# Decomposing graphs into edges and triangles

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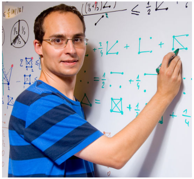
Florian Pfender Jan Volec

Atlanta Lecture Series
March 30, 2019

Use Acroread, no smooths

Adam Blumenthal



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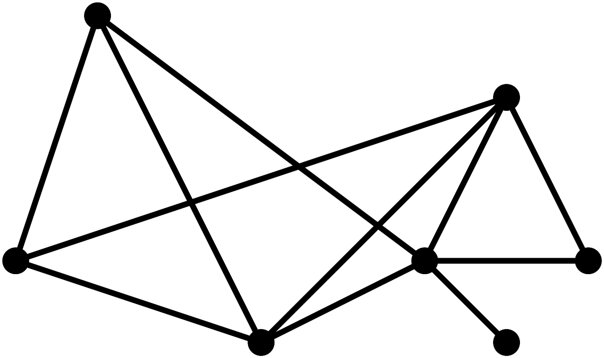

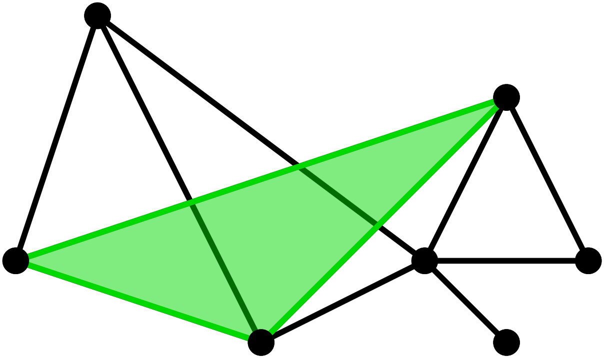

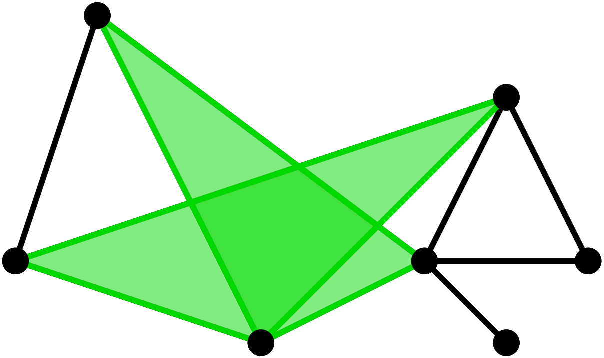

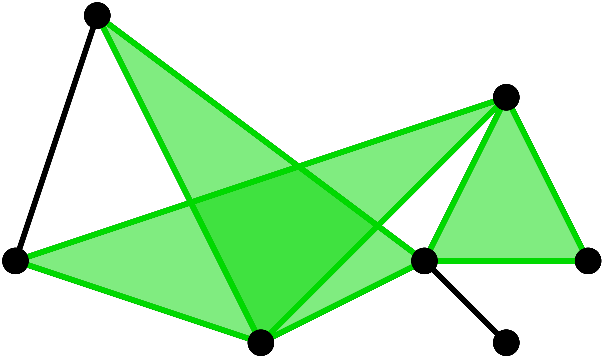

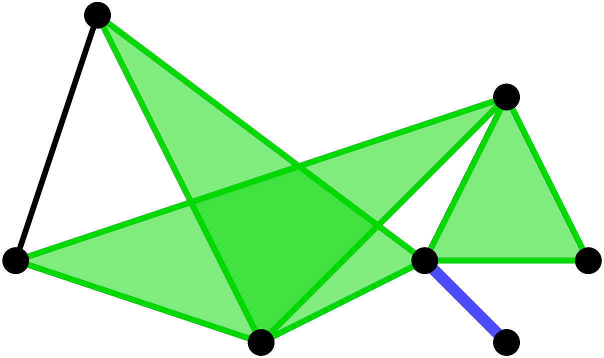

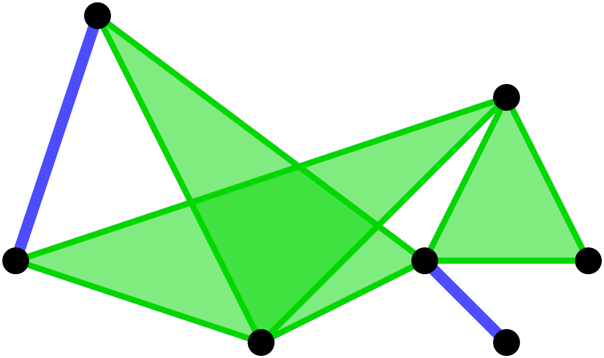

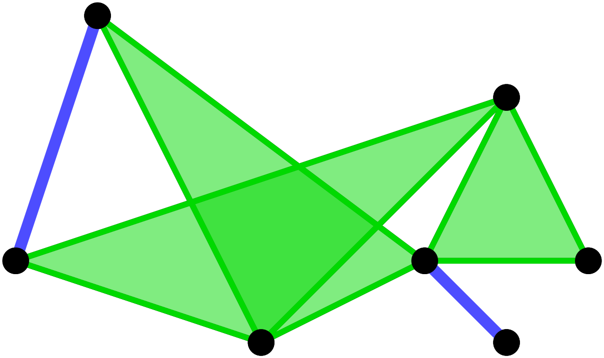

Motivation: Designing experiments.

# Some history

**Theorem (Erdős, Goodman, Pósa (1966))**

*The edges of any graph $G$ of order $n$ can be decomposed into at most $\lfloor \frac{n^2}{4} \rfloor$ cliques.*

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Quick check:

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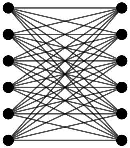
Quick check:

- No triangles $\rightarrow$ $K_{\frac{n}{2},\frac{n}{2}}$ can be decomposed into $\frac{n^2}{4}$ edges;



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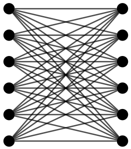
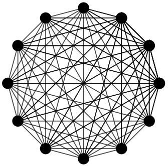
- No triangles $\rightarrow K_{\frac{n}{2},\frac{n}{2}}$ can be decomposed into $\frac{n^2}{4}$ edges;
- "All triangles" $\rightarrow K_n$ can be decomposed into $\frac{1}{3}\binom{n}{2} \approx \frac{n^2}{6}$ triangles.





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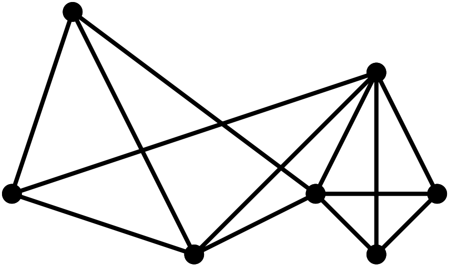
Theorem (Chung (1981); Győri, Kostochka (1980))

*The edges of any graph* $G$ *of order* $n$ *can be decomposed into cliques* $C_1, \ldots, C_\ell$ *with* $\sum_i |C_i| \leq \frac{n^2}{2}$.



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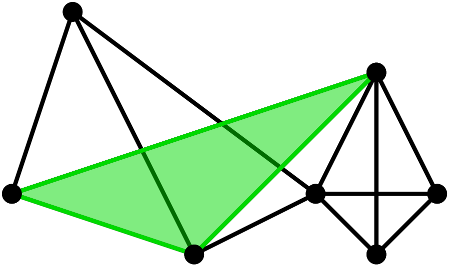

$$3 \leq \frac{7^2}{2}$$

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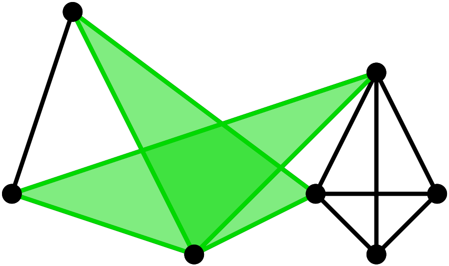

$$3 + 3 \leq \frac{7^2}{2}$$

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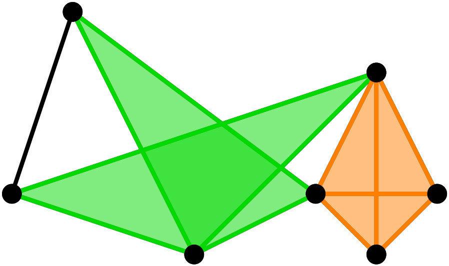

$$4 + 3 + 3 \leq \frac{7^2}{2}$$

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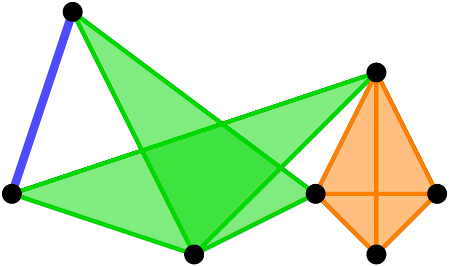

$$2 + 4 + 3 + 3 \le \frac{7^2}{2}$$

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In general:

- Assign cost $c_r$ to a clique $K_r$.
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Theorem (Győri and Tuza (1987))

*The edges of any graph $G$ of order $n$ can be decomposed into edges and triangles $C_1, \dots, C_\ell$ with $\sum_i |C_i| \leq \frac{9n^2}{16}$.*

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*The edges of any graph $G$ of order $n$ can be decomposed into edges and triangles $C_1, \ldots, C_\ell$ with $\sum_i |C_i| \leq \frac{n^2}{2} + o(n^2)$.*

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$cost = 2\#K_2 + 3\#K_3$

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**Theorem (Kráľ, L., Martins, Pehova 2019)**

