

# COLORING COUNT CONES OF PLANAR GRAPHS

Zdeněk Dvořák    Bernard Lidický

CanaDAM  
Vancouver, BC  
May 28, 2019



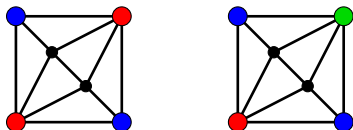
# MOTIVATION

## THEOREM (4CT)

*Every planar graph is 4-colorable.*

## PROBLEM

*Is there a polynomial-time algorithm to decide if a precoloring of a 4-face extends? (all other faces are triangles)*



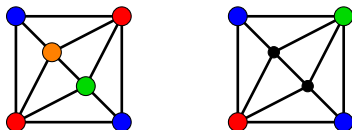
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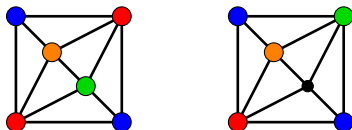
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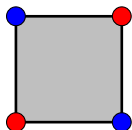
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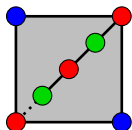
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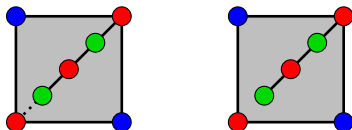
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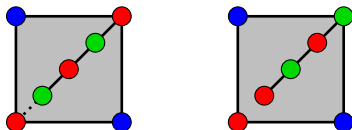
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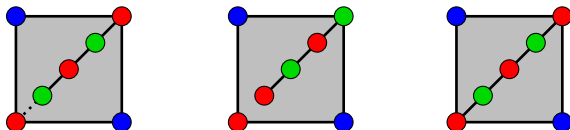
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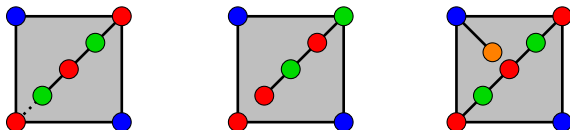
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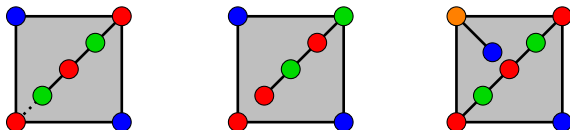
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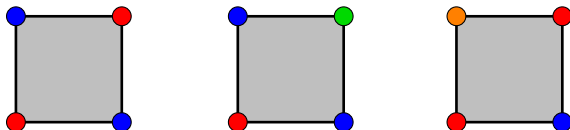
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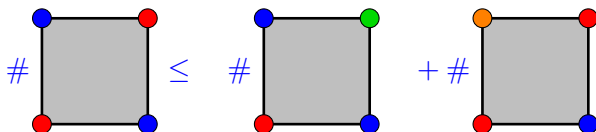
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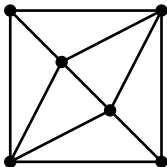
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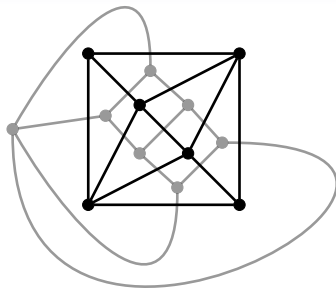
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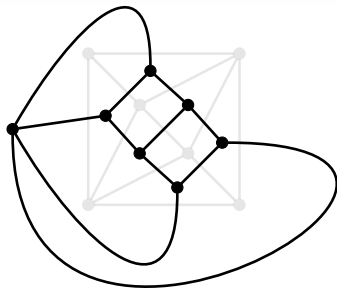
# DUAL



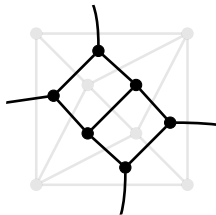
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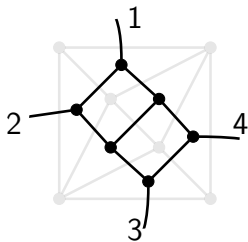
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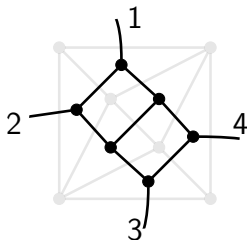
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$G$  is a *near cubic plane graph*, 3-edge-coloring of  $G$   
 $\psi$  precoloring of half edges of  $G$

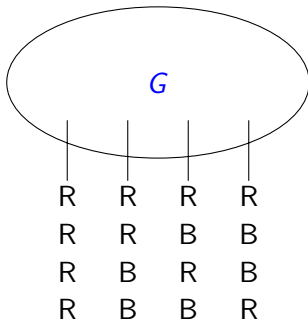
$$n_G(\psi) := \# \text{ extensions of } \psi \text{ to } G$$

Our goal is to “describe” vectors

$$(n_G(\psi_1), n_G(\psi_2), n_G(\psi_3), \dots)$$

$$n_G(\psi) := \# \text{ extensions of } \psi \text{ to } G$$

For  $G$  with  $d$  half edges if  $n_G(\psi) \neq 0$  then  
 $|\psi^{-1}(R)| \equiv |\psi^{-1}(G)| \equiv |\psi^{-1}(B)| \equiv d \pmod{2}$ .

