Flag Algebra Methods (hopefully easy basics)

Bernard Lidický

6th Lake Michigan Workshop on Combinatorics and Graph Theory

Apr 7, 2019

Outline

- Flag Algebras "definitions"
- First try for Mantel's theorem
- More automatic approach
- Additional constraints
- maybe break
- Define flag algebras
- Graph sequences and homomorphisms
- Turán's Theorem (in limit)
- Finally Mantel's Theorem (for real)
- mega break
- Applications

Thanks to Florian Pfender and Jan Volec

Flag algebras

Seminal paper:

Razborov, Flag Algebras, *Journal of Symbolic Logic* **72** (2007), 1239–1282.

David P. Robbins Prize by AMS for Razborov in 2013

Flag algebras

Seminal paper:
Razborov, Flag Algebras, *Journal of Symbolic Logic* **72** (2007), 1239–1282.
David P. Robbins Prize by AMS for Razborov in 2013

Example (Goodman, Razborov)

If density of edges is at least $\rho > 0$, what is the minimum density of triangles?

- Designed to attack extremal problems.
- Works well if constraints as well as desired value can be computed by checking small subgraphs (or average over small subgraphs).
- The results are for the limit as graphs get very large.

Applications (early incomplete list)

Author	Year	Application/Result
Razborov	2008	edge density vs. triangle density
Hladký, Kráľ, Norin	2009	Bounds for the Caccetta-Haggvist conjectur
Razborov	2010	On 3-hypergraphs with forbidden 4-vertex co
Hatami, Hladký,Kráľ,Norine,Razborov / Grzesik	2011	Erdős Pentagon problem
Hatami, Hladký, Kráľ, Norin, Razborov	2012	Non-Three-Colourable Common Graphs Exist
Balogh, Hu, L., Liu / Baber	2012	4-cycles in hypercubes
Das, Huang, Ma, Naves, Sudakov	2013	minimum number of k-cliques
Baber, Talbot	2013	A Solution to the 2/3 Conjecture
Falgas-Ravry, Vaughan	2013	Turán density of many 3-graphs
Cummings, Kráľ, Pfender, Sperfeld, Treglown, Young	2013	Monochromatic triangles in 3-edge colored
Kramer, Martin, Young	2013	Boolean lattice
Balogh, Hu, L., Pikhurko, Udvari, Volec	2013	Monotone permutations
Norin, Zwols	2013	New bound on Zarankiewicz's conjecture
Huang, Linial, Naves, Peled, Sudakov	2014	3-local profiles of graphs
Balogh, Hu, L., Pfender, Volec, Young	2014	Rainbow triangles in 3-edge colored graphs
Balogh, Hu, L., Pfender	2014	Induced density of C_5
Goaoc, Hubard, de Verclos, Séréni, Volec	2014	Order type and density of convex subsets
Coregliano, Razborov	2015	Tournaments
Alon, Naves, Sudakov	2015	Phylogenetic trees
...	...	...

Applications to graphs, oriented graphs, hypergraphs, hypercubes, permutations, crossing number of graphs, order types, discrete geometry, Ramsey numbers, phylogenetic trees...

Example extremal problem

Theorem (Mantel 1907)

Every n-vertex triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

Problem

Maximize a graph parameter (# of edges) over a class of graphs (triangle-free).

- local condition and global parameter
- threshold
- bound and extremal example

Example extremal problem

Theorem (Mantel 1907)

Every n-vertex triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

Problem

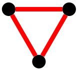
Maximize a graph parameter (# of edges) over a class of graphs (triangle-free).

- local condition and global parameter
- threshold
- bound and extremal example

We will use colors for edges and non-edges.

Flag algebras definitions

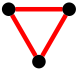
Let G be a 2-edge-colored complete graph on n vertices.



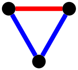
The probability that three random vertices in G span a red triangle, i.e. $\# \triangledown / \binom{n}{3}$.

Flag algebras definitions

Let G be a 2-edge-colored complete graph on n vertices.



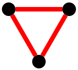
The probability that three random vertices in G span a red triangle, i.e. $\#$$/\binom{n}{3}$.



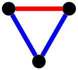
The probability that three random vertices in G span a graph isomorphic to a triangle with one red and two blue edges.

Flag algebras definitions

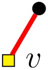
Let G be a 2-edge-colored complete graph on n vertices.



The probability that three random vertices in G span a red triangle, i.e. $\#$$/\binom{n}{3}$.



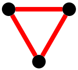
The probability that three random vertices in G span a graph isomorphic to a triangle with one red and two blue edges.