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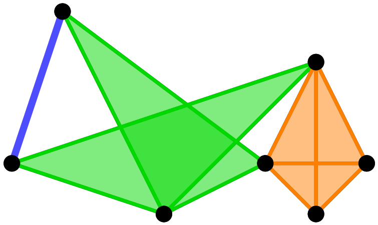

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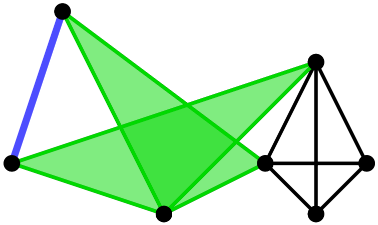

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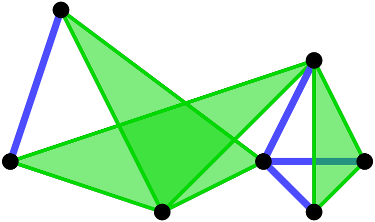

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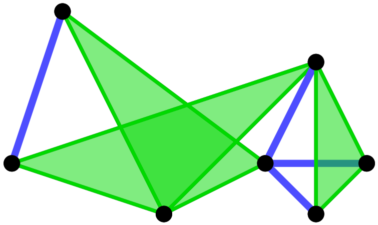

### Proof outline

1. Obtain a fractional decomposition into edges and triangles. (flag algebras method)

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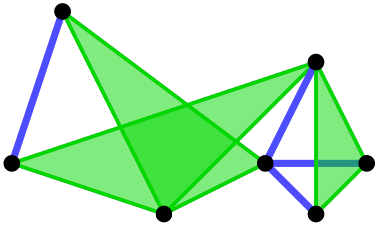


### Proof outline

1. Obtain a fractional decomposition into edges and triangles. (flag algebras method)
2. Fractional to full decomposition. (regularity method)

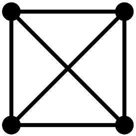
$cost = 2\#K_2 + 3\#K_3$

# Fractional decomposition

### Definition

A *decomposition* of a graph $G$ into triangles $\mathcal{T}$ and edges $\mathcal{E}$ is an assignment $w : \mathcal{T} \cup \mathcal{E} \to \{ 0, 1 \}$ such that for each $e \in E(G)$:

$$\sum_{T \supseteq e} w(T) + \sum_{e \in \mathcal{E}} w(e) = 1.$$



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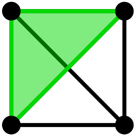

3

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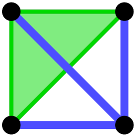

$3 + 2 + 2 + 2 = 9$

$cost = 2\#K_2 + 3\#K_3$

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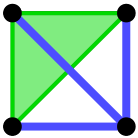

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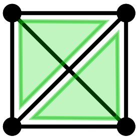

$\frac{1}{2}(3+3 \qquad )$

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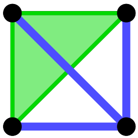

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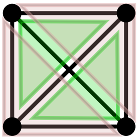

$\frac{1}{2}(3 + 3 + 3 + 3) = 6$

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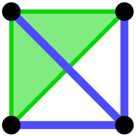

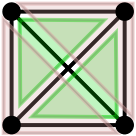

$3 + 2 + 2 + 2 = 9$ $\qquad$ $\frac{1}{2}(3 + 3 + 3 + 3) = 6$

$K_4$ doesn't have a triangle decomposition but it has a fractional triangle decomposition.

$cost = 2\#K_2 + 3\#K_3$

# Finding a fractional decomposition

## Definition

For a graph $G$, let

$$\pi_3(G) = \text{min cost of a triangle-edge decomposition of } G$$

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Clearly, $\pi_{3,f}(G) \leq \pi_3(G)$.

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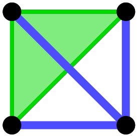

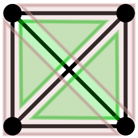

$\pi_3(K_4) = 9$ $\pi_{3,f}(K_4) = 6$

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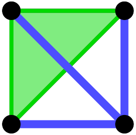

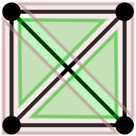

$\pi_3(K_4) = 9$ $\pi_{3,f}(K_4) = 6$

Our Theorem first step: $\pi_{3,f}(G) \leq \frac{n^2}{2} + o(n^2)$

$cost = 2\#K_2 + 3\#K_3$

# Finding a fractional decomposition

## Key lemma (using flag algebras)

Let $G$ be a (large) graph and $W$ be a uniformly chosen random subset of $7$ vertices of $G$. Then

$$E[\pi_{3,f}(G[W])] \leq 21 + o(1).$$

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Fractional decomposition of $G$:

- sum optimal decompositions for all $W \in \binom{V}{7}$
- divide by $\binom{n-2}{5}$

Each edge is in $\binom{n-2}{5}$ $W$s so it is a fractional decomposition of $G$.

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$$\begin{aligned}\pi_{3,f}(G) &\leq \frac{1}{\binom{n-2}{5}} \sum_W \pi_{3,f}(G[W]) \\ &\leq \frac{1}{\binom{n-2}{5}} \binom{n}{7} (21 + o(1)) = \frac{n^2}{2} + o(n^2).\end{aligned}$$

# Main result 1

Theorem (Kráľ, L., Martins, Pehova 2019)

For any sufficiently large graph $G$, $\pi_3(G) \leq \frac{n^2}{2} + o(n^2)$.

Proof

✓ Obtain a fractional decomposition into edges and triangles $\pi_{3,f}(G) \leq \frac{n^2}{2} + o(n^2)$ (flag algebra methods).

$cost = 2\#K_2 + 3\#K_3$

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PROOF

- ✓ Obtain a fractional decomposition into edges and triangles $\pi_{3,f}(G) \leq \frac{n^2}{2} + o(n^2)$ (flag algebra methods).
- Fractional to full decomposition (regularity method).

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# Fractional to full decomposition

## Definition

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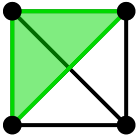
$\nu_H(G) =$max size of $H$-packing of $G$,



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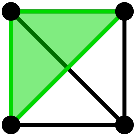

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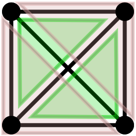

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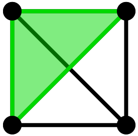

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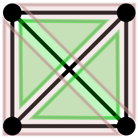

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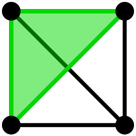

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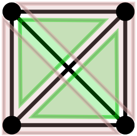

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### Theorem (Haxell, Rödl 2001)

For any graph $H$ and an $n$-vertex graph $G$ we have

$$\nu_H^f(G) \leq \nu_H(G) + o(n^2).$$

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### Corollary

$$\pi_3(G) \leq \pi_{3,f}(G) + o(n^2) \leq \frac{n^2}{2} + o(n^2)$$

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Corollary

$$\pi_3(G) \leq \pi_{3,f}(G) + o(n^2) \leq \frac{n^2}{2} + o(n^2)$$

Theorem (Yuster 2004)

For a fixed family $\mathcal{F}$ of graphs and an $n$ vertex graph $G$ we have

$$\nu_{\mathcal{F}}^f(G) \leq \nu_{\mathcal{F}}(G) + o(n^2).$$

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# Recap

**Theorem (Kráľ, L., Martins, Pehova 2019)**

For any sufficiently large graph $G$

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Examples:

$\pi_3(K_{\frac{n}{2},\frac{n}{2}}) = \frac{n^2}{2}$

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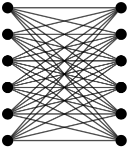

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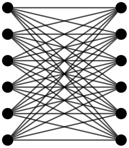
$\pi_3(K_n) = ?$



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# DECOMPOSING $K_n$

Conditions for triangle decomposition

- every vertex has even degree
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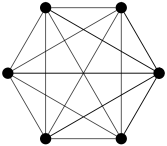

$e = 15, d = 5$

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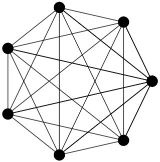

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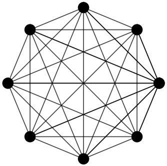

$e = 28, d = 7$

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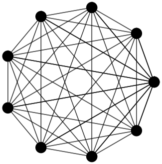

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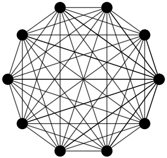

$e = 45, d = 9$

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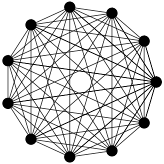

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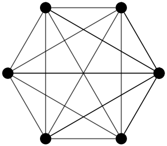

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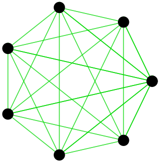

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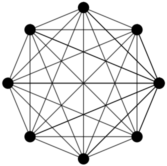

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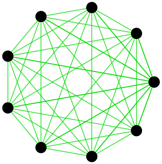

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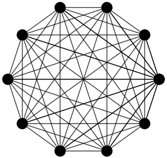

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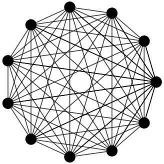

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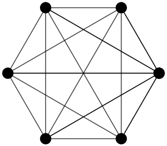

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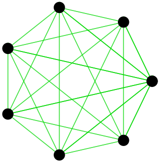

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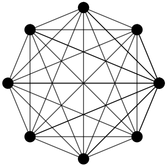

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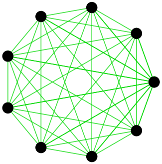

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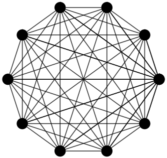

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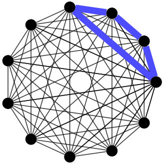

$e = 51, d = 8, 10$

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# Decomposing $K_n$

Conditions for triangle decomposition

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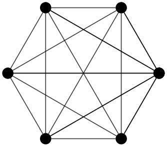
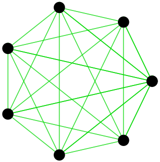
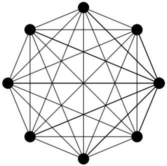