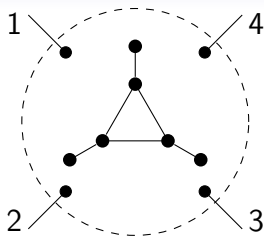


$$n_{RRRR} = n_{GGGG} = n_{BBBB}$$

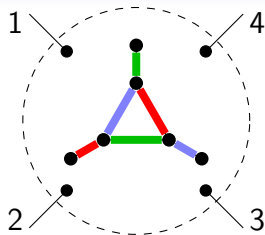
Goal: Describe vectors

$$(n_{RRRR}, n_{RRBB}, n_{RBRB}, n_{RBBR}).$$

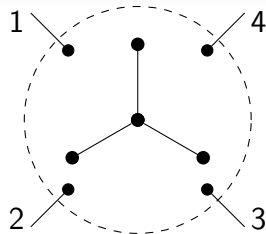
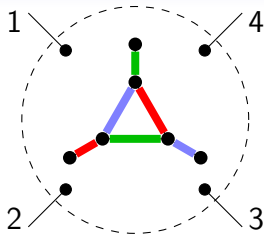
# REDUCTIONS WITH FIXED $\psi$



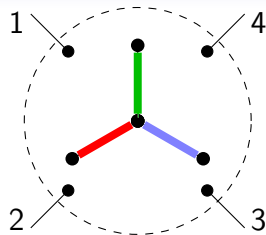
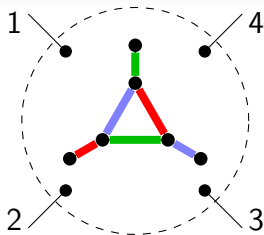
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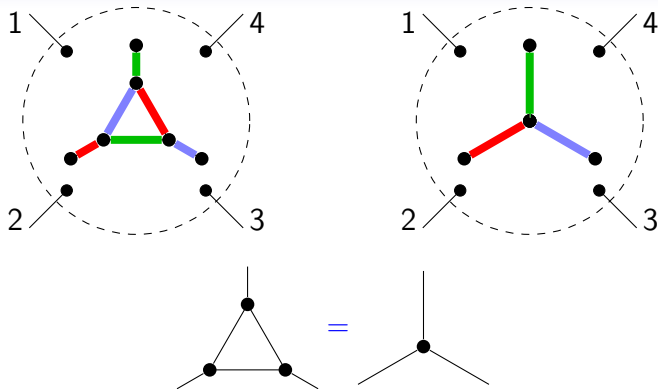
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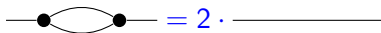
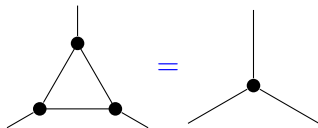
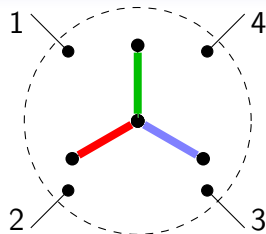
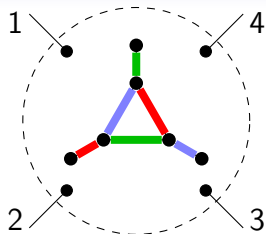
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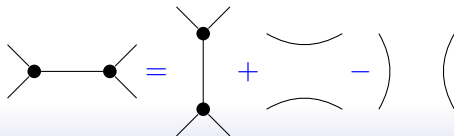
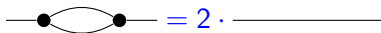
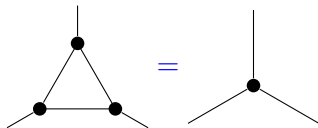
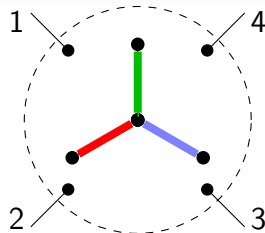
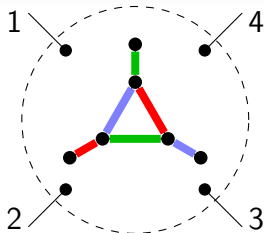
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$$\begin{aligned}
 & \text{Diagram 1} = \text{Diagram 2} + \text{Diagram 3} - \text{Diagram 4} \\
 & \text{Diagram 2} = \text{Diagram 5} + 2 \text{Diagram 6} - \text{Diagram 7} \\
 & \text{Diagram 5} + 2 \text{Diagram 6} - \left[ \text{Diagram 5} + \text{Diagram 6} - \text{Diagram 8} \right] \\
 & = \text{Diagram 6} + \text{Diagram 8}
 \end{aligned}$$

The diagrams are as follows:
 

- Diagram 1:** A square with four vertices, each having two external lines extending outwards.
- Diagram 2:** A triangle with three vertices, each having two external lines extending outwards.
- Diagram 3:** Two vertices connected by a horizontal line, with a curved line above them.
- Diagram 4:** Two vertices connected by a horizontal line, with two curved lines on the right side.
- Diagram 5:** Two vertices connected by a vertical line, each having two external lines extending outwards.
- Diagram 6:** Two vertices connected by a horizontal line, with two curved lines above them.
- Diagram 7:** Two vertices connected by a horizontal line, each having two external lines extending outwards.
- Diagram 8:** Two vertices connected by a horizontal line, with two curved lines on the right side.