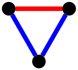
The probability that a random vertex other than v is connected to v by a red edge, i.e., the red degree of v divided by $n-1$.

Flag algebras definitions

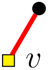
Let G be a 2-edge-colored complete graph on n vertices.



The probability that three random vertices in G span a red triangle, i.e. $\#$$/\binom{n}{3}$.



The probability that three random vertices in G span a graph isomorphic to a triangle with one red and two blue edges.

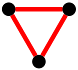


The probability that a random vertex other than v is connected to v by a red edge, i.e., the red degree of v divided by $n-1$.

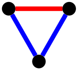
 $+$  $=$

Flag algebras definitions

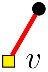
Let G be a 2-edge-colored complete graph on n vertices.



The probability that three random vertices in G span a red triangle, i.e. $\#$  $/\binom{n}{3}$.



The probability that three random vertices in G span a graph isomorphic to a triangle with one red and two blue edges.

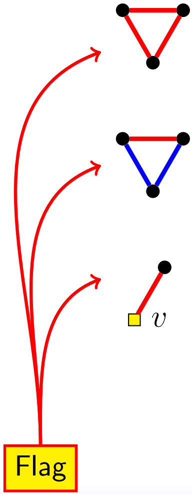


The probability that a random vertex other than v is connected to v by a red edge, i.e., the red degree of v divided by $n-1$.

 $+$  $= 1$

Flag algebras definitions

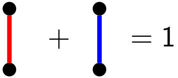
Let G be a 2-edge-colored complete graph on n vertices.

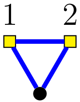


The probability that three random vertices in G span a red triangle, i.e. $\#$  $/\binom{n}{3}$.

The probability that three random vertices in G span a graph isomorphic to a triangle with one red and two blue edges.

The probability that a random vertex other than v is connected to v by a red edge, i.e., the red degree of v divided by $n-1$.

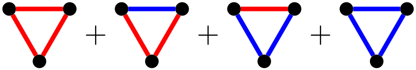




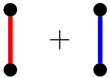
Type - flag induced by labeled vertices

Flag algebras identities

Let G be a 2-edge-colored complete graph on n vertices.

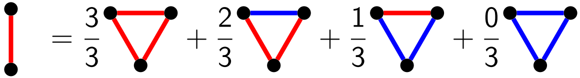
 $= 1$

Same kind as

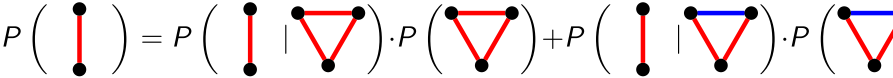
 $= 1.$

Flag algebras identities

Let G be a 2-edge-colored complete graph on n vertices.

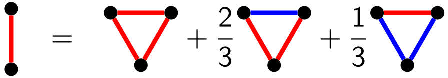


Expanded version:

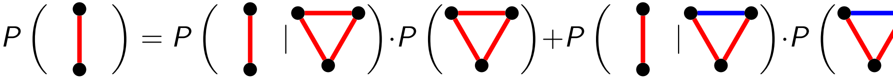


Flag algebras identities

Let G be a 2-edge-colored complete graph on n vertices.

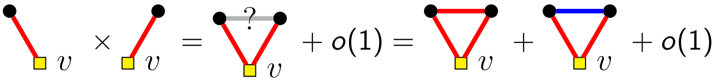


Expanded version:



Flag algebras identities

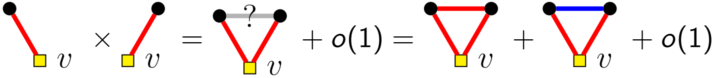
Let G be a 2-edge-colored complete graph on n vertices.

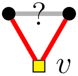


$o(1)$ as $|V(G)| \to \infty$ (will be omitted on next slides)

Flag algebras identities

Let G be a 2-edge-colored complete graph on n vertices.



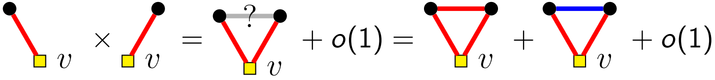


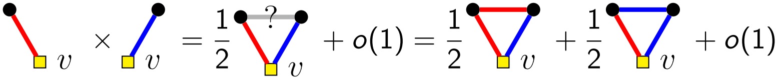
: The probability of choosing two different vertices …

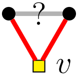
$o(1)$ as $|V(G)| \to \infty$ (will be omitted on next slides)

Flag algebras identities

Let G be a 2-edge-colored complete graph on n vertices.