$e = 15, d = 5$ $\quad$ $e = 21, d = 6$ $\quad$ $e = 28, d = 7$

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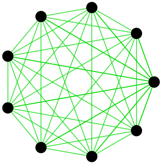
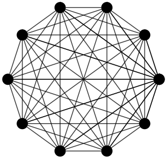
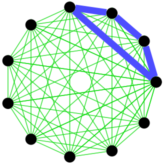

$e = 36, d = 8$ $\quad$ $e = 45, d = 9$ $\quad$ $e = 51, d = 8, 10$

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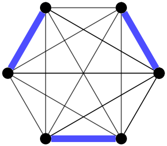

$e = 12, d = 4$

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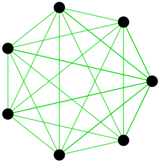

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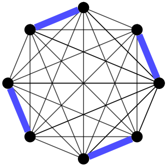

$e = 24, d = 6$

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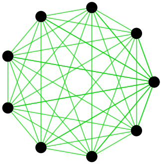

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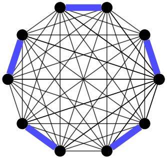
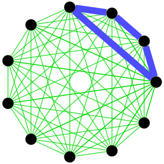


$e = 40, d = 8$

$n = 10 \equiv 4$



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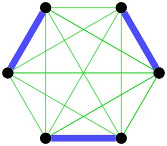

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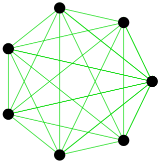

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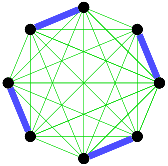

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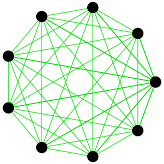

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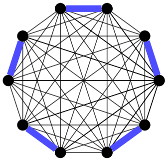

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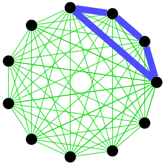

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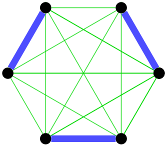

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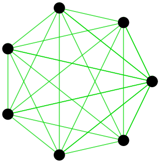

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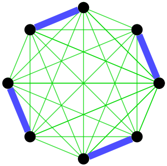

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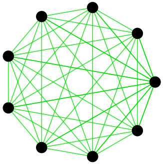
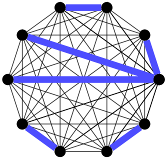


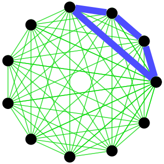
$e = 36, d = 8$

$n = 9 \equiv 3$



$e = 39, d = 6, 8$

$n = 10 \equiv 4$



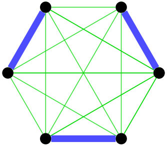
$e = 51, d = 8, 10$

$n = 11 \equiv 5$

# Decomposing $K_n$

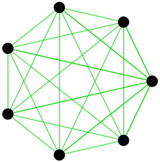
Conditions for triangle decomposition

- every vertex has even degree
- number of edges is divisible by 3



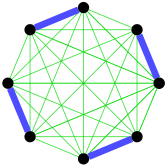
$e = 12, d = 4$

$n = 6 \equiv 0$



$e = 21, d = 6$

$n = 7 \equiv 1$



$e = 24, d = 6$

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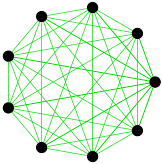

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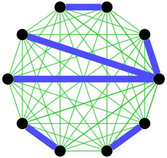

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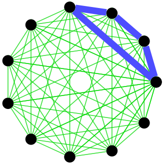

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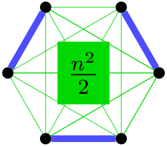


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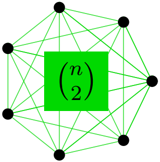


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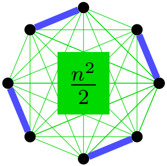


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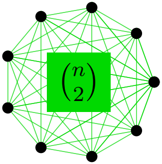


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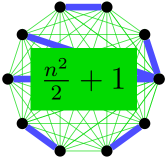


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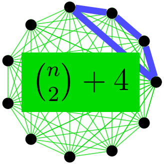


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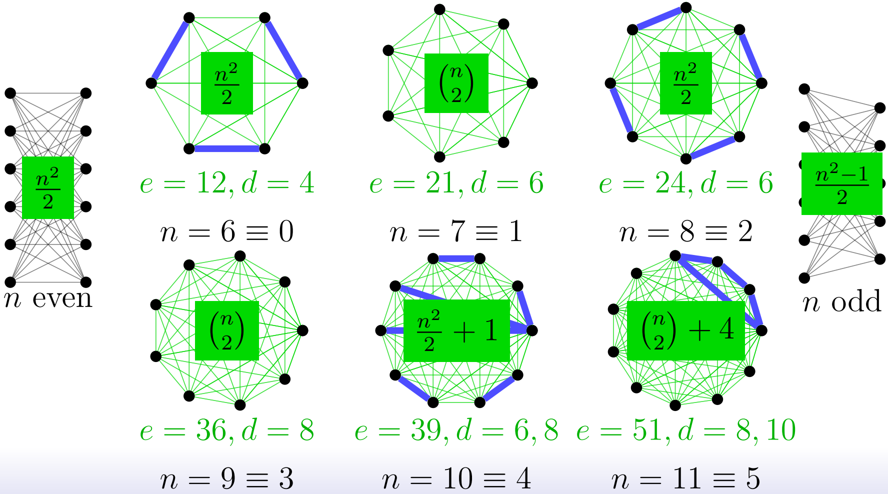
# Decomposing $K_n$

Conditions for triangle decomposition
- every vertex has even degree
- number of edges is divisible by $3$



# Decomposing $K_n$

Conditions for triangle decomposition
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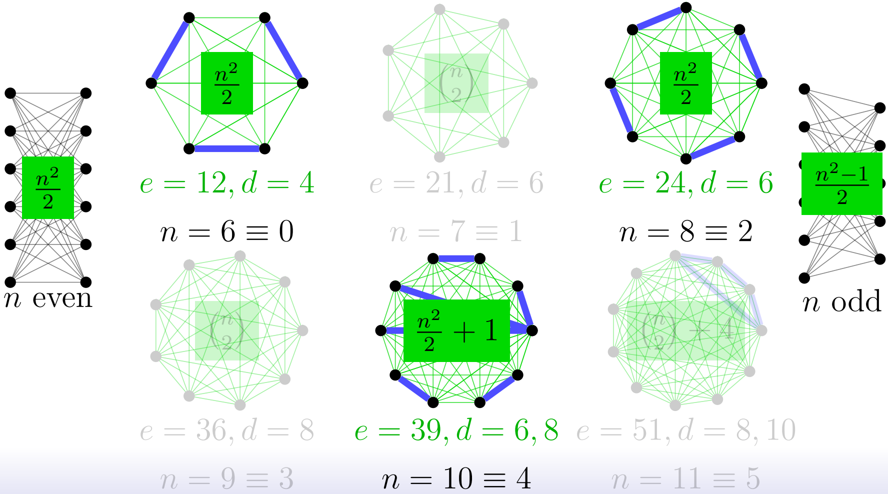
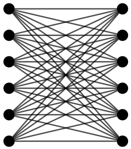


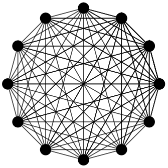
# Main Result 2

**Theorem** (Blumenthal, L., Pikhurko, Pehova, Pfender, Volec)

*For sufficiently large $n$,*

$$\pi_3(G) \leq \begin{cases} \frac{n^2}{2} & \textit{if} \quad n \equiv 0,2 \mod 6 \quad \ldots K_{\frac{n}{2},\frac{n}{2}} \textit{ and } K_n, \\ \frac{n^2-1}{2} & \textit{if} \quad n \equiv 1,3,5 \mod 6 \quad \ldots K_{\frac{n-1}{2},\frac{n+1}{2}}, \\ \frac{n^2}{2}+1 & \textit{if} \quad n \equiv 4 \mod 6 \quad \ldots K_n. \end{cases}$$



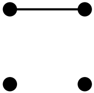
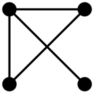
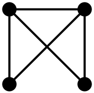


Note $\pi_3(K_5) = 14 > \frac{n^2}{2} + 1$.
The theorem cannot be extended to all $n$ without adding exception(s).

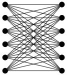
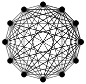
# Extremal examples and Stability

If $\pi_{3,f}(G) \leq \left(\frac{1}{2} - \varepsilon\right) n^2$ then $\pi_3(G) < \frac{1}{2}n^2$ by Yuster/Haxell,Rödl.

If $\pi_{3,f}(G) \geq \left(\frac{1}{2} - \varepsilon\right) n^2$, by flag algebra methods the following graphs

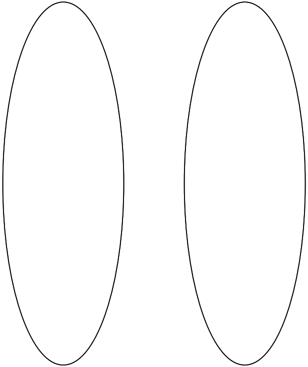
$\mathcal{F} = \Bigg\{$  ,  ,  $\Bigg\}$

have density at most $\delta$, where $\delta \to 0$ as $\varepsilon \to 0$.
By Induced removal lemma, $G$ is $\mathcal{F}$-free up to $\delta' n^2$ edges.

Hence $G$ is  or  up to $\delta' n^2$ edges.

# Exact result $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$

$G$ is close to $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$, show $\pi_3(G) \le \pi_3(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil})$.



