\]

\[
e = 102, \ d = 12, 14 \\
n = 15 \equiv 3
\]

\[
e = 101, \ d = 12, 14 \\
n = 15 \equiv 3
\]

\[
\pi_3(K_n) = \binom{n}{2} \\
\pi_3(K_n^-) = \binom{n}{2} + 1 \\
\pi_3(K_n^+) = \\
\]

\[
\text{cost} = 2\#K_2 + 3\#K_3
\]
\( K_n \) WITHOUT A MATCHING

\( n \equiv 1, 3 \pmod{6} \) means \( K_n \) is triangle divisible

\[
\binom{n}{2}, \quad \binom{n}{2} + 1, \quad \binom{n}{2} + 2
\]

\( e = 105, d = 14 \)
\( n = 15 \equiv 3 \)

\( e = 102, d = 12, 14 \)
\( n = 15 \equiv 3 \)

\( e = 99, d = 12, 14 \)
\( n = 15 \equiv 3 \)

\[
\pi_3(K_n) = \binom{n}{2}, \quad \pi_3(K_n^-) = \binom{n}{2} + 1, \quad \pi_3(K_n^-) =
\]

\( \text{cost} = 2\#K_2 + 3\#K_3 \)
$K_n$ WITHOUT A MATCHING

$n \equiv 1, 3 \pmod{6}$ means $K_n$ is triangle divisible

\[
\pi_3(K_n) = \binom{n}{2}
\]
\[
\pi_3(K_n^-) = \binom{n}{2} + 1
\]
\[
\pi_3(K_n^-) =
\]

\[
e = 105, d = 14
n = 15 \equiv 3
\]
\[
e = 102, d = 12, 14
n = 15 \equiv 3
\]
\[
e = 99, d = 12, 14
n = 15 \equiv 3
\]

\[
\text{cost} = 2\#K_2 + 3\#K_3
\]
\( K_n \) WITHOUT A MATCHING

\( n \equiv 1, 3 \pmod{6} \) means \( K_n \) is triangle divisible

\[ e = 105, \ d = 14 \]
\[ n = 15 \equiv 3 \]

\[ \pi_3(K_n) = \binom{n}{2} \]

\[ \pi_3(K_n^-) = \binom{n}{2} + 1 \]

\[ \pi_3(K_n^-) = \binom{n}{2} + 2 \]

\[ \text{cost} = 2\#K_2 + 3\#K_3 \]
$K_n$ WITHOUT A MATCHING

$n \equiv 1, 3 \pmod{6}$ means $K_n$ is triangle divisible

\[
\begin{align*}
\binom{n}{2} & = 105, \quad d = 14, \\
\binom{n}{2} + 1 & = 102, \quad d = 12, 14, \\
\binom{n}{2} + 2 & = 99, \quad d = 12, 14
\end{align*}
\]

\[
\begin{align*}
\pi_3(K_n) & = \binom{n}{2}, \\
\pi_3(K_n^-) & = \binom{n}{2} + 1, \\
\pi_3(K_n^+) & = \binom{n}{2} + 2
\end{align*}
\]

\[\text{cost} = 2\#K_2 + 3\#K_3\]
**K\(_n\) WITHOUT A MATCHING**

\(n \equiv 1, 3 \mod 6\) means \(K\(_n\)\) is triangle divisible

\[
\begin{align*}
\binom{n}{2} & : e = 105, d = 14 \\
\binom{n}{2} + 1 & : e = 102, d = 12, 14 \\
\binom{n}{2} + 2 & : e = 99, d = 12, 14
\end{align*}
\]

\(n = 15 \equiv 3\)

\(\pi_3(K\(_n\)) = \binom{n}{2}\)

\(\pi_3(K\(_n^\sim\)) = \binom{n}{2} + 1\)

\(\pi_3(K\(_n^\sim\)) = \binom{n}{2} + 2\)

\[
\begin{align*}
e & = 102, d = 13, 14 \\
n & = 15 \equiv 3
\end{align*}
\]
$K_n$ WITHOUT A MATCHING