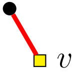
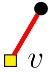
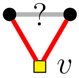
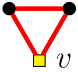
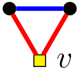


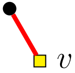
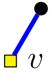
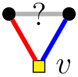
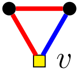
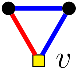
: The probability of choosing two different vertices ...

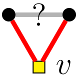
$o(1)$ as $|V(G)| \to \infty$ (will be omitted on next slides)

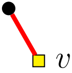
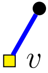
Flag algebras identities

Let G be a 2-edge-colored complete graph on n vertices.

 $\times$  $=$  $+\, o(1) =$  $+$  $+\, o(1)$

 $\times$  $= \frac{1}{2}$  $+\, o(1) = \frac{1}{2}$  $+ \frac{1}{2}$  $+\, o(1)$

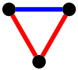
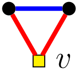
 : The probability of choosing two different vertices ...

 $\times$  : The probability that choosing two vertices u_1, u_2 other than v gives red vu_1 and blue vu_2.

$o(1)$ as $|V(G)| \to \infty$ (will be omitted on next slides)

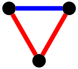
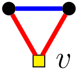
Flag algebras identities

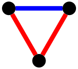
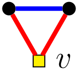
Let G be a 2-edge-colored complete graph on n vertices.

$\frac{1}{3}$  $= \frac{1}{|V(G)|} \sum_{v \in V(G)}$  v

Flag algebras identities

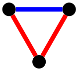
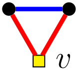
Let G be a 2-edge-colored complete graph on n vertices.

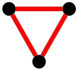
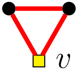
$\frac{1}{3}$  $= \frac{1}{|V(G)|} \sum_{v \in V(G)}$  v

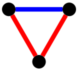
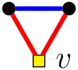
 $\binom{n}{3} = \sum_{v \in V(G)}$  $v \binom{n-1}{2}$

Flag algebras identities

Let G be a 2-edge-colored complete graph on n vertices.

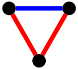
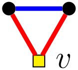
$\frac{1}{3}$  $= \frac{1}{|V(G)|} \sum_{v \in V(G)}$  v

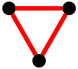
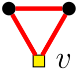
 $= \frac{1}{|V(G)|} \sum_{v \in V(G)}$  v

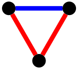
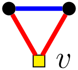
 $\binom{n}{3} = \sum_{v \in V(G)}$  $v \binom{n-1}{2}$

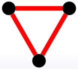
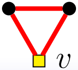
Flag algebras identities

Let G be a 2-edge-colored complete graph on n vertices.

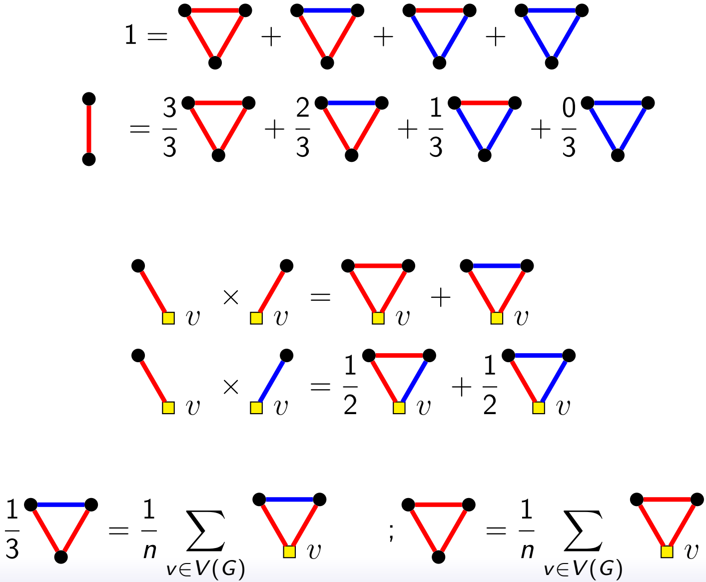
$\frac{1}{3}$  $= \frac{1}{|V(G)|} \sum_{v \in V(G)}$  v

 $= \frac{1}{|V(G)|} \sum_{v \in V(G)}$  v

 $\binom{n}{3} = \sum_{v \in V(G)}$  $v \binom{n-1}{2}$

 $\binom{n}{3} = \frac{1}{3} \sum_{v \in V(G)}$  $v \binom{n-1}{2}$

Identities Summary



First try for Mantel's theorem

- How to use the equations to prove something
- Gives bounds as well as helps with extremal examples

Example - Mantel's Theorem version 1

Theorem (Mantel 1907)

A triangle-free n-vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2}\binom{n}{2}$ *edges.*

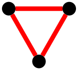
Assume edges are red and non-edges are blue.