- take maxcut ($|E(G)| \ge \frac{n^2}{4}$)

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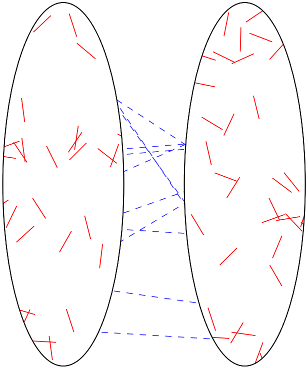

- take maxcut ($|E(G)| \ge \frac{n^2}{4}$)
- extra edges, missing edges

# Exact result $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$

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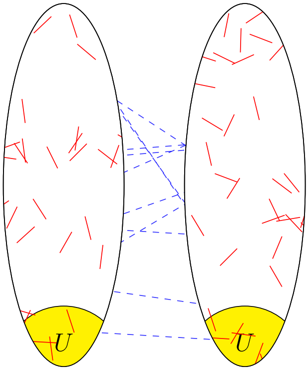


- take maxcut ($|E(G)| \geq \frac{n^2}{4}$)
- extra edges, missing edges
- high extra degree vertices $U$

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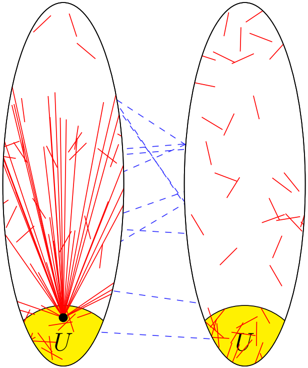


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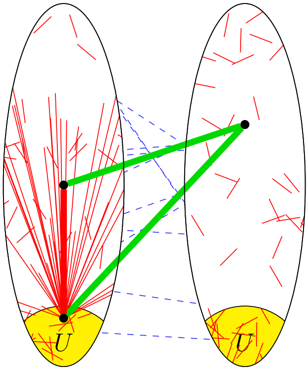
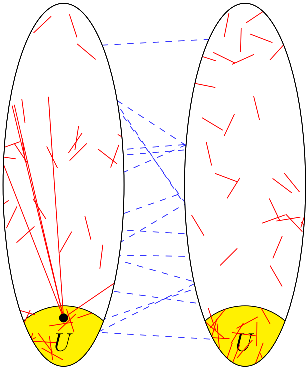



- take maxcut ($|E(G)| \geq \frac{n^2}{4}$)
- extra edges, missing edges
- high extra degree vertices $U$
- triangles with 1 extra edge with $U$

# Exact result $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$

$G$ is close to $K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}$, show $\pi_3(G) \leq \pi_3(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil})$.

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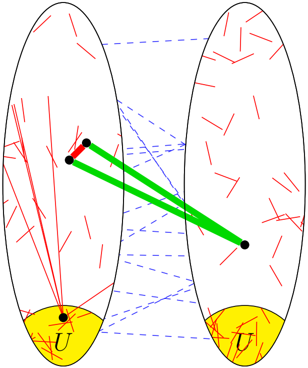


- take maxcut ($|E(G)| \geq \frac{n^2}{4}$)
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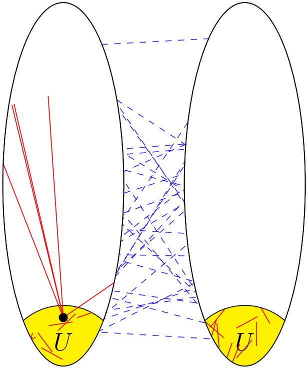


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- extra edges, missing edges
- high extra degree vertices $U$
- triangles with 1 extra edge with $U$
- other triangles with extra edges
- rest taken as $K_2$s

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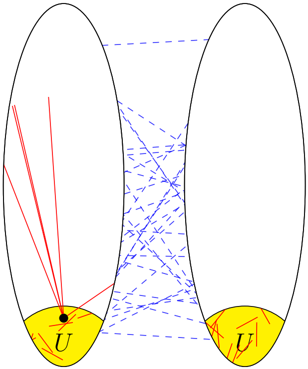


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- other triangles with extra edges
- rest taken as $K_2$s

### Theorem (Győri 1988)

*If $G$ is a graph with $n$ vertices and $n^2/4 + k$ edges, where $n \to \infty$ and $k = o(n^2)$, then it has at least $k - O(k^2/n^2)$ edge-disjoint triangles.*

## Exact result $K_n$

Theorem (Barber, Kuhn, Lo, Osthus; Dross; Gustavsson)
*Every large graph $G$ on $n$ vertices, where $|E(G)|$ is a multiple of 3 and all vertices have even degree at least $(9/10 + o(1))n$ has a triangle decomposition.*

# Exact result $K_n$

**Theorem (Barber, Kuhn, Lo, Osthus; Dross; Gustavsson)**
*Every large graph $G$ on $n$ vertices, where $|E(G)|$ is a multiple of 3 and all vertices have even degree at least $(9/10 + o(1))n$ has a triangle decomposition.*

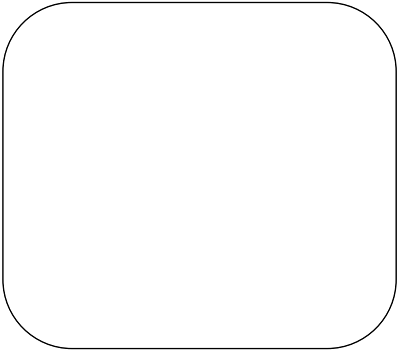
$K_n$ proof overview

- special treatment for low degree vertices
- make all degrees even and $|E(G)|$ divisible by $3$
- apply Theorem

Desired conclusion:
If $G$ is close to $K_n$ then $\pi_3(G) \leq \pi_3(K_n)$.

# Exact result $K_n$

$G$ is close to $K_n$ and maximize try to show $\pi_3(G) \leq \pi_3(K_n)$.



- $\delta(G) \geq n/8$

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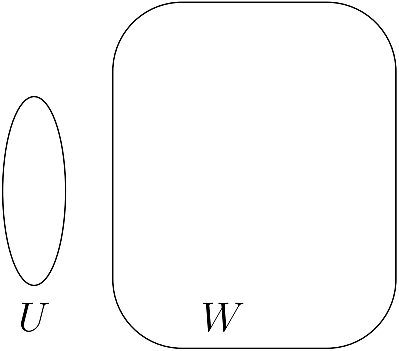
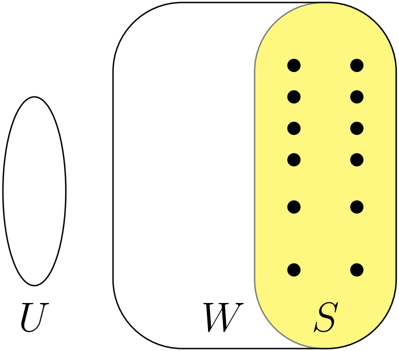



- $\delta(G) \geq n/8$
- degree $< 0.99n$ go to $U$, rest in $W$

# Exact result $K_n$

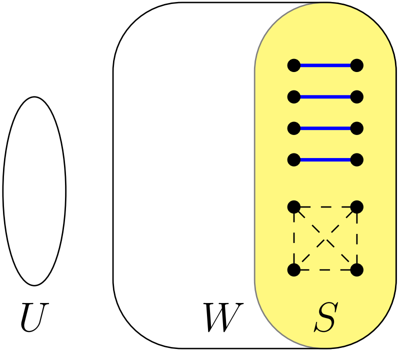
$G$ is close to $K_n$ and maximize try to show $\pi_3(G) \leq \pi_3(K_n)$.




- $\delta(G) \geq n/8$
- degree $< 0.99n$ go to $U$, rest in $W$
- $S \subseteq W$ with odd degree in $G$

# EXACT RESULT $K_n$

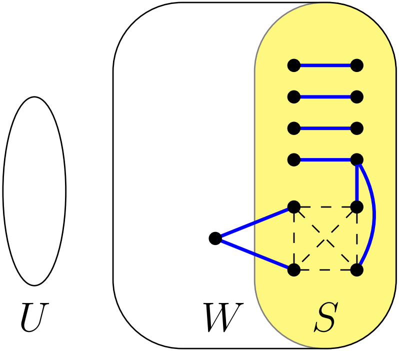
$G$ is close to $K_n$ and maximize try to show $\pi_3(G) \le \pi_3(K_n)$.




- $\delta(G) \ge n/8$
- degree $< 0.99n$ go to $U$, rest in $W$
- $S \subseteq W$ with odd degree in $G$
- make degrees in $S$ even

# Exact result $K_n$

$G$ is close to $K_n$ and maximize try to show $\pi_3(G) \leq \pi_3(K_n)$.

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# EXACT RESULT $K_n$

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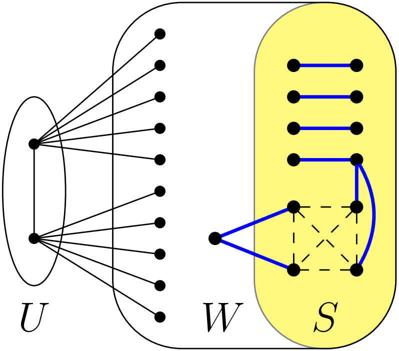


- $\delta(G) \geq n/8$
- degree $< 0.99n$ go to $U$, rest in $W$
- $S \subseteq W$ with odd degree in $G$
- make degrees in $S$ even
- for $u$ in $U$ cover $N_u[W]$ by triangles

# Exact result $K_n$

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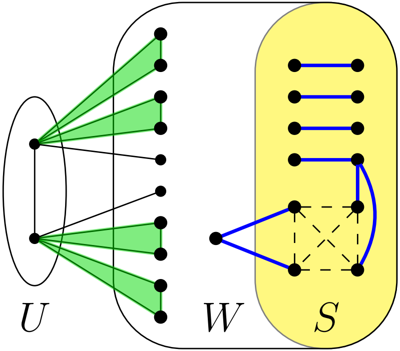


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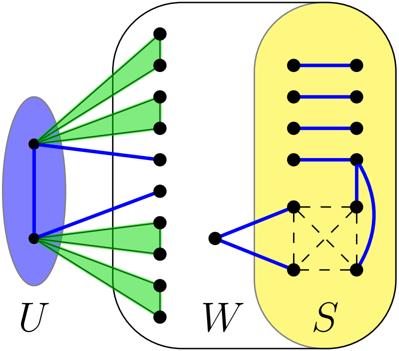
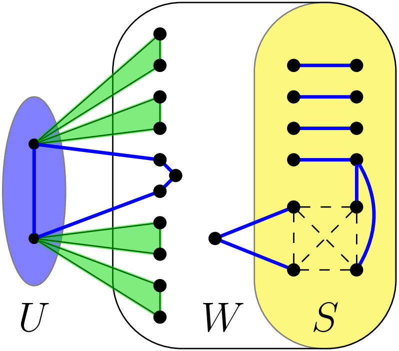



- $\delta(G) \geq n/8$
- degree $< 0.99n$ go to $U$, rest in $W$
- $S \subseteq W$ with odd degree in $G$
- make degrees in $S$ even
- for $u$ in $U$ cover $N_u[W]$ by triangles
- cover $U$ edges using $K_2$s.