## REPRESENTATION AS A LINEAR SUBSPACE

Let  $\mathcal{G}_4$  be vectors  $(n_{RRRR}, n_{RRBB}, n_{RBRB}, n_{RBBR})$  of all graphs with 4 half-edges.

$$\begin{pmatrix} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \end{pmatrix} \in \mathcal{G}_4 \subset \mathcal{L}$$

$\mathcal{G}_d$  is in a linear combination of vectors corresponding to forests.

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$$\begin{array}{c}
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 \text{1} \quad \text{4} \\
 \diagdown \quad \diagup \\
 \text{---} \text{ } \text{ } \text{---} \\
 \diagup \quad \diagdown \\
 \text{2} \quad \text{3}
 \end{array}
 \begin{array}{c}
 G \\
 \\
 (?)
 \end{array}
 \in \mathcal{G}_4 \subset \mathcal{L}
 \end{array}
 \left(
 \begin{array}{c}
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 \text{1} \quad \text{4} \\
 \diagdown \quad \diagup \\
 \text{---} \text{ } \text{---} \\
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 \text{2} \quad \text{3}
 \end{array}
 \begin{array}{c}
 (1, 0, 0, 1)
 \end{array},
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 \diagdown \quad \diagup \\
 \text{---} \text{ } \text{---} \\
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 \end{array}
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 \end{array}
 \begin{array}{c}
 (0, 1, 1, 0)
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 \right)$$

$\mathcal{G}_d$  is in a linear combination of vectors corresponding to forests.

Can one do better and find a cone?

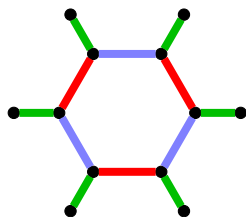
(linear combinations with non-negative coefficients preserve positive coordinates )

# BEST CONES WE FOUND!

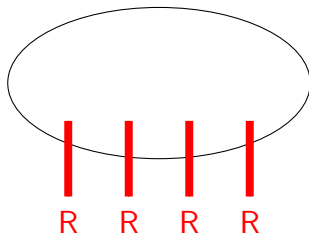


# KEMPE CHAIN RELATIONS

Kempe chains are paths and cycles.



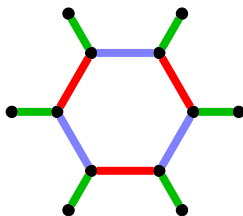
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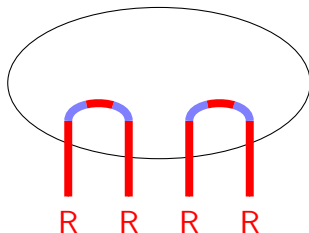
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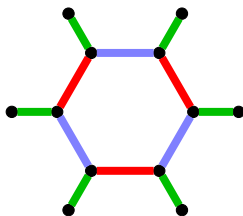
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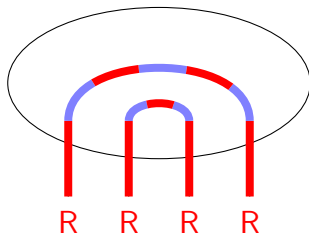
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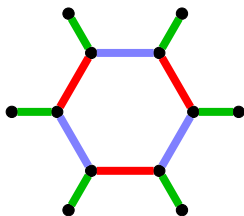
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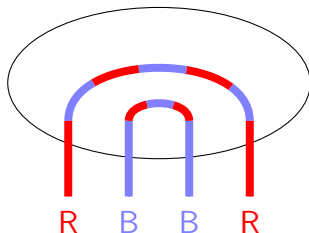
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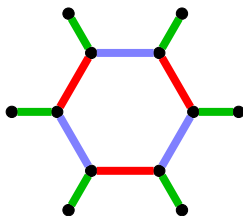


$$n_{RRRR} = n \begin{array}{c} \oplus \\ \oplus \end{array} + n \begin{array}{c} \oplus \\ \oplus \end{array}$$

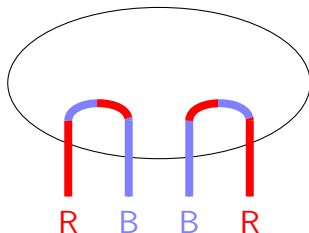
$$n_{RBBR} = \quad + n \begin{array}{c} \oplus \\ \oplus \end{array}$$

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$$n_{RRRR} = n \oplus \oplus + n \oplus \oplus \oplus$$

$$n_{RBBR} = n \ominus \ominus + n \oplus \oplus \oplus$$

## Resulting system of equations

$$n_{RRRR} = n \oplus \oplus + n \overbrace{\oplus}^{+}$$

$$n_{RRBB} = n \oplus \oplus + n \overbrace{\oplus}^{-}$$

$$n_{RBRB} = n \ominus \ominus + n \overbrace{\ominus}^{-}$$

$$n_{RBBR} = n \ominus \ominus + n \overbrace{\oplus}^{+}$$

and all  $\geq 0$ .

