Coloring Count Cones of Planar Graphs

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CanaDAM
Vancouver, BC
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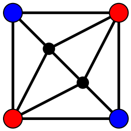
Motivation

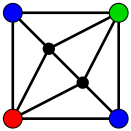
Theorem (4CT)

Every planar graph is 4-colorable.

Problem

Is there a polynomial-time algorithm to decide if a precoloring of a 4-face extends? (all other faces are triangles)





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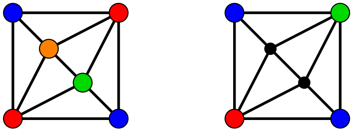

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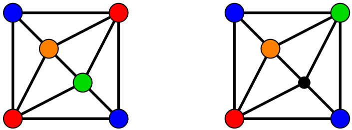

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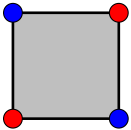

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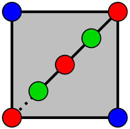

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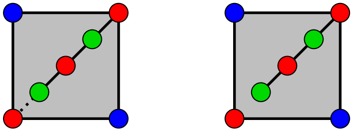

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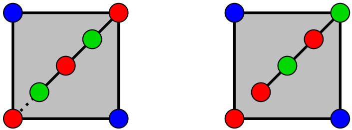

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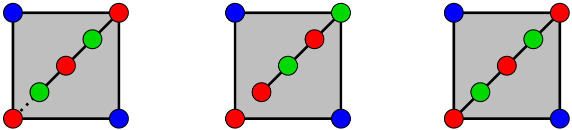

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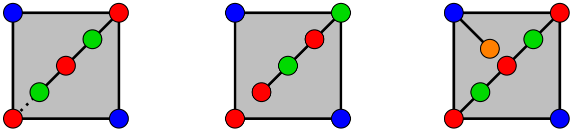

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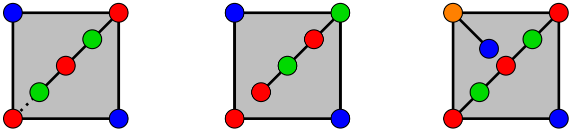

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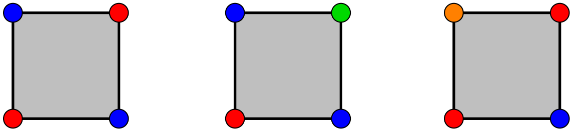

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Number of extensions satisfies some constraints.

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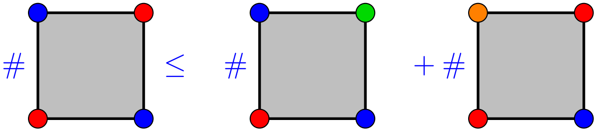

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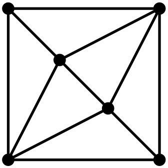
Dual



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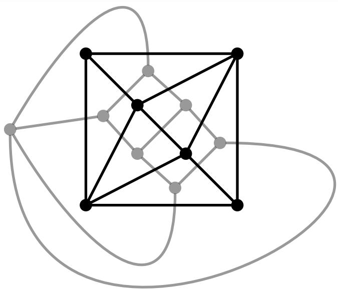

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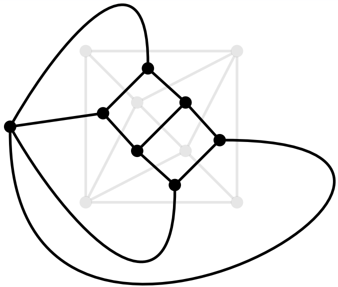

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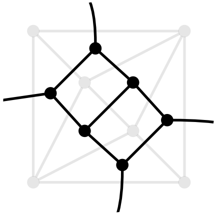

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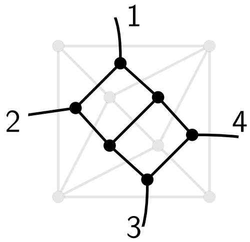

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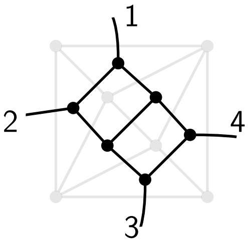


G is a *near cubic plane graph*, 3-edge-coloring of G
ψ precoloring of half edges of G

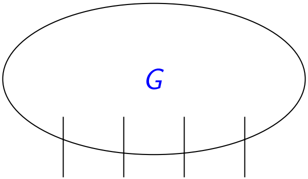
$$n_G(\psi) := \#\text{ extensions of } \psi \text{ to } G$$

Our goal is to "describe" vectors

$$(n_G(\psi_1), n_G(\psi_2), n_G(\psi_3), \ldots)$$

$n_G(\psi) := \#$ extensions of ψ to G

For G with d half edges if $n_G(\psi) \neq 0$ then
$|\psi^{-1}(R)| \equiv |\psi^{-1}(G)| \equiv |\psi^{-1}(B)| \equiv d \pmod 2$.