$n \equiv 1, 3 \pmod{6}$ means $K_n$ is triangle divisible

\[
\binom{n}{2} \equiv 3 \pmod{3}
\]

\[e = 105, d = 14\]
\[n = 15 \equiv 3\]

\[\pi_3(K_n) = \binom{n}{2}\]
\[\pi_3(K^-_n) = \binom{n}{2} + 1\]
\[\pi_3(K^+_n) = \binom{n}{2} + 2\]

\[e = 102, d = 12, 14\]
\[n = 15 \equiv 3\]

\[e = 99, d = 12, 14\]
\[n = 15 \equiv 3\]

\[\text{cost} = 2#K_2 + 3#K_3\]
\( K_n \) WITHOUT A MATCHING

\( n \equiv 1, 3 \pmod{6} \) means \( K_n \) is triangle divisible

\[
\begin{align*}
\binom{n}{2} & = 105, \; d = 14 \\
n & = 15 \equiv 3 \\
\pi_3(K_n) & = \binom{n}{2} \\
\pi_3(K_n^-) & = \binom{n}{2} + 1 \\
\pi_3(K_n^-) & = \binom{n}{2} + 2
\end{align*}
\]

\[
\begin{align*}
\binom{n}{2} + 1 & = 102, \; d = 12, 14 \\
n & = 15 \equiv 3 \\
\pi_3(K_n -) & = \binom{n}{2} + 1 \\
\pi_3(K_n^-) & = \binom{n}{2} + 2 \\
\end{align*}
\]

\[
\begin{align*}
\binom{n}{2} + 2 & = 99, \; d = 12, 14 \\
n & = 15 \equiv 3 \\
\pi_3(K_n^-) & = \binom{n}{2} + 2
\end{align*}
\]

\[\text{cost} = 2\#K_2 + 3\#K_3\]

\[
\begin{align*}
e & = 99, \; d = 12, 14 \\
n & = 15 \equiv 3
\end{align*}
\]
$K_n$ WITHOUT A MATCHING

$n \equiv 1, 3 \pmod{6}$ means $K_n$ is triangle divisible

\[
\binom{n}{2} \quad \binom{n}{2} + 1 \quad \binom{n}{2} + 2
\]

\[
e = 105, d = 14 \\
n = 15 \equiv 3
\]

\[
e = 102, d = 12, 14 \\
n = 15 \equiv 3
\]

\[
e = 99, d = 12, 14 \\
n = 15 \equiv 3
\]

\[
\pi_3(K_n) = \binom{n}{2} \\
\pi_3(K_n^-) = \binom{n}{2} + 1 \\
\pi_3(K_n^+) = \binom{n}{2} + 2
\]

\[
e = 99, d = 12, 14 \\
n = 15 \equiv 3
\]

\[
\text{cost} = 2#K_2 + 3#K_3
\]
THEOREM (Blumenthal, L., Pikhurko, Pehova, Pfender, Volec)

For sufficiently large \( n \),

\[
\pi_3(G) \leq \begin{cases} 
\frac{n^2}{2} & \text{if } n \equiv 0, 2 \mod 6 \ldots K\frac{n}{2}, \frac{n}{2} \text{ and } K_n, \\
\frac{n^2-1}{2} & \text{if } n \equiv 1, 3, 5 \mod 6 \ldots K\frac{n-1}{2}, \frac{n+1}{2}, \\
\frac{n^2}{2} + 1 & \text{if } n \equiv 4 \mod 6 \ldots K_n.
\end{cases}
\]

Can we find a better upper bound on \( \pi_3(G) \) if we also know the edge-density of \( G \)?
Possible values of $\pi_3(G)$

\[
\pi_3(G)/n^2
\]

\[
\frac{e(G)}{\binom{n}{2}}
\]

Only numerical upper bound. Not clear how the curve looks like. Can you find a lower bound?
Possible values of $\pi_3(G)$

\[
\pi_3(G)/n^2
\]

Only numerical upper bound. Not clear how the curve looks like. Can you find a lower bound?
Possible values of $\pi_3(G)$

$\pi_3(G)/n^2$ vs. $e(G)/(\binom{n}{2})$
Possible values of $\pi_3(G)$

\[
\frac{\pi_3(G)}{n^2}
\]

Only numerical upper bound. Not clear how the curve looks like. Can you find a lower bound?
Possible values of $\pi_3(G)$