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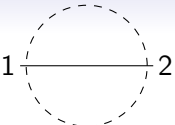
$$n_{RRBB} = n \oplus \oplus + n \begin{array}{c} \ominus \\ \ominus \end{array}$$

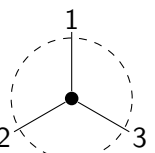
$$n_{RBRB} = n \ominus \ominus + n \begin{array}{c} \ominus \\ \ominus \end{array}$$

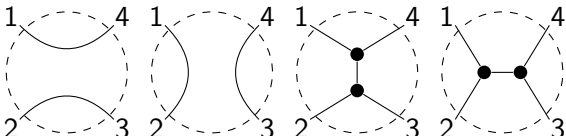
$$n_{RBBR} = n \ominus \ominus + n \begin{array}{c} \oplus \\ \oplus \end{array}$$

and all  $\geq 0$ . Solution:

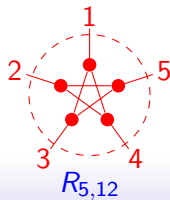
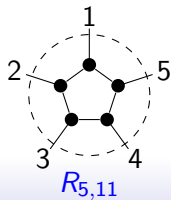
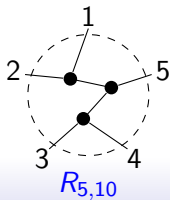
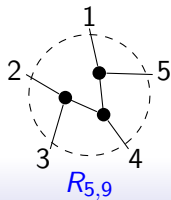
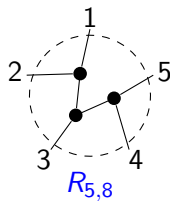
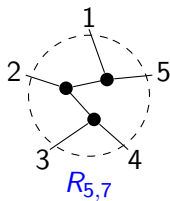
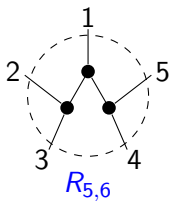
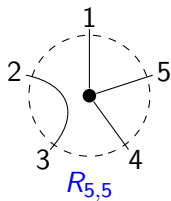
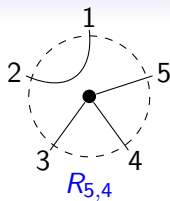
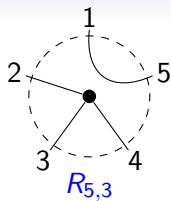
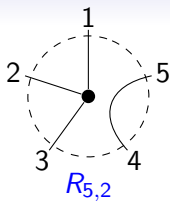
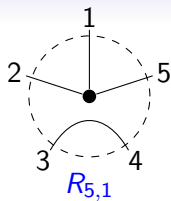
$$\mathcal{G}_4 \subseteq \text{Cone} \left( \begin{array}{cccc} \begin{array}{c} \text{Diagram 1: Dashed circle with two internal arcs connecting vertices 1-4 and 2-3.} \\ (1, 0, 0, 1) \end{array} & \begin{array}{c} \text{Diagram 2: Dashed circle with two internal arcs connecting vertices 1-2 and 3-4.} \\ (1, 1, 0, 0) \end{array} & \begin{array}{c} \text{Diagram 3: Dashed circle with two internal arcs connecting vertices 1-3 and 2-4, meeting at a central point.} \\ (0, 1, 1, 0) \end{array} & \begin{array}{c} \text{Diagram 4: Dashed circle with two internal arcs connecting vertices 1-2 and 3-4, meeting at a central point.} \\ (0, 0, 1, 1) \end{array} \end{array} \right)$$

$$\mathcal{G}_2 \subseteq \text{Cone} \left( 1 \text{ --- } 2 \right) =: K_2$$


$$\mathcal{G}_3 \subseteq \text{Cone} \left( \begin{array}{c} 1 \\ | \\ \bullet \\ / \quad \backslash \\ 2 \quad 3 \end{array} \right) =: K_3$$


$$\mathcal{G}_4 \subseteq \text{Cone} \left( \begin{array}{cccc} \begin{array}{c} 1 \quad 4 \\ \text{---} \\ 2 \quad 3 \end{array} & \begin{array}{c} 1 \quad 4 \\ | \\ 2 \quad 3 \end{array} & \begin{array}{c} 1 \quad 4 \\ \bullet \\ \bullet \\ / \quad \backslash \\ 2 \quad 3 \end{array} & \begin{array}{c} 1 \quad 4 \\ \bullet \quad \bullet \\ / \quad \backslash \quad / \quad \backslash \\ 2 \quad 3 \end{array} \end{array} \right) =: K_4$$


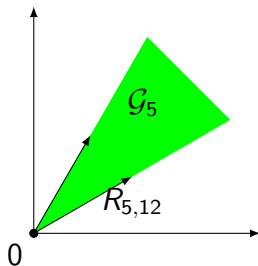
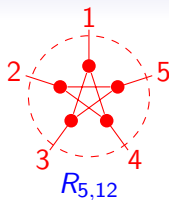
# RAYS FOR $\mathcal{G}_5$ CONE



## LEMMA

The following claims are equivalent.