# Exact result $K_n$

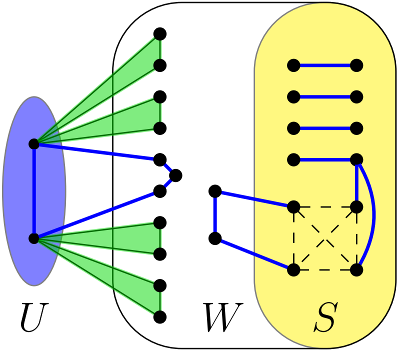
$G$ is close to $K_n$ and maximize try to show $\pi_3(G) \leq \pi_3(K_n)$.




- $\delta(G) \geq n/8$
- degree $< 0.99n$ go to $U$, rest in $W$
- $S \subseteq W$ with odd degree in $G$
- make degrees in $S$ even
- for $u$ in $U$ cover $N_u[W]$ by triangles
- cover $U$ edges using $K_2$s.
- make rest triangle divisible

# Exact result $K_n$

$G$ is close to $K_n$ and maximize try to show $\pi_3(G) \leq \pi_3(K_n)$.




- $\delta(G) \geq n/8$
- degree $< 0.99n$ go to $U$, rest in $W$
- $S \subseteq W$ with odd degree in $G$
- make degrees in $S$ even
- for $u$ in $U$ cover $N_u[W]$ by triangles
- cover $U$ edges using $K_2$s.
- make rest triangle divisible

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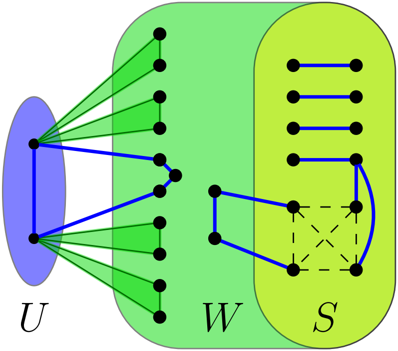


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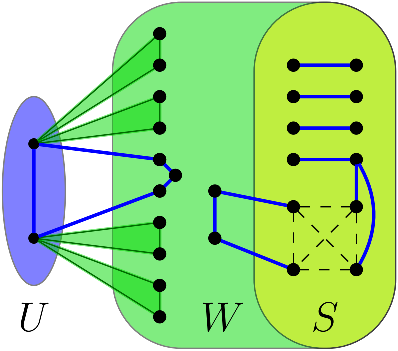


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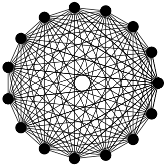
- $G$ is $K_n$ or
- $G$ is $K_n$ without a matching of size $2 \pmod 3$ and $n \equiv 1, 3 \pmod 6$.

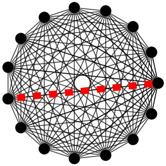
## $K_n$ WITHOUT A MATCHING

$n \equiv 1, 3 \pmod 6$ means $K_n$ is triangle divisible



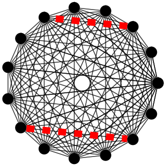
$e = 105, d = 14$

$n = 15 \equiv 3$



$e = 104, d = 13, 14$

$n = 15 \equiv 3$



$e = 103, d = 13, 14$

$n = 15 \equiv 3$

$\pi_3(K_n) =$
$\pi_3(K_n^-) =$
$\pi_3(K_n^=) =$

cost $= 2\#K_2 + 3\#K_3$

# $K_n$ WITHOUT A MATCHING

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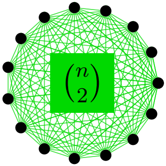


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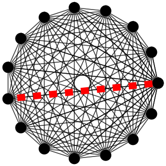

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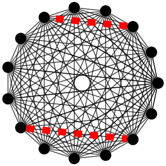

$e = 103, d = 13, 14$

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$\pi_3(K_n) = \binom{n}{2}$
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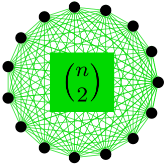


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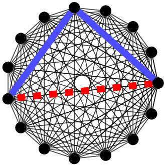

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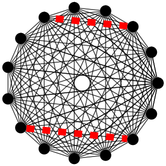

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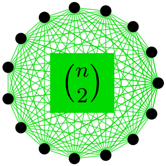


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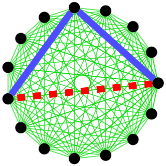

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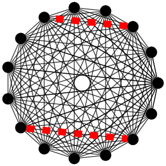

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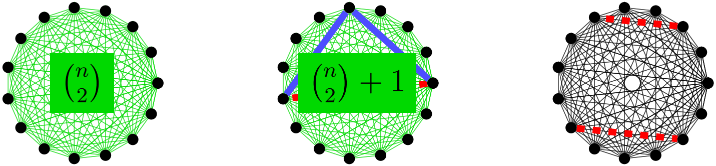


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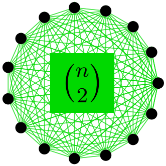


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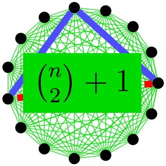


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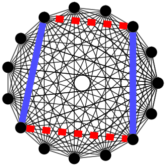

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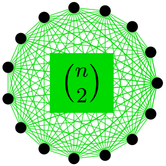


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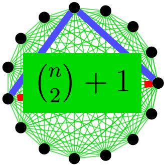


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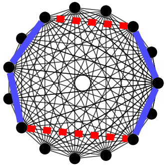

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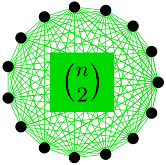


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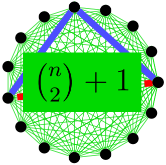


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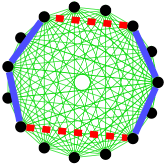

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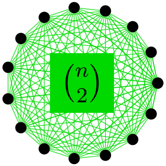


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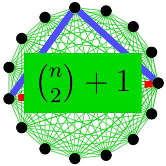


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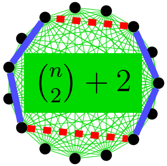


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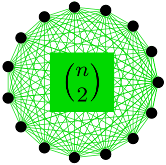


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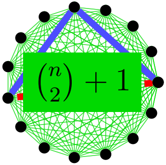


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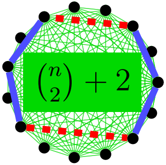


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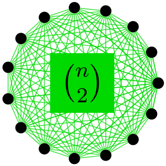


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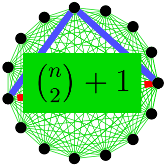


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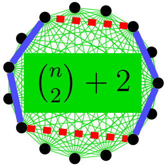


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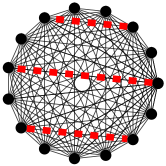

$e = 102, d = 13, 14$

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$cost = 2\#K_2 + 3\#K_3$

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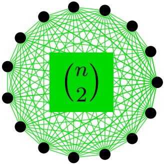
$n \equiv 1, 3 \pmod 6$ means $K_n$ is triangle divisible

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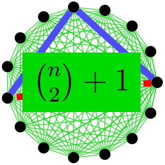


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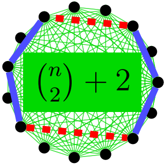


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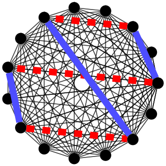

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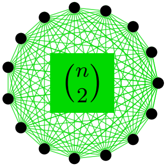
# $K_n$ WITHOUT A MATCHING

$n \equiv 1, 3 \pmod 6$ means $K_n$ is triangle divisible

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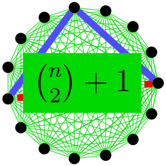


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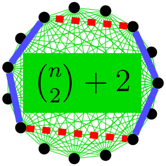


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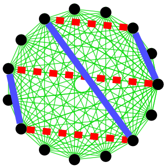

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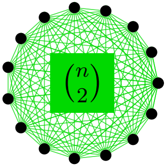
*cost* $= 2\#K_2 + 3\#K_3$

# $K_n$ without a matching

$n \equiv 1, 3 \pmod 6$ means $K_n$ is triangle divisible

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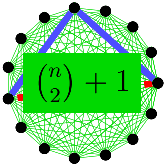


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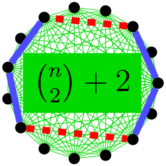


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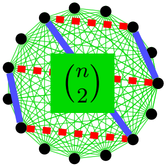