R R R R
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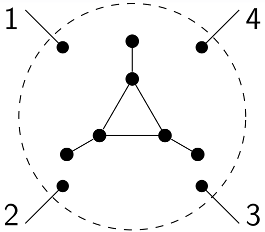
$n_{RRRR} = n_{GGGG} = n_{BBBB}$
Goal: Describe vectors

$$(n_{RRRR}, n_{RRBB}, n_{RBRB}, n_{RBBR}).$$

Reductions with fixed ψ



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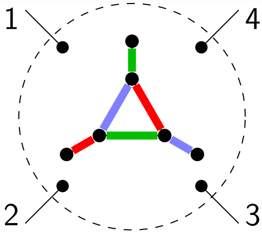

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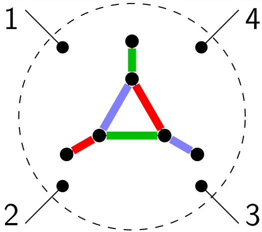

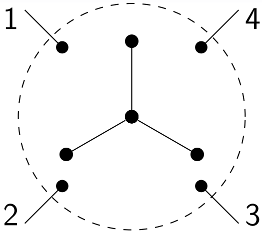

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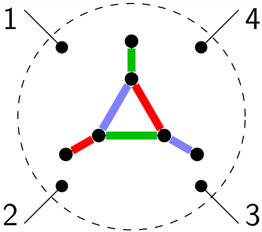

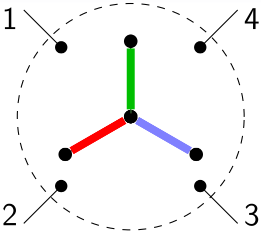

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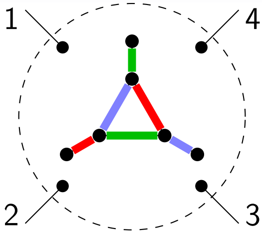

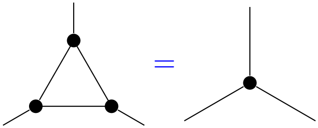

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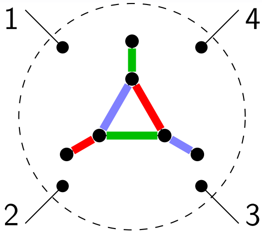

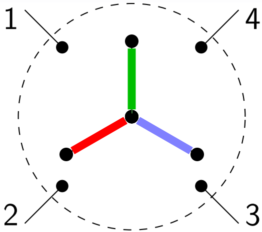

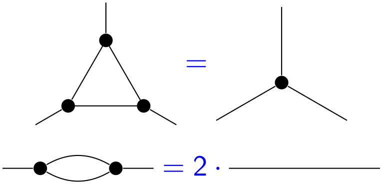

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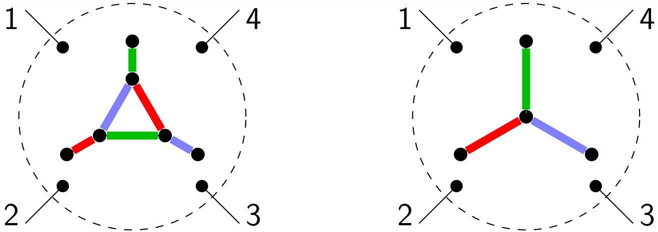
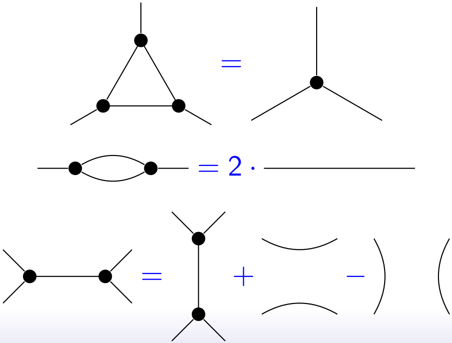
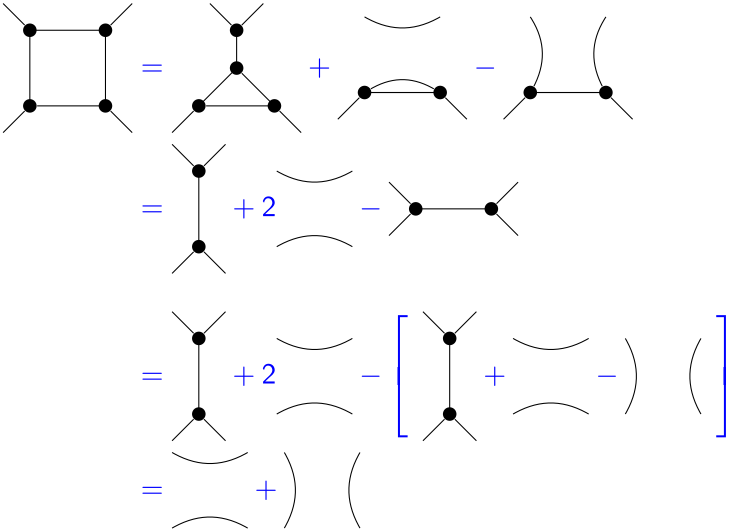
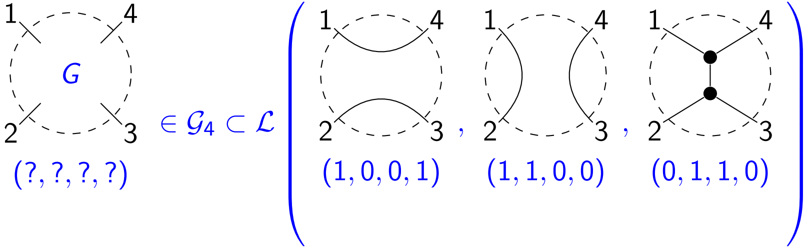