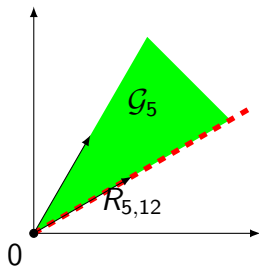
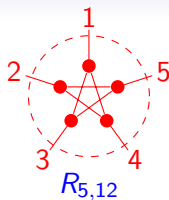
- (a) Every planar cubic 2-edge-connected graph is 3-edge-colorable. (4CT)
- (b) For every plane near-cubic graph  $G$  with 5 half-edges, if  $n_G \in \text{ray}(R_{5,12})$ , then  $n_G = \mathbf{0}$ .



## LEMMA

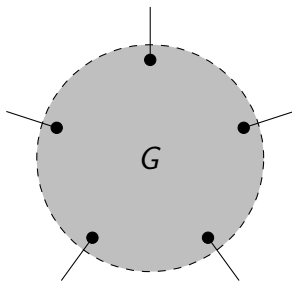
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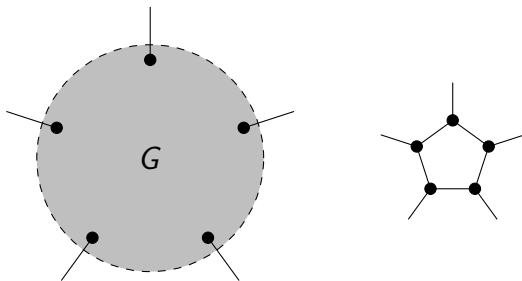
Sketch  $(a) \implies (b)$

Let  $G$  have  $n_G \in \text{ray}(R_{5,12})$ . Goal  $n_G = 0$ .



Sketch (a)  $\implies$  (b)

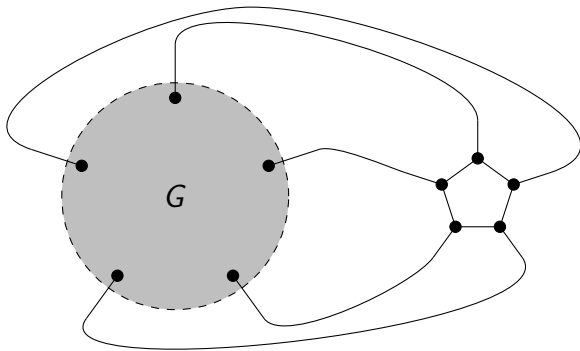
Let  $G$  have  $n_G \in \text{ray}(R_{5,12})$ . Goal  $n_G = 0$ .



Glue  $G$  with  $C_5$  to  $G$  as  $G \oplus C_5$ .

Sketch (a)  $\Rightarrow$  (b)

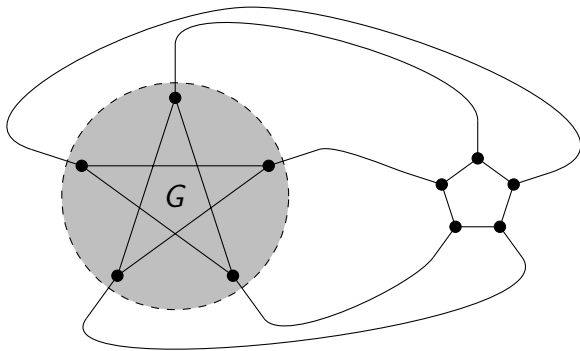
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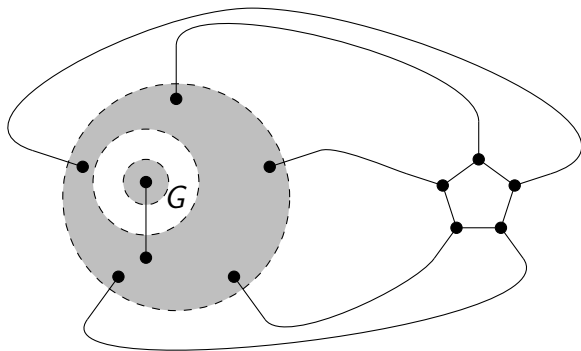


Glue  $G$  with  $C_5$  to  $G$  as  $G \oplus C_5$ .

$G \oplus C_5$  is not 3-edge-colorable (Petersen graph).

Sketch (a)  $\implies$  (b)

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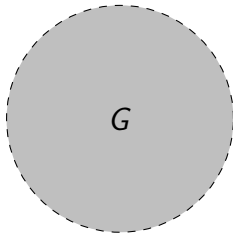
$G \oplus C_5$  is not 3-edge-colorable (Petersen graph).

By (a),  $G$  has a bridge.

$G$  no precoloring extends so  $n_G = 0$

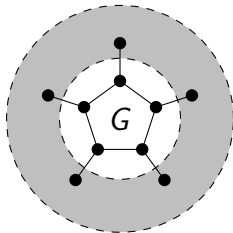
Sketch  $(b) \implies (a)$

Let  $G$  be a smallest plane 2-edge-connected graph that is not 3-edge-colorable. Assume  $(b)$  and show  $G$  is 3-edge-colorable.