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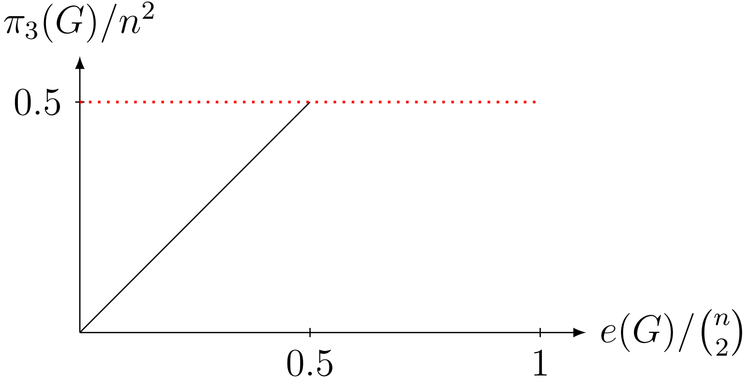
**Theorem** (Blumenthal, L., Pikhurko, Pehova, Pfender, Volec)
*For sufficiently large $n$,*

$$\pi_3(G) \leq \begin{cases} \frac{n^2}{2} & \textit{if} \quad n \equiv 0, 2 \mod 6 \quad \ldots K_{\frac{n}{2},\frac{n}{2}} \textit{ and } K_n, \\ \frac{n^2-1}{2} & \textit{if} \quad n \equiv 1, 3, 5 \mod 6 \quad \ldots K_{\frac{n-1}{2},\frac{n+1}{2}}, \\ \frac{n^2}{2}+1 & \textit{if} \quad n \equiv 4 \mod 6 \quad \ldots K_n. \end{cases}$$

Can we find a better upper bound on $\pi_3(G)$ if we also know the edge-density of $G$?

# Possible values of $\pi_3(G)$



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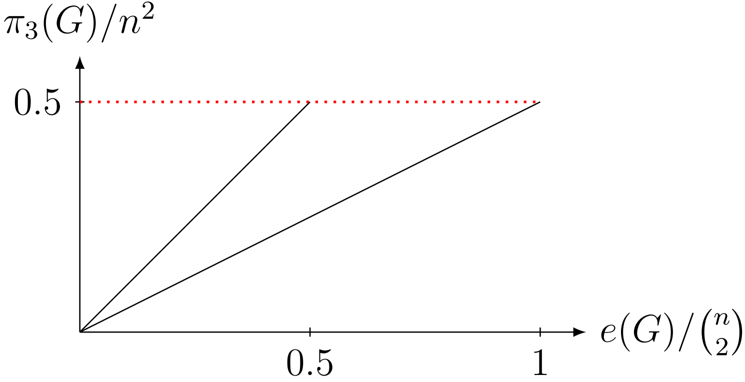

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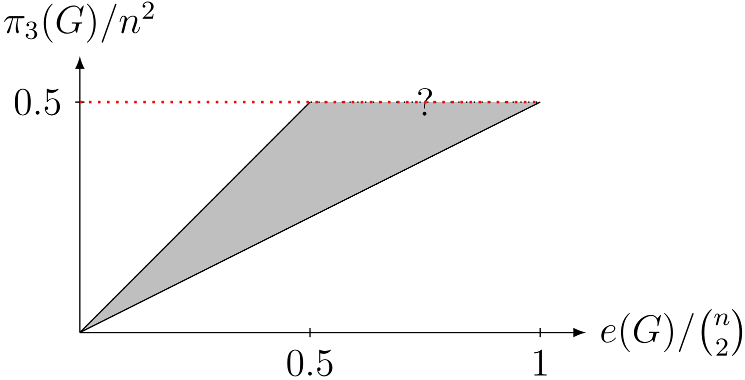

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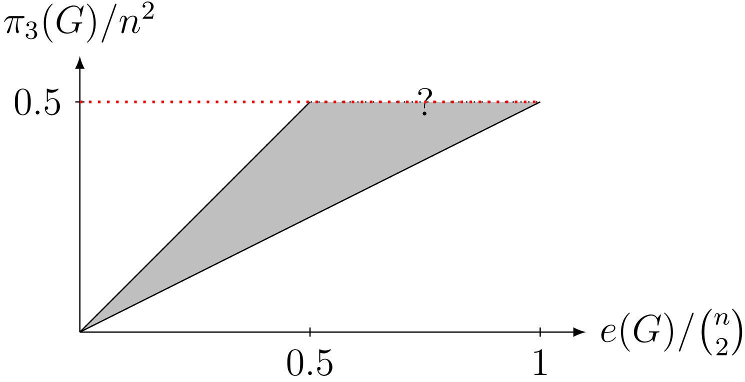

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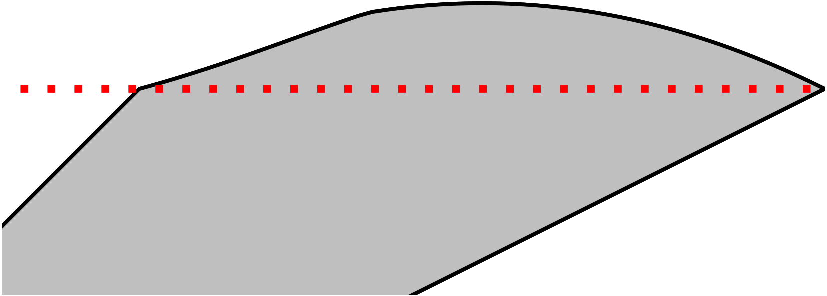
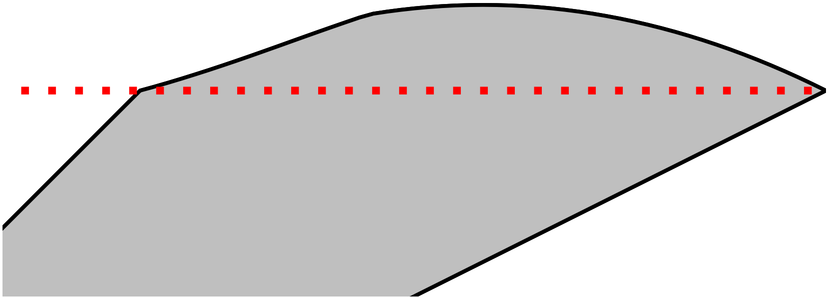


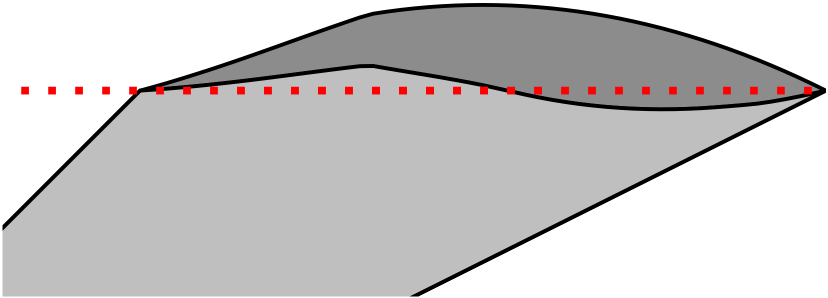
Flag algebras on 4 vertices

# Possible values of $\pi_3(G)$



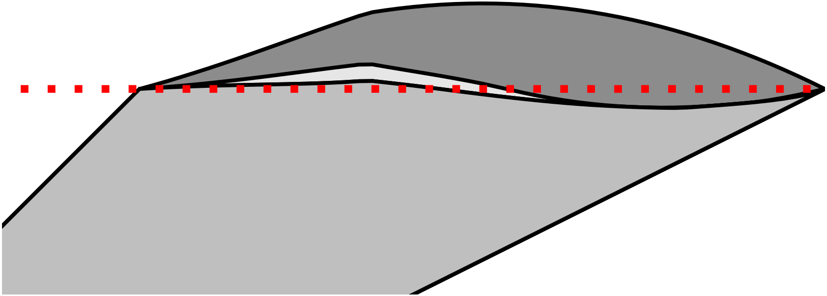
Flag algebras on $5$ vertices

# Possible values of $\pi_3(G)$



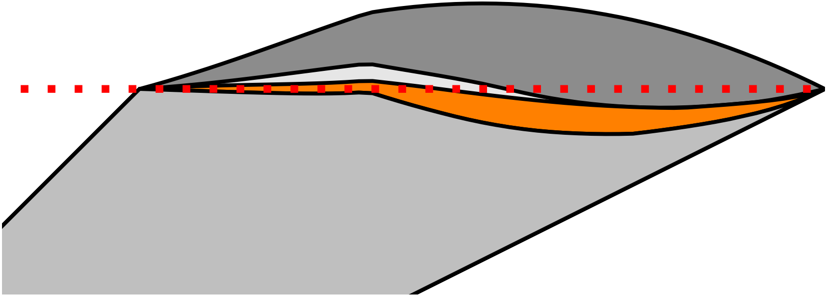
Flag algebras on $6$ vertices

# Possible values of $\pi_3(G)$



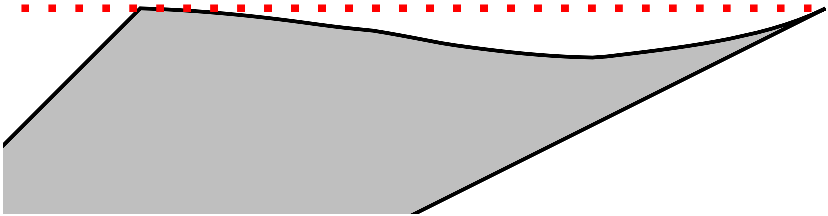
Flag algebras on 7 vertices

# Possible values of $\pi_3(G)$



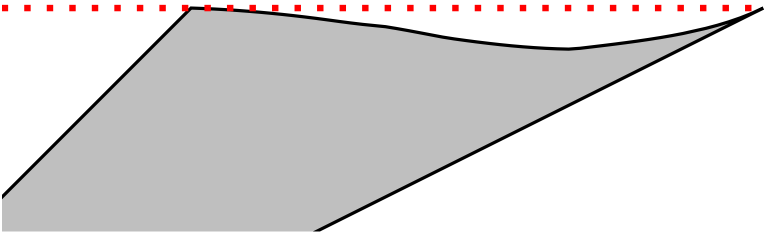
Flag algebras on 8 vertices

# Possible values of $\pi_3(G)$



Flag algebras on $8$ vertices

# Possible values of $\pi_3(G)$



Flag algebras on 8 vertices

Only numerical upper bound. Not clear how the curve looks like.