Flag algebras on 4 vertices
Possible values of $\pi_3(G)$

Flag algebras on 5 vertices
Possible values of $\pi_3(G)$

Flag algebras on 6 vertices
POSSIBLE VALUES OF $\pi_3(G)$

Flag algebras on 7 vertices
Possible values of $\pi_3(G)$

Flag algebras on 8 vertices
Possible values of $\pi_3(G)$

Flag algebras on 8 vertices
Flag algebras on 8 vertices
Only numerical upper bound. Not clear how the curve looks like. Can you find a lower bound?
Extending the result

\[ \pi_3(G) := 2\#K_2 + 3\#K_3 = 2e(G) - 3\nu_{K_3}(G) \]

Recall \( \nu_{K_3}(G) \) is a size of a maximum triangle packing.
Extending the result

π₃(G) := 2#K₂ + 3#K₃ = 2e(G) − 3ν₃(G)

π₃^α(G) := 2#K₂ + α#K₃ = 2e(G) − (6 − α)ν₃(G)

Recall ν₃(G) is a size of a maximum triangle packing.
Extending the result

\[ \pi_3(G) := 2 \# K_2 + 3 \# K_3 = 2e(G) - 3\nu_{K_3}(G) \]

\[ \pi^\alpha_3(G) := 2 \# K_2 + \alpha \# K_3 = 2e(G) - (6 - \alpha)\nu_{K_3}(G) \]

Notice \( \pi_3^\geq 6(G) = 2e(G) \).

Recall \( \nu_{K_3}(G) \) is a size of a maximum triangle packing.
Extending the result

\[ \pi_3(G) := 2\#K_2 + 3\#K_3 = 2e(G) - 3\nu_{K_3}(G) \]

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Notice \( \pi_3^\geq 6(G) = 2e(G) \).

What if \( \alpha < 6 \)? Which graphs maximize \( \pi_3^\alpha \)?

Recall \( \nu_{K_3}(G) \) is a size of a maximum triangle packing.
Extending the result

\[ \pi_3(G) := 2\#K_2 + 3\#K_3 = 2e(G) - 3\nu_{K_3}(G) \]

\[ \pi_3^{\alpha}(G) := 2\#K_2 + \alpha\#K_3 = 2e(G) - (6 - \alpha)\nu_{K_3}(G) \]

Notice \( \pi_3^{>6}(G) = 2e(G). \)

What if \( \alpha < 6? \) Which graphs maximize \( \pi_3^{\alpha}? \)

We solved \( \pi_3^{\alpha} \) for \( \alpha = 3. \)

Recall \( \nu_{K_3}(G) \) is a size of a maximum triangle packing.
Extending the result

\[ \pi_3(G) := 2\#K_2 + 3\#K_3 = 2e(G) - 3\nu_{K_3}(G) \]

\[ \pi_3^\alpha(G) := 2\#K_2 + \alpha\#K_3 = 2e(G) - (6 - \alpha)\nu_{K_3}(G) \]

**Observation**

If \( \pi_3^\alpha(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}) > \pi_3^\alpha(G) \) then \( \pi_3^\beta(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}) > \pi_3^\beta(G) \) for all \( \alpha > \beta \).
Extending the result

\[
\pi_3(G) := 2\#K_2 + 3\#K_3 = 2e(G) - 3\nu_{K_3}(G)
\]

\[
\pi_3^\alpha(G) := 2\#K_2 + \alpha\#K_3 = 2e(G) - (6 - \alpha)\nu_{K_3}(G)
\]

Observation

If \( \pi_3^\alpha(K_{\lfloor \frac{n}{2} \rfloor}, \lceil \frac{n}{2} \rceil) > \pi_3^\alpha(G) \) then \( \pi_3^\beta(K_{\lfloor \frac{n}{2} \rfloor}, \lceil \frac{n}{2} \rceil) > \pi_3^\beta(G) \) for all \( \alpha > \beta \).