Sketch (b)  $\Rightarrow$  (a)

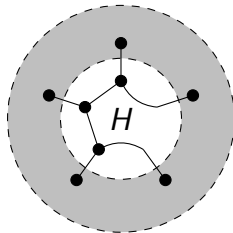
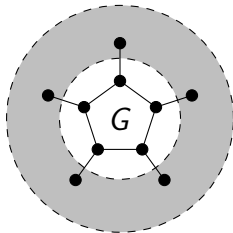
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Find a 5-face  $C_5$

Sketch  $(b) \Rightarrow (a)$

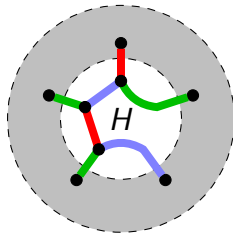
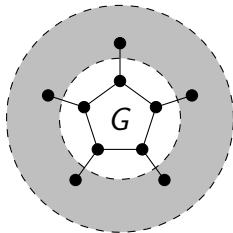
Let  $G$  be a smallest plane 2-edge-connected graph that is not 3-edge-colorable. Assume  $(b)$  and show  $G$  is 3-edge-colorable.



Find a 5-face  $C_5$ , replace it by a path

Sketch (b)  $\Rightarrow$  (a)

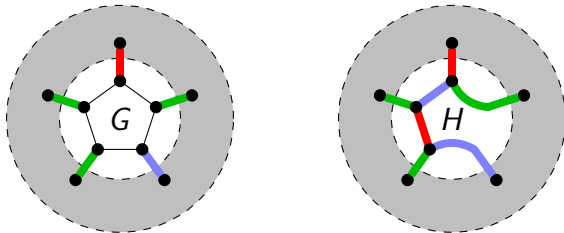
Let  $G$  be a smallest plane 2-edge-connected graph that is not 3-edge-colorable. Assume (b) and show  $G$  is 3-edge-colorable.



Find a 5-face  $C_5$ , replace it by a path, now  $H$  is 3-edge-colorable.

Sketch (b)  $\Rightarrow$  (a)

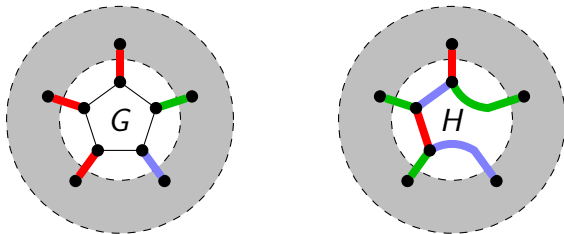
Let  $G$  be a smallest plane 2-edge-connected graph that is not 3-edge-colorable. Assume (b) and show  $G$  is 3-edge-colorable.



Find a 5-face  $C_5$ , replace it by a path, now  $H$  is 3-edge-colorable.  
 $G - C_5$  has a 3-edge-coloring.

Sketch (b)  $\Rightarrow$  (a)

Let  $G$  be a smallest plane 2-edge-connected graph that is not 3-edge-colorable. Assume (b) and show  $G$  is 3-edge-colorable.

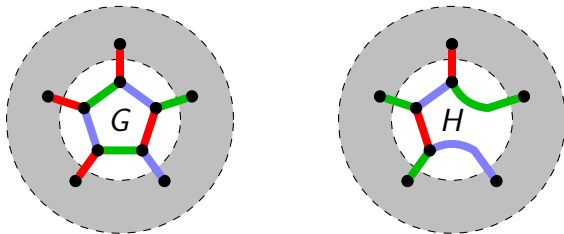


Find a 5-face  $C_5$ , replace it by a path, now  $H$  is 3-edge-colorable.  $G - C_5$  has a 3-edge-coloring.

Since  $n_{G-C_5} \notin \text{ray}(R_{5,12})$ ,  $n_{G-C_5}$  is positive at entries of another ray. Hence there is a 3-edge-coloring of  $G - C_5$  that extends to  $C_5$  (after checking cases).

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### LEMMA

Every plane near-cubic graph  $G$  with 5 half-edges satisfies

$$n_G \in \text{Cone}(R_{5,1}, \dots, R_{5,12}) \setminus \text{ray}^+(R_{5,12}).$$

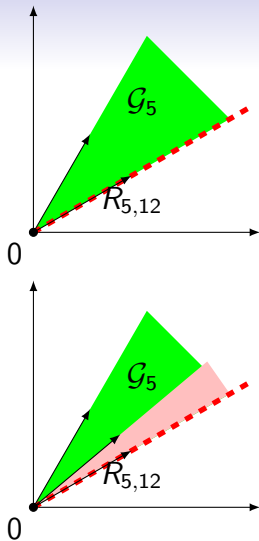
### CONJECTURE (DVOŘÁK, L.)

Every plane near-cubic graph  $G$  with 5 half-edges satisfies

$$n_G \in \text{Cone}(R_{5,1}, \dots, R_{5,11}).$$

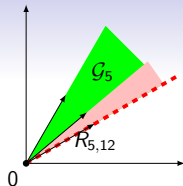
### THEOREM (DVOŘÁK, L.)

Any counterexample to the conjecture has at least 29 vertices.



## CONJECTURE (DVOŘÁK, L.)

Every plane near-cubic graph  $G$  with 5 half-edges satisfies  $n_G \in \text{Cone}(R_{5,1}, \dots, R_{5,11}) := K_5$ .