Representation as a linear subspace

Let $\mathcal{G}_4$ be vectors $(n_{RRRR}, n_{RRBB}, n_{RBRB}, n_{RBBR})$ of all graphs with 4 half-edges.



$\mathcal{G}_d$ is in a linear combination of vectors corresponding to forests.

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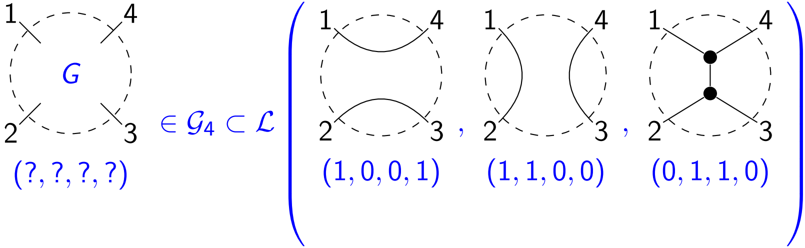
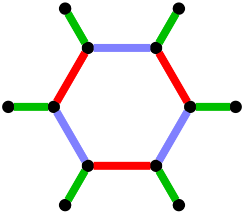


$\mathcal{G}_d$ is in a linear combination of vectors corresponding to forests.
Can one do better and find a cone?
(linear combinations with non-negative coefficients preserve positive coordinates)

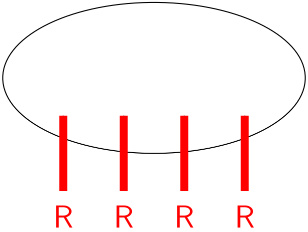
Best cones we found!

Kempe chain relations

Kempe chains are paths and cycles.



$n_G(\psi) := \#$ extensions of ψ to G



$n_{RRRR} =$

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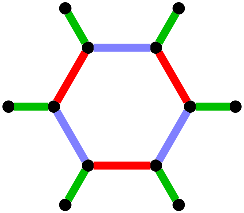

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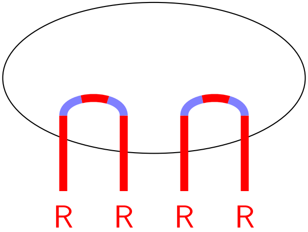

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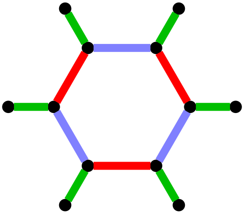

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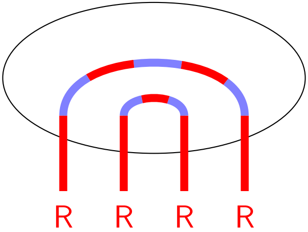
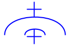


$n_{RRRR} = n$  $+ \; n$ 

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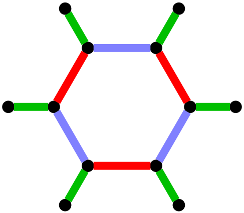

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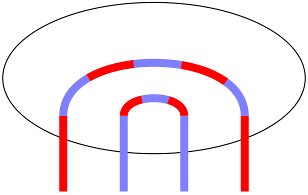

$$n_{RRRR} = n \qquad + \quad n$$

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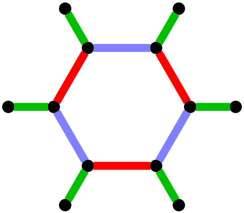