Conclusion

For every \( \alpha < 3 \), sufficiently large \( n \) and every graph \( G \) on \( n \) vertices

\[
\pi_3^\alpha(G) \leq \pi_3^\alpha(K_{\lfloor \frac{n}{2} \rfloor}, \lceil \frac{n}{2} \rceil).
\]
Extending the result

\[ \pi_3(G) := 2\#K_2 + 3\#K_3 = 2e(G) - 3\nu_{K_3}(G) \]

\[ \pi_3^\alpha(G) := 2\#K_2 + \alpha\#K_3 = 2e(G) - (6 - \alpha)\nu_{K_3}(G) \]

Observation

If \( \pi_3^\alpha(K_n) > \pi_3^\alpha(G) \) then \( \pi_3^\beta(K_n) > \pi_3^\beta(G) \) for all \( \alpha < \beta \).
**Extending the result**

\[ \pi_3(G) := 2\#K_2 + 3\#K_3 = 2e(G) - 3\nu_{K_3}(G) \]

\[ \pi_3^\alpha(G) := 2\#K_2 + \alpha\#K_3 = 2e(G) - (6 - \alpha)\nu_{K_3}(G) \]

**Observation**
If \( \pi_3^\alpha(K_n) > \pi_3^\alpha(G) \) then \( \pi_3^\beta(K_n) > \pi_3^\beta(G) \) for all \( \alpha < \beta \).

**Conclusion**
For every \( \alpha > 3 \), sufficiently large \( n \) and every graph \( G \) on \( n \) vertices

\[ \pi_3^\alpha(G) \leq \pi_3^\alpha(K_n) \]

unless \( n \equiv 1, 3 \pmod{6} \). The exception is if \( 3 < \alpha < 4 \), then

\[ \pi_3^\alpha(G) \leq \pi_3^\alpha(K_n^\equiv) \].
If $\alpha > 4$ then $\pi_3^\alpha(G) < \pi_3^\alpha(K_n)$ for all $n$.

Let $G'$ be obtained from $G$ by adding an edge.

$$\alpha \geq 4$$

$$\pi_3^\alpha(G')$$
If $\alpha > 4$ then $\pi_3^\alpha(G) < \pi_3^\alpha(K_n)$ for all $n$. Let $G'$ be obtained from $G$ by adding an edge.
If $\alpha > 4$ then $\pi_3^\alpha(G) < \pi_3^\alpha(K_n)$ for all $n$.
Let $G'$ be obtained from $G$ by adding an edge.
If $\alpha > 4$ then $\pi_3^\alpha(G) < \pi_3^\alpha(K_n)$ for all $n$.
Let $G'$ be obtained from $G$ by adding an edge.

$\pi_3^\alpha(G')$

$\pi_3^\alpha(G') + 2$

$\pi_3^\alpha(G')$

$\alpha \geq 4$

cost = $2\#K_2 + \alpha\#K_3$
$\alpha \geq 4$

If $\alpha > 4$ then $\pi_3^\alpha(G) < \pi_3^\alpha(K_n)$ for all $n$.
Let $G'$ be obtained from $G$ by adding an edge.

\[
\begin{align*}
\pi_3^\alpha(G) & \quad \pi_3^\alpha(G') + 2 \\
\pi_3^\alpha(G') & \quad \pi_3^\alpha(G') - 4 + \alpha
\end{align*}
\]

$\text{cost} = 2\#K_2 + \alpha\#K_3$
\( \alpha \geq 4 \)

If \( \alpha > 4 \) then \( \pi_3^\alpha(G) < \pi_3^\alpha(K_n) \) for all \( n \).
Let \( G' \) be obtained from \( G \) by adding an edge.

If \( \alpha = 4 \), then maximizers are

\[
K_n, K_n^-, K_n^=\]

\[\text{cost} = 2\#K_2 + \alpha\#K_3\]
**Theorem (Blumenthal, L., Pikhurko, Pehova, Pfender, Volec)**

For every $\alpha \geq 0$ exists $n_0$ such that for all graphs $G$ on $n > n_0$ the following cases hold.

- If $\alpha < 3$ then $\pi_3^\alpha(G) \leq \pi_3^\alpha(K_{\lfloor n/2 \rfloor}, \lceil n/2 \rceil)$.
- If $\alpha = 3$ then $\pi_3^\alpha(G) \leq \max\{\pi_3^\alpha(K_n), \pi_3^\alpha(K_{\lfloor n/2 \rfloor}, \lceil n/2 \rceil)\}$
- If $3 < \alpha < 4$, then $\pi_3^\alpha(G) \leq \max\{\pi_3^\alpha(K_n), \pi_3^\alpha(K_n^-)\}$.
- If $\alpha = 4$, then $\pi_3^\alpha(G) \leq \max\{\pi_3^\alpha(K_n), \pi_3^\alpha(K_n^-), \pi_3^\alpha(K_n^-^-)\}$.
- If $4 < \alpha$, then $\pi_3^\alpha(G) \leq \pi_3^\alpha(K_n)$.