## THEOREM (DVOŘÁK, L.)

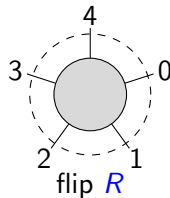
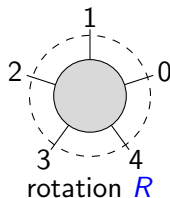
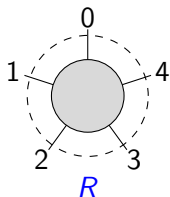
Any counterexample to the conjecture has at least 29 vertices.

Plan (other than checking all graphs on 28 vertices)

- Start with cones  $K_2, \dots, K_5$  for graphs with up to 5 half-edges without  $R_{5,12}$ .  
(We hope  $\mathcal{G}_5 \subset K_5$ .)
- Generate cones  $K_6$  and  $K_7$  for graphs with 6 and 7 half-edges by combining rays from cones  $K_2, \dots, K_7$ .  
(Not necessarily  $\mathcal{G}_6 \subset K_6$  and  $\mathcal{G}_7 \subset K_7$ .)
- Graphs with at most 28 vertices and 5 half-edges are covered by  $K_5$ .

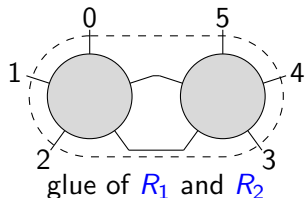
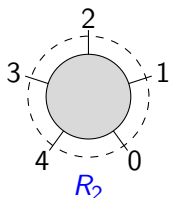
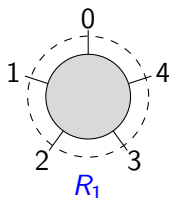
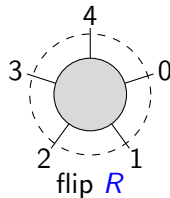
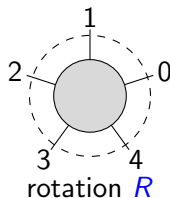
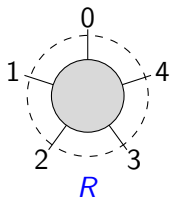
# OPERATIONS ON RAYS

Cones  $K_2, \dots, K_7$  defined by rays. Closed under the following:



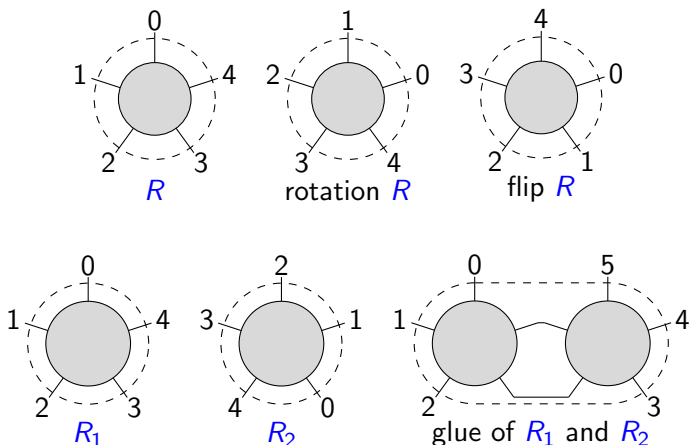
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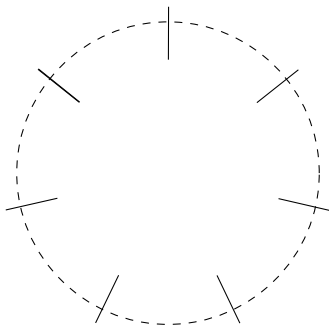
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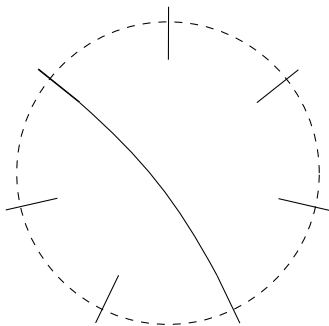
$K_6$  has 102 rays,  $K_7$  has 22,605 rays

“Human readable proof” has 48GB and 100,405,321 lines.

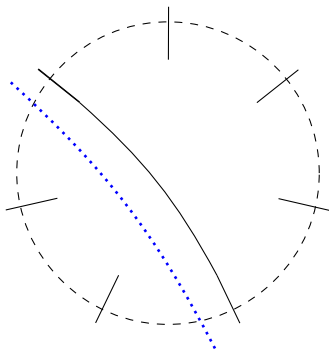
Suppose  $G'$  with 5 half-edges is smallest such that  $n_{G'} \notin K_5$ .  
 Then there is  $G$  with  $n_G \notin K_7$  looking like