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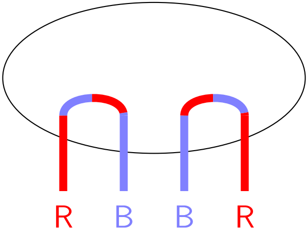

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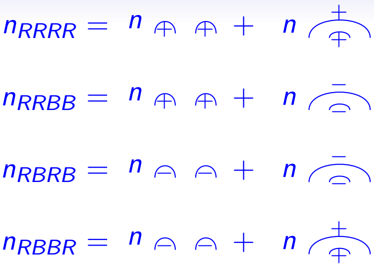
Resulting system of equations



and all ≥ 0.

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$$n_{RRRR} = n + n$$

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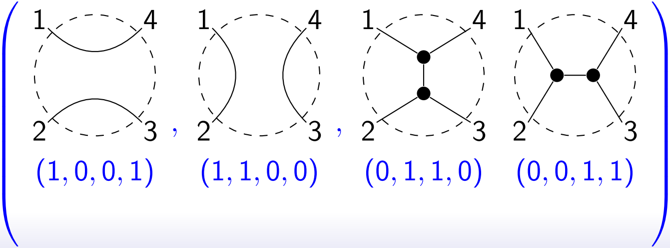
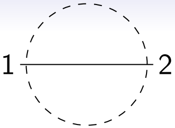
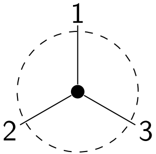
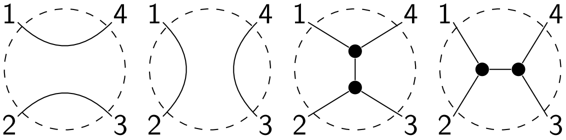
and all ≥ 0. Solution:

$\mathcal{G}_4 \subseteq \mathcal{C}one$

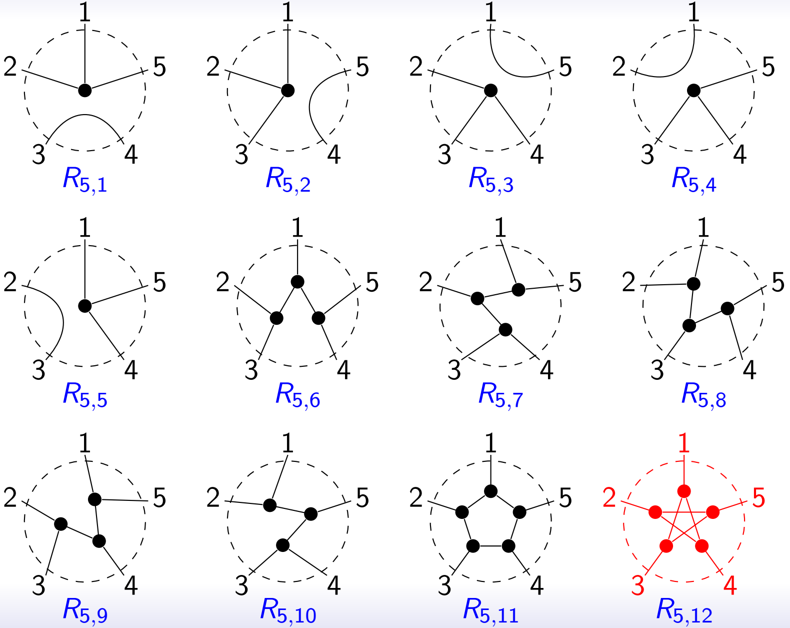


$\mathcal{G}_2 \subseteq \mathcal{C}one \Bigg($  $\Bigg) =: K_2$

$\mathcal{G}_3 \subseteq \mathcal{C}one \Bigg($  $\Bigg) =: K_3$

$\mathcal{G}_4 \subseteq \mathcal{C}one \Bigg($  $\Bigg) =: K_4$

Rays for $\mathcal{G}_5$ cone

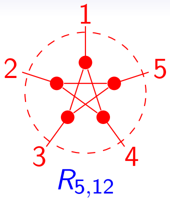


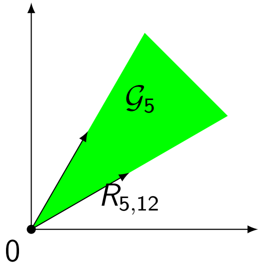
Lemma

The following claims are equivalent.