Moreover, these are the only possible extremal examples.
Problem (Erdős)

Assuming that each complete subgraph $K_i$ has weight $i - 1$ ($i = 2, 3, \ldots$), prove that every graph $G$ on $n$ vertices admits a partition into $K_i$s of total weight $w(G)$ at most $n^2/4 = 0.25n^2$. 
**Problem (Erdős)**

Assuming that each complete subgraph $K_i$ has weight $i - 1$ ($i = 2, 3, \ldots$), prove that every graph $G$ on $n$ vertices admits a partition into $K_i$s of total weight $w(G)$ at most $n^2/4 = 0.25n^2$. Using only $K_2$ and $K_3$ gives $w(K_n) \approx n^2/3$. 
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Using only $K_2$ and $K_3$ gives $w(K_n) \approx n^2 / 3$.
Flag algebras using $K_2, K_3, K_4$ give upper bound of $0.27256n^2$. 

\[
\frac{w(G)}{n^2} \leq \frac{1}{4} \\
\frac{e(G)}{\binom{n}{2}} \leq 1 
\]
Problem (Erdős)

Assuming that each complete subgraph $K_i$ has weight $i - 1$ ($i = 2, 3, \ldots$), prove that every graph $G$ on $n$ vertices admits a partition into $K_i$s of total weight $w(G)$ at most $n^2/4 = 0.25n^2$.

Using only $K_2$ and $K_3$ gives $w(K_n) \approx n^2/3$.

Flag algebras using $K_2, K_3, K_4$ give upper bound of $0.27256n^2$.

Flag algebras using $K_2, \ldots, K_7$ give upper bound of $0.27256n^2$. 

\[
\begin{align*}
\frac{w(G)}{n^2} & \quad K_2, \ldots, K_7 \\
\frac{1}{4} & \quad \frac{1}{2} \\
& \quad e(G) / \binom{n}{2}
\end{align*}
\]
Why Flag Algebras might struggle?

**Problem (Simpler)**

Assuming that each complete subgraph $K_i$ has weight $i - 1$ ($i = 2, 3, \ldots$), prove that every $K_4$-free graph $G$ on $n$ vertices admits a partition into $K_i$ of total weight $w(G)$ at most $n^2/4 = 0.25n^2$.

$$w(G)/n^2$$

$$w(G) := \#K_2 + 2\#K_3$$
Why Flag Algebras might struggle?

Problem (Simpler)

Assuming that each complete subgraph $K_i$ has weight $i - 1$ ($i = 2, 3, \ldots$), prove that every $K_4$-free graph $G$ on $n$ vertices admits a partition into $K_i$ of total weight $w(G)$ at most $n^2/4 = 0.25n^2$.

$$w(G)/n^2$$

$$w(G) := \#K_2 + 2\#K_3$$
**Why Flag Algebras might struggle?**

Complete 3-partite graph $T_3(n)$

Since $T_3(n)$ is (almost) triangle decomposable,

$$w(T_3(n)) \approx \frac{2}{9} n^2 \approx 0.22222n^2$$
Why Flag Algebras might struggle?

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Flag algebras approach:

For every $X \in \binom{V}{3}$ take $\frac{w_f(G[X])}{\binom{n-2}{1}}$ decomposition and sum these.
Why Flag Algebras might struggle?

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Expected cost of decomposition on 3 vertices

$$w(G) := \#K_2 + 2\#K_3$$
**Why Flag Algebras might struggle?**

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For every $X \in \binom{V}{3}$ take $\frac{w_f(G[X])}{\binom{n-2}{1}}$ decomposition and sum these.