Can you find a lower bound?

# Extending the result

$$\pi_3(G) := 2\#K_2 + 3\#K_3 = 2e(G) - 3\nu_{K_3}(G)$$

Recall $\nu_{K_3}(G)$ is a size of a maximum triangle packing.

# Extending the result

$$\pi_3(G) := 2\#K_2 + 3\#K_3 = 2e(G) - 3\nu_{K_3}(G)$$

$$\pi_3^{\alpha}(G) := 2\#K_2 + \alpha\#K_3 = 2e(G) - (6-\alpha)\nu_{K_3}(G)$$

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Notice $\pi_3^{\geq 6}(G) = 2e(G)$.

Recall $\nu_{K_3}(G)$ is a size of a maximum triangle packing.

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What if $\alpha < 6$? Which graphs maximize $\pi_3^{\alpha}$?

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Notice $\pi_3^{\geq 6}(G) = 2e(G)$.

What if $\alpha < 6$? Which graphs maximize $\pi_3^\alpha$?

We solved $\pi_3^\alpha$ for $\alpha = 3$.

Recall $\nu_{K_3}(G)$ is a size of a maximum triangle packing.

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$$\pi_3(G) := 2\#K_2 + 3\#K_3 = 2e(G) - 3\nu_{K_3}(G)$$

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Observation

*If* $\pi_3^{\alpha}(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}) > \pi_3^{\alpha}(G)$ *then* $\pi_3^{\beta}(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}) > \pi_3^{\beta}(G)$ *for all* $\alpha > \beta$.

# Extending the result

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**Conclusion**

*For every* $\alpha < 3$*, sufficiently large* $n$ *and every graph* $G$ *on* $n$ *vertices*

$$\pi_3^{\alpha}(G) \leq \pi_3^{\alpha}(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil}).$$

## Extending the result

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Observation

*If* $\pi_3^{\alpha}(K_n) > \pi_3^{\alpha}(G)$ *then* $\pi_3^{\beta}(K_n) > \pi_3^{\beta}(G)$ *for all* $\alpha < \beta$.

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**Observation**

*If $\pi_3^\alpha(K_n) > \pi_3^\alpha(G)$ then $\pi_3^\beta(K_n) > \pi_3^\beta(G)$ for all $\alpha < \beta$.*

**Conclusion**

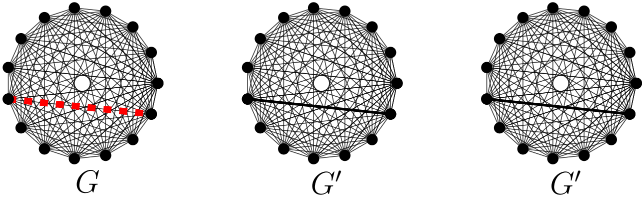
*For every $\alpha > 3$, sufficiently large $n$ and every graph $G$ on $n$ vertices*

$$\pi_3^\alpha(G) \leq \pi_3^\alpha(K_n)$$

*unless $n \equiv 1, 3 \pmod 6$. The exception is if $3 < \alpha < 4$, then $\pi_3^\alpha(G) \leq \pi_3^\alpha(K_n^=)$.*

# $\alpha \geq 4$

If $\alpha > 4$ then $\pi_3^\alpha(G) < \pi_3^\alpha(K_n)$ for all $n$.
Let $G'$ be obtained from $G$ by adding an edge.




$\pi_3^\alpha(G)$

$cost = 2\#K_2 + \alpha\#K_3$

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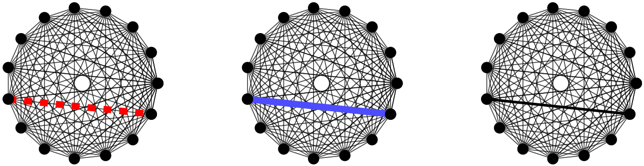

$G$ $\quad$ $G'$ $\quad$ $G'$

$\pi_3^\alpha(G)$ $\quad$ $\pi_3^\alpha(G) + 2$

$cost = 2\#K_2 + \alpha\#K_3$

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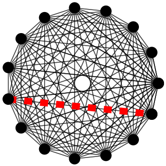
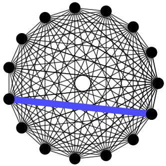


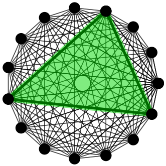
$G$

$\pi_3^\alpha(G)$



$G'$

$\pi_3^\alpha(G) + 2$



$G'$

cost $= 2\#K_2 + \alpha\#K_3$

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If $\alpha > 4$ then $\pi_3^\alpha(G) < \pi_3^\alpha(K_n)$ for all $n$.
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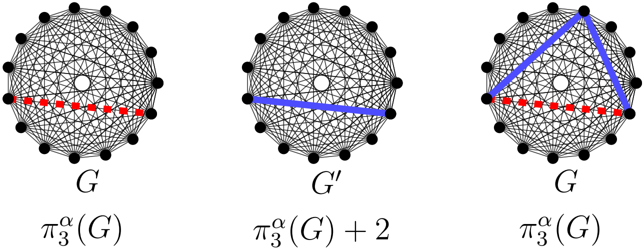


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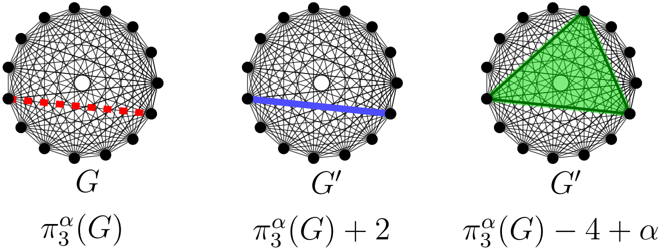


$cost = 2\#K_2 + \alpha\#K_3$

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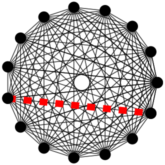
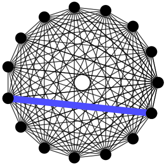


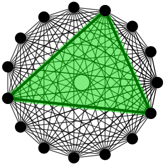
$G$

$\pi_3^\alpha(G)$



$G'$

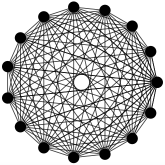
$\pi_3^\alpha(G) + 2$



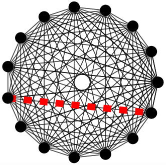
$G'$

$\pi_3^\alpha(G) - 4 + \alpha$

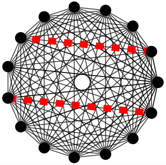
If $\alpha = 4$, then maximizers are



$K_n$



$K_n^-$



$K_n^=$

$cost = 2\#K_2 + \alpha\#K_3$

# $\alpha$ Summary

**Theorem (Blumenthal, L., Pikhurko, Pehova, Pfender, Volec)**

*For every $\alpha \geq 0$ exists $n_0$ such that for all graphs $G$ on $n > n_0$ the following cases hold.*

- *If $\alpha < 3$ then $\pi_3^\alpha(G) \leq \pi_3^\alpha(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil})$.*
- *If $\alpha = 3$ then $\pi_3^\alpha(G) \leq \max\{\pi_3^\alpha(K_n), \pi_3^\alpha(K_{\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil})\}$*
- *If $3 < \alpha < 4$, then $\pi_3^\alpha(G) \leq \max\{\pi_3^\alpha(K_n), \pi_3^\alpha(K_n^=)\}$.*
- *If $\alpha = 4$, then $\pi_3^\alpha(G) \leq \max\{\pi_3^\alpha(K_n), \pi_3^\alpha(K_n^-), \pi_3^\alpha(K_n^=)\}$.*
- *If $4 < \alpha$, then $\pi_3^\alpha(G) \leq \pi_3^\alpha(K_n)$.*

*Moreover, these are the only possible extremal examples.*

*cost* $= 2\#K_2 + \alpha\#K_3$

## Problem (Erdős)

*Assuming that each complete subgraph* $K_i$ *has weight* $i-1$ $(i = 2, 3, \ldots)$, *prove that every graph* $G$ *on* $n$ *vertices admits a partition into* $K_i$*s of total weight* $w(G)$ *at most* $n^2/4 = 0.25n^2$.

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Using only $K_2$ and $K_3$ gives $w(K_n) \approx n^2/3$.

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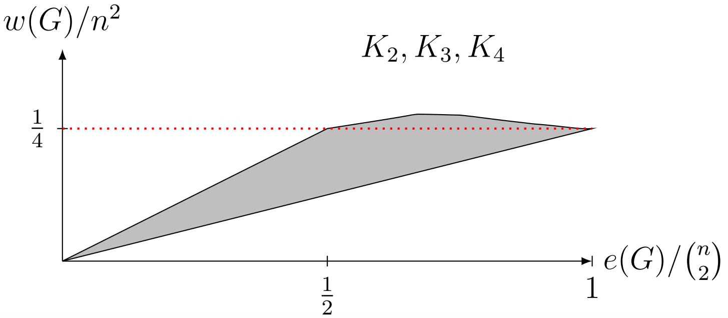
Using only $K_2$ and $K_3$ gives $w(K_n) \approx n^2/3$.
Flag algebras using $K_2, K_3, K_4$ give upper bound of $0.27256n^2$.