(a) *Every planar cubic 2-edge-connected graph is 3-edge-colorable. (4CT)*

(b) *For every plane near-cubic graph G with 5 half-edges, if $n_G \in ray(R_{5,12})$, then $n_G = \mathbf{0}$.*





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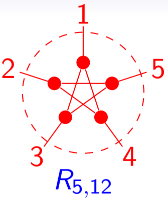
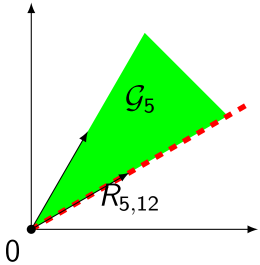
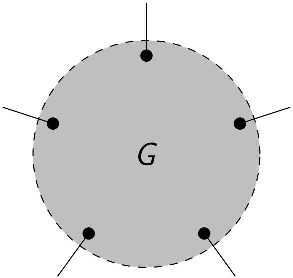




Sketch $(a) \implies (b)$

Let G have $n_G \in ray(R_{5,12})$. Goal $n_G = 0$.



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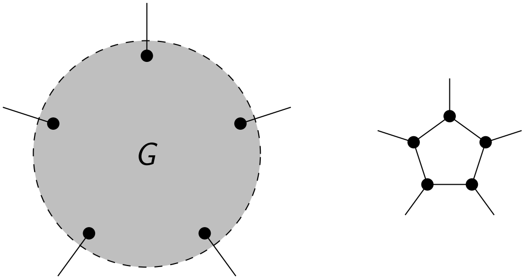

Glue G with C_5 to G as $G \oplus C_5$.

Sketch $(a) \implies (b)$

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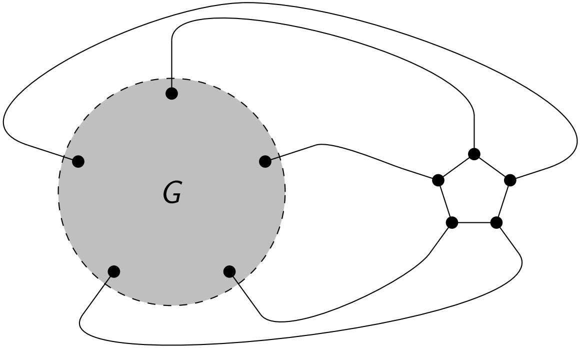

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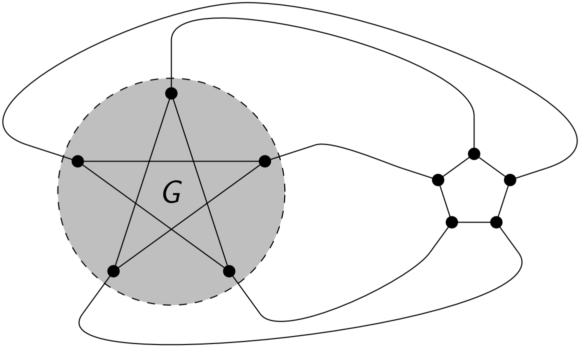

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$G \oplus C_5$ is not 3-edge-colorable (Petersen graph).

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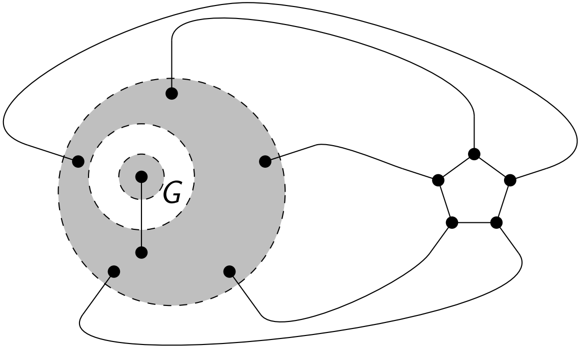
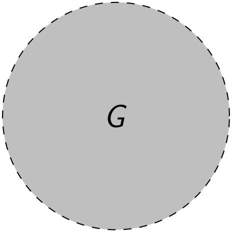


Glue G with C_5 to G as $G \oplus C_5$.
$G \oplus C_5$ is not 3-edge-colorable (Petersen graph).
By (a), G has a bridge.
G no precoloring extends so $n_G = 0$

Sketch $(b) \implies (a)$

Let G be a smallest plane 2-edge-connected graph that is not 3-edge-colorable. Assume (b) and show G is 3-edge-colorable.