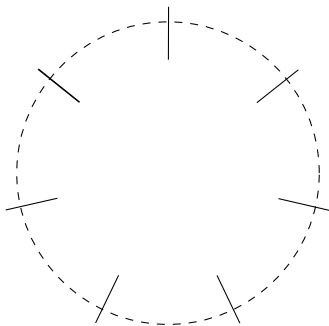
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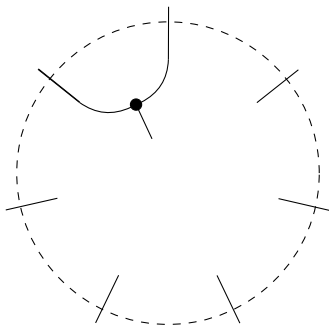
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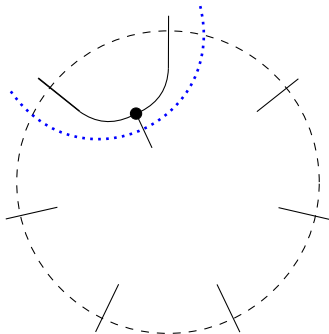
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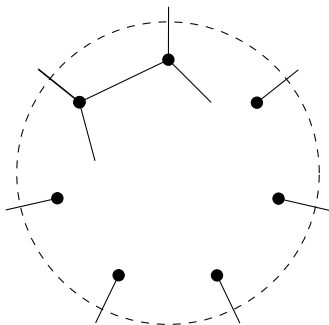
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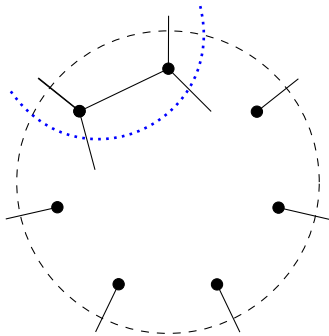
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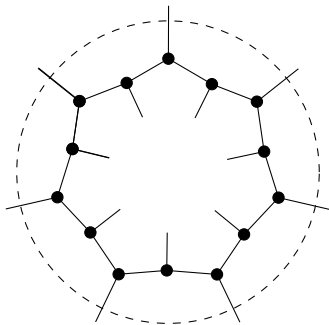
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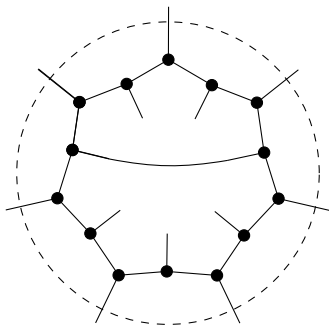
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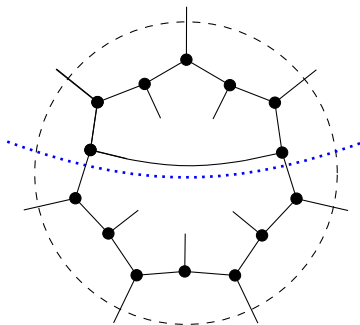
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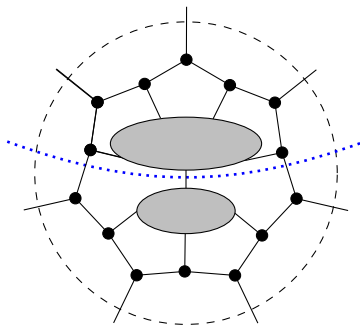
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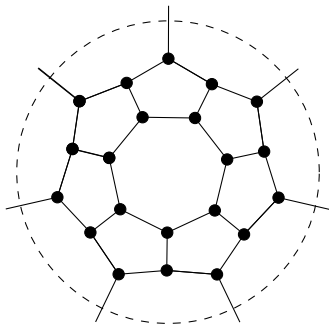
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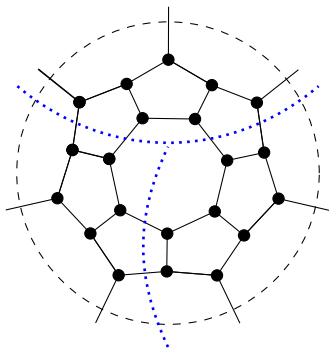
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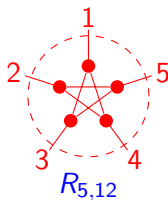
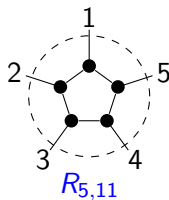
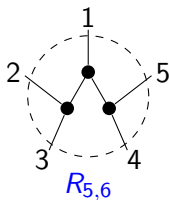
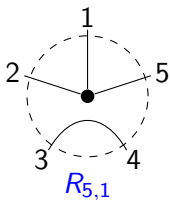
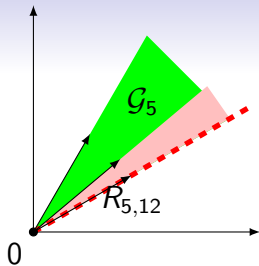


We generate also some part of  $K_8$  to allow this cut.  
 Eventually gives  $|V(G')| \geq 29$ .

# CONJECTURE (DVOŘÁK, L.)

Every plane near-cubic graph  $G$  with 5 half-edges satisfies

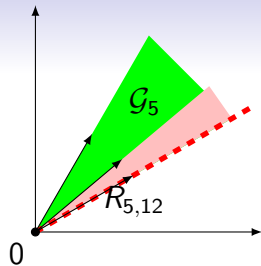
$n_G \in \text{Cone}(R_{5,1}, \dots, R_{5,11})$ .



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Thank you for your attention.