Expected cost of decomposition on 3 vertices

$$0 \quad \frac{1}{9} \quad + \quad 1 \quad 0 \quad + \quad 2 \quad \frac{6}{9} \quad + \quad 2 \quad \frac{2}{9} \quad = \quad 16/9$$

$$w(G) := \#K_2 + 2\#K_3$$
**Why Flag Algebras might struggle?**

Complete 3-partite graph $T_3(n)$

Since $T_3(n)$ is (almost) triangle decomposable,

$$w(T_3(n)) \approx \frac{2}{9}n^2 \approx 0.22222n^2$$

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For every $X \in \binom{V}{3}$ take $\frac{w_f(G[X])}{\binom{n-2}{1}}$ decomposition and sum these.

Expected cost of decomposition on 3 vertices

\[
\begin{align*}
0 & + \frac{1}{9} & + 1 & \cdot & + 2 & \frac{6}{9} & \cdot & + 2 & \frac{2}{9} \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{align*}
\]

This gives

$$w(T_3(n)) \leq 0.2963n^2$$

Conclusion: Our formulation using FA is not strong enough.
Problem (Pyber 1991)

Can the edge set of every \( n \)-vertex graph be covered with triangles of weight 3 and edges of weight 2 such that their total weight is at most \( \left\lfloor \frac{n^2}{2} \right\rfloor \)?
**Problem (Pyber 1991)**

Can the edge set of every $n$-vertex graph be covered with triangles of weight 3 and edges of weight 2 such that their total weight is at most $\left\lfloor \frac{n^2}{2} \right\rfloor$?

**Theorem (Blumenthal, L., Pikhurko, Pehova, Pfender, Volec)**

For sufficiently large $n$,

$$\pi_3(G) \leq \begin{cases} 
\frac{n^2}{2} & \text{if } n \equiv 0, 2 \mod 6 \quad \ldots K_{\frac{n}{2}, \frac{n}{2}} \text{ and } K_n, \\
\frac{n^2-1}{2} & \text{if } n \equiv 1, 3, 5 \mod 6 \quad \ldots K_{\frac{n-1}{2}, \frac{n+1}{2}}, \\
\frac{n^2}{2} + 1 & \text{if } n \equiv 4 \mod 6 \quad \ldots K_n.
\end{cases}$$

\[\text{cost} = 2\#K_2 + 3\#K_3\]
Pyber’s problem

\[ n = 6k + 4, \text{ find a covering of } G = K_n \text{ of cost } \leq \frac{n^2}{2}. \]

Triangle decomposition: even degrees, \(|E(G)|\) divisible by 3

\[ d(v) = 6k + 3 \]
\[ e(G) = 3(6k^2 + 7k + 2) \]

\[ \text{cost} = 2K_2 + 3K_3 \]
Pyber's problem

\( n = 6k + 4 \), find a covering of \( G = K_n \) of cost \( \leq \frac{n^2}{2} \).

Triangle decomposition: even degrees, \(|E(G)|\) divisible by 3

\[
d(v) = 6k + 2
\]
\[
e(G) = 3(6k^2 + 7k + 2) - 3k - 2
\]

\[\text{cost} = 2\#K_2 + 3\#K_3\]
Pyber’s problem

\[ n = 6k + 4, \] find a covering of \( G = K_n \) of cost \( \leq \frac{n^2}{2} \).

Triangle decomposition: even degrees, \( |E(G)| \) divisible by 3

\[ d(v) = 6k+2 \text{ or } 6k \]
\[ e(G) = 3(6k^2 + 7k + 2) - 3k - 3 \]
\[ cost = \frac{n^2}{2} + 1 \]
Pyber’s problem

\[ n = 6k + 4, \text{ find a covering of } G = K_n \text{ of cost } \leq \frac{n^2}{2}. \]

Triangle decomposition: even degrees, \(|E(G)|\) divisible by 3

\[ d(v) = 6k+2 \text{ or } 6k \]
\[ e(G) = 3(6k^2 + 7k + 2) - 3k - 3 \]
\[ cost = \frac{n^2}{2} + 1 - 4 + 3 \]
Pyber’s problem

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Triangle decomposition: even degrees, \(|E(G)|\) divisible by 3

\[ d(v) = 6k + 2 \text{ or } 6k \]
\[ e(G) = 3(6k^2 + 7k + 2) - 3k - 3 \]
\[ cost = \frac{n^2}{2} + 1 - 4 + 3 \]

Thank you for your attention