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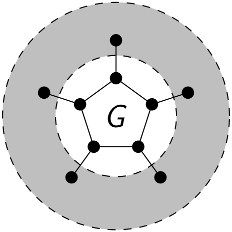

Find a 5-face C_5

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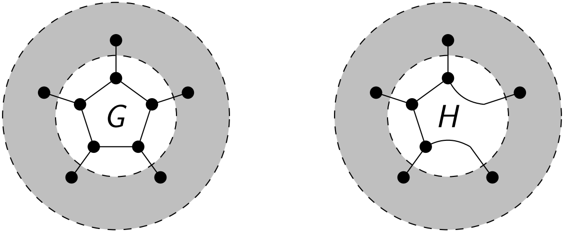

Find a 5-face C_5, replace it by a path

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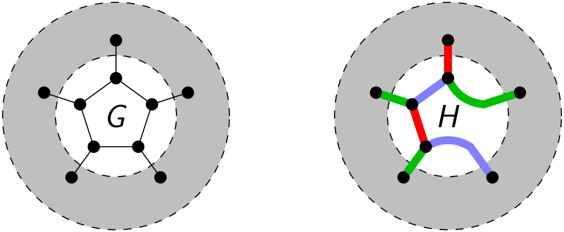

Find a 5-face C_5, replace it by a path, now H is 3-edge-colorable.

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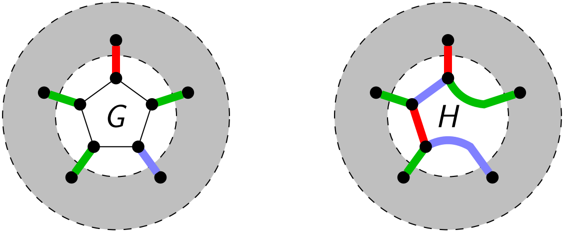

Find a 5-face C_5, replace it by a path, now H is 3-edge-colorable.
$G - C_5$ has a 3-edge-coloring.

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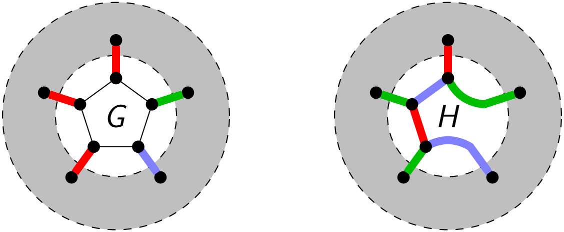

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Since $n_{G-C_5} \notin ray(R_{5,12})$, n_{G-C_5} is positive at entries of another ray. Hence there is a 3-edge-coloring of $G - C_5$ that extends to C_5 (after checking cases).

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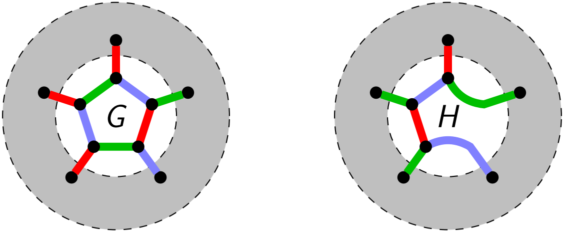

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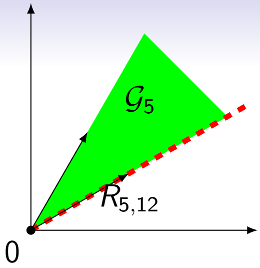
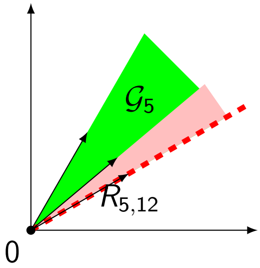
LEMMA

Every plane near-cubic graph G with 5 half-edges satisfies $n_G \in \mathit{Cone}\,(R_{5,1}, \ldots, R_{5,12}) \setminus \mathit{ray}^+(R_{5,12})$.



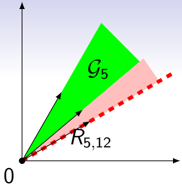
CONJECTURE (DVOŘÁK, L.)

Every plane near-cubic graph G with 5 half-edges satisfies $n_G \in \mathit{Cone}\,(R_{5,1}, \ldots, R_{5,11})$.



THEOREM (DVOŘÁK, L.)

Any counterexample to the conjecture has at least 29 vertices.



Conjecture (Dvořák, L.)

Every plane near-cubic graph G with 5 half-edges satisfies $n_G \in Cone\left(R_{5,1}, \dots, R_{5,11}\right) := K_5$.

Theorem (Dvořák, L.)

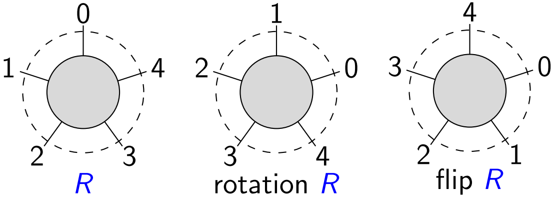
Any counterexample to the conjecture has at least 29 vertices.