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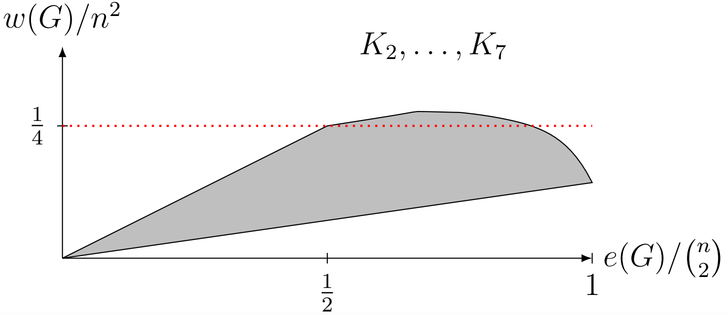
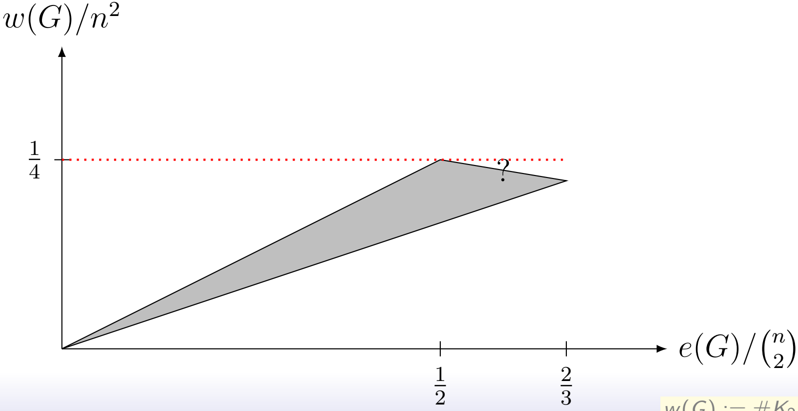
Flag algebras using $K_2, \ldots, K_7$ give upper bound of $0.27256n^2$.



# Why Flag Algebras might struggle?

## Problem (Simpler)

*Assuming that each complete subgraph* $K_i$ *has weight* $i-1$ $(i = 2, 3, \ldots)$, *prove that every* $K_4$*-free graph* $G$ *on* $n$ *vertices admits a partition into* $K_i$ *of total weight* $w(G)$ *at most* $n^2/4 = 0.25n^2$.




$w(G) := \#K_2 + 2\#K_3$

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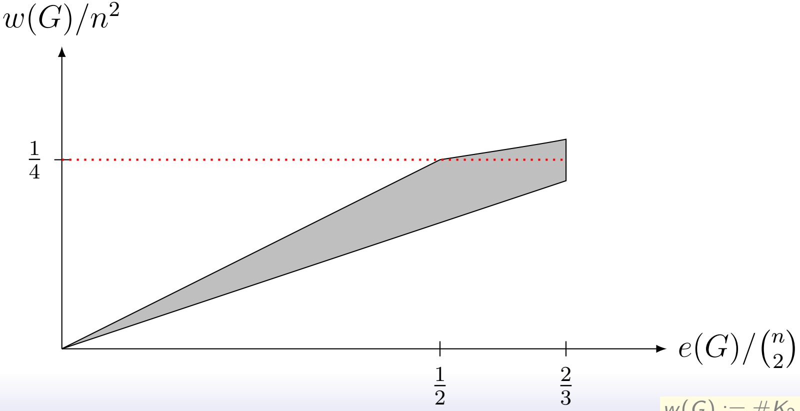


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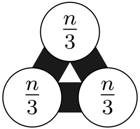
# Why Flag Algebras might struggle?

Complete 3-partite graph $T_3(n)$

Since $T_3(n)$ is (almost) triangle decomposable,

$$w(T_3(n)) \approx \frac{2}{9}n^2 \approx 0.22222n^2$$

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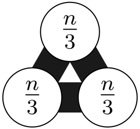


Flag algebras approach:

For every $X \in \binom{V}{3}$ take $\frac{w_f(G[X])}{\binom{n-2}{1}}$ decomposition and sum these.

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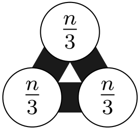


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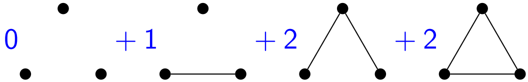
For every $X \in \binom{V}{3}$ take $\frac{w_f(G[X])}{\binom{n-2}{1}}$ decomposition and sum these.

Expected cost of decomposition on 3 vertices

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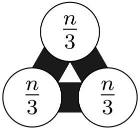


$$w(T_3(n)) \approx \frac{2}{9}n^2 \approx 0.22222n^2$$

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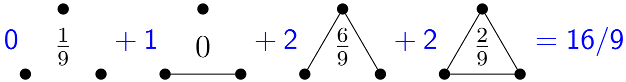

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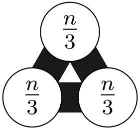


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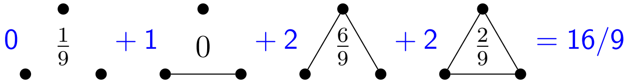



This gives

$$w(T_3(n)) \leq 0.2963n^2$$

Conclusion: Our formulation using FA is not strong enough.

$w(G) := \#K_2 + 2\#K_3$

# One More Problem

### Problem (Pyber 1991)

Can the edge set of every $n$-vertex graph be covered with triangles of weight $3$ and edges of weight $2$ such that their total weight is at most $\lfloor \frac{n^2}{2} \rfloor$?

# One More Problem

### Problem (Pyber 1991)

Can the edge set of every $n$-vertex graph be covered with triangles of weight $3$ and edges of weight $2$ such that their total weight is at most $\lfloor \frac{n^2}{2} \rfloor$?

### Theorem (Blumenthal, L., Pikhurko, Pehova, Pfender, Volec)

*For sufficiently large $n$,*

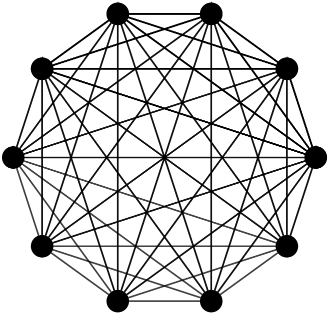
$$\pi_3(G) \leq \begin{cases} \frac{n^2}{2} & \textit{if} \quad n \equiv 0,2 \mod 6 \quad \ldots K_{\frac{n}{2},\frac{n}{2}} \textit{ and } K_n, \\ \frac{n^2-1}{2} & \textit{if} \quad n \equiv 1,3,5 \mod 6 \quad \ldots K_{\frac{n-1}{2},\frac{n+1}{2}}, \\ \frac{n^2}{2}+1 & \textit{if} \quad n \equiv 4 \mod 6 \quad \ldots K_n. \end{cases}$$

$cost = 2\#K_2 + 3\#K_3$

# Pyber's problem

$n = 6k + 4$, find a covering of $G = K_n$ of cost $\leq \frac{n^2}{2}$.

Triangle decomposition: even degrees, $|E(G)|$ divisible by $3$



$d(v) = 6k{+}3$

$e(G) = 3(6k^2 + 7k + 2)$

$cost = 2\#K_2 + 3\#K_3$

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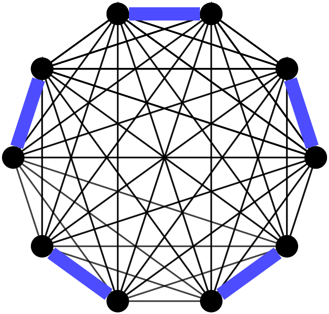

$d(v) = 6k{+}2$

$e(G) = 3(6k^2 + 7k + 2) - 3k - 2$

$cost = 2\#K_2 + 3\#K_3$

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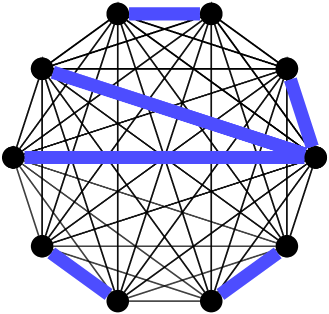

$d(v) = 6k{+}2$ or $6k$
$e(G) = 3(6k^2 + 7k + 2) - 3k - 3$
$cost = \frac{n^2}{2} + 1$

$cost = 2\#K_2 + 3\#K_3$

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$n = 6k + 4$, find a covering of $G = K_n$ of cost $\leq \frac{n^2}{2}$.
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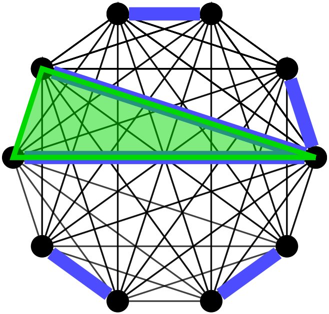
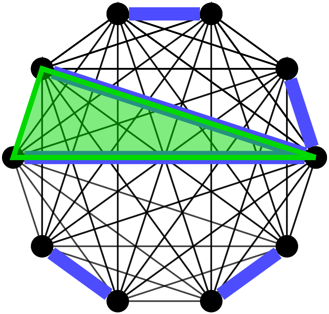


$d(v) = 6k{+}2$ or $6k$
$e(G) = 3(6k^2 + 7k + 2) - 3k - 3$
$cost = \frac{n^2}{2} + 1 - 4 + 3$

$cost = 2\#K_2 + 3\#K_3$

# Pyber's problem

$n = 6k + 4$, find a covering of $G = K_n$ of cost $\leq \frac{n^2}{2}$.
Triangle decomposition: even degrees, $|E(G)|$ divisible by $3$



$d(v) = 6k{+}2$ or $6k$
$e(G) = 3(6k^2 + 7k + 2) - 3k - 3$
$cost = \frac{n^2}{2} + 1 - 4 + 3$

**Thank you for your attention**

$cost = 2\#K_2 + 3\#K_3$