Plan (other than checking all graphs on 28 vertices)

- Start with cones $K_2, \dots, K_5$ for graphs with up to 5 half-edges without $R_{5,12}$.
  (We hope $\mathcal{G}_5 \subset K_5$.)
- Generate cones K_6 and K_7 for graphs with 6 and 7 half-edges by combining rays from cones $K_2, \dots, K_7$.
  (Not necessarily $\mathcal{G}_6 \subset K_6$ and $\mathcal{G}_7 \subset K_7$.)
- Graphs with at most 28 vertices and 5 half-edges are covered by K_5.

Operations on rays

Cones $K_2, \ldots, K_7$ defined by rays. Closed under the following:



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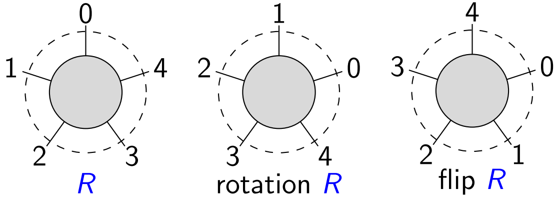

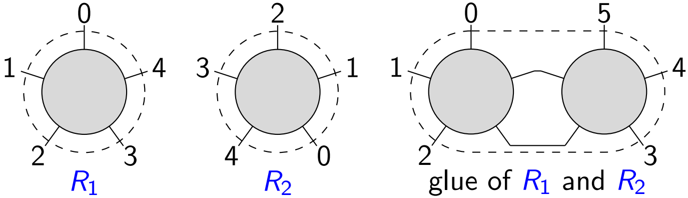

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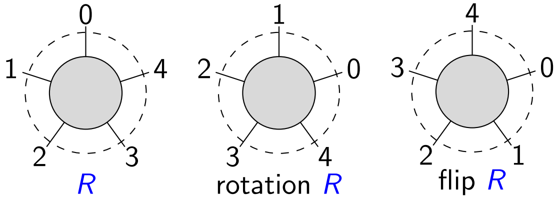
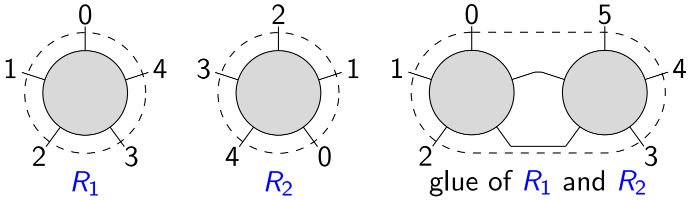
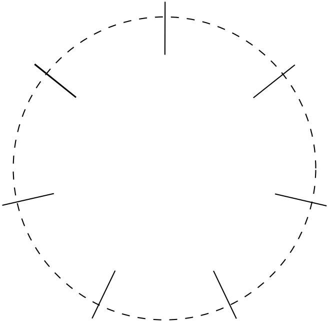




K_6 has 102 rays, K_7 has 22,605 rays

"Human readable proof" has 48GB and 100,405,321 lines.

Suppose G' with 5 half-edges is smallest such that $n_{G'} \notin K_5$.
Then there is G with $n_G \notin K_7$ looking like



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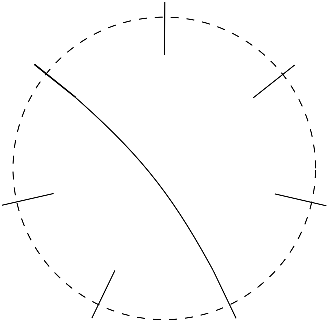

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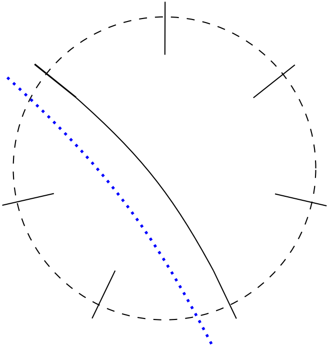

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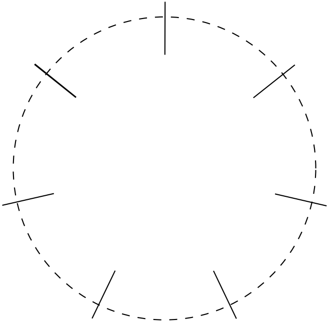

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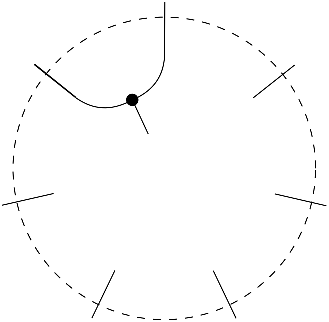

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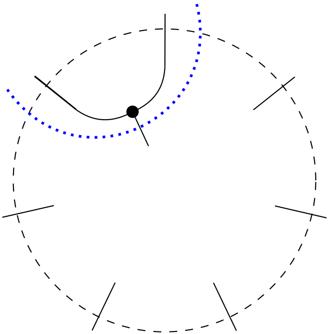

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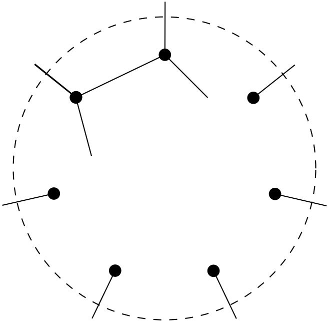

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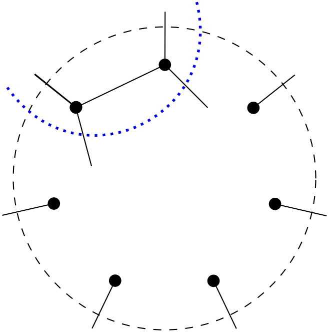

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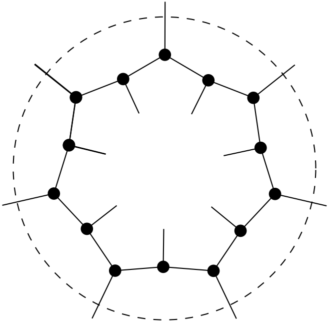

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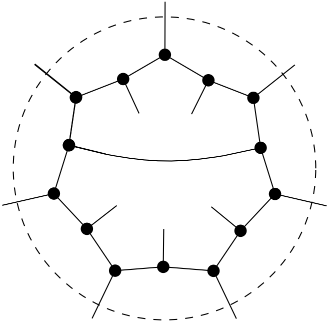

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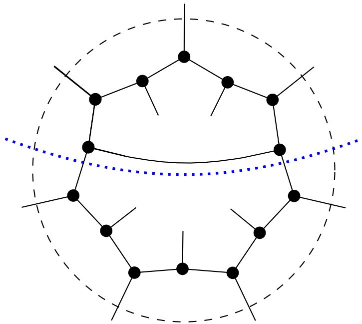

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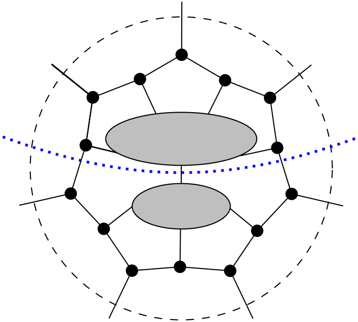

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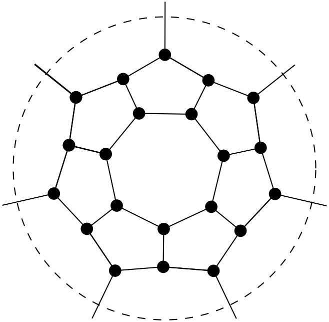

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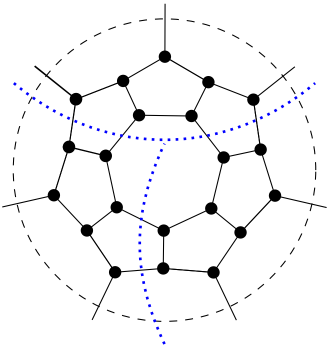
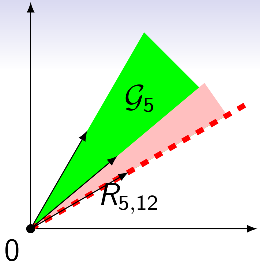
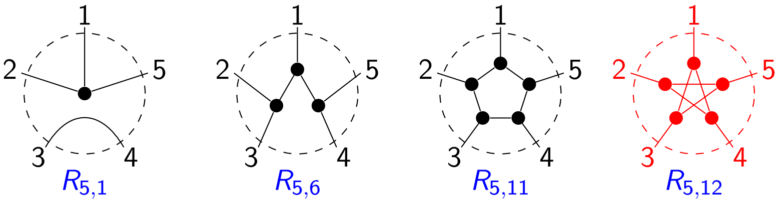


We generate also some part of K_8 to allow this cut.
Eventually gives $|V(G')| \geq 29$.

Conjecture (Dvořák, L.)

Every plane near-cubic graph G with 5 half-edges satisfies $n_G \in \mathit{Cone}\,(R_{5,1}, \ldots, R_{5,11})$.





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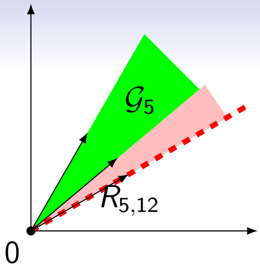

Thank you for your attention.