Example - Mantel's Theorem version 1

Theorem (Mantel 1907)

A triangle-free n-vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2}\binom{n}{2}$ *edges.*

Assume edges are red and non-edges are blue.

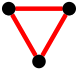
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

Example - Mantel's Theorem version 1

Theorem (Mantel 1907)

A triangle-free n-vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2}\binom{n}{2}$ edges.

Assume edges are red and non-edges are blue.

Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

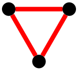
$0 \leq$ $\Bigg(1 - 2$  $v \Bigg)^2$

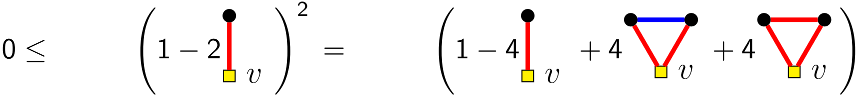
Example - Mantel's Theorem version 1

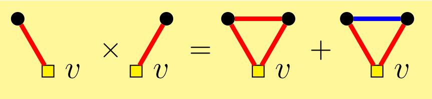
Theorem (Mantel 1907)

A triangle-free n-vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2}\binom{n}{2}$ *edges.*

Assume edges are red and non-edges are blue.

Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)



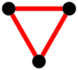


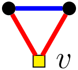
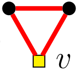
Example - Mantel's Theorem version 1

Theorem (Mantel 1907)

A triangle-free n-vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2}\binom{n}{2}$ *edges.*

Assume edges are red and non-edges are blue.

Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

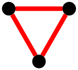
$0 \leq \frac{1}{n}\sum_{v}\Big(1 - 2$  $v\Big)^2 = \frac{1}{n}\sum_{v}\Big(1 - 4$  $v + 4$  $v + 4$  $v\Big)$

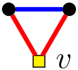
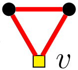
Example - Mantel's Theorem version 1

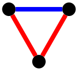
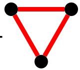
Theorem (Mantel 1907)

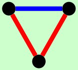
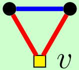
A triangle-free n-vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2}\binom{n}{2}$ *edges.*

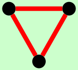
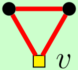
Assume edges are red and non-edges are blue.

Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$0 \leq \frac{1}{n}\sum_v \Big(1 - 2$  $v\Big)^2 = \frac{1}{n}\sum_v \Big(1 - 4$  $v + 4$  $v + 4$  $v\Big)$

$= 1 - 4$  $+ \frac{4}{3}$  $+ 4$ 

$\frac{1}{3}$  $= \frac{1}{n}\sum_{v \in V(G)}$  v

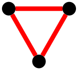
 $= \frac{1}{n}\sum_{v \in V(G)}$  v

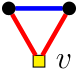
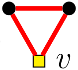
Example - Mantel's Theorem version 1

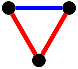
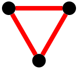
Theorem (Mantel 1907)

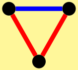
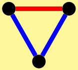
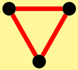
A triangle-free n-vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2}\binom{n}{2}$ *edges.*

Assume edges are red and non-edges are blue.

Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$0 \leq \frac{1}{n}\sum_{v} \Big(1 - 2$  $v\Big)^2 = \frac{1}{n}\sum_{v}\Big(1 - 4$  $v + 4$  $v + 4$  $v\Big)$

$= 1 - 4$  $+ \frac{4}{3}$  $+ 4$ 

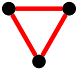
 $= \frac{2}{3}$  $+ \frac{1}{3}$  $+$ 

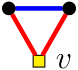
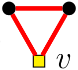
Example - Mantel's Theorem version 1

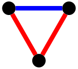
Theorem (Mantel 1907)

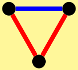
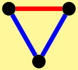
A triangle-free n-vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2}\binom{n}{2}$ *edges.*

Assume edges are red and non-edges are blue.

Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$0 \leq \frac{1}{n}\sum_{v}\Big(1 - 2$  $v\Big)^2 = \frac{1}{n}\sum_{v}\Big(1 - 4$  $v + 4$  $v + 4$  $v\Big)$

$= 1 - 4$  $+ \frac{4}{3}$ 

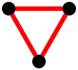
 $= \frac{2}{3}$  $+ \frac{1}{3}$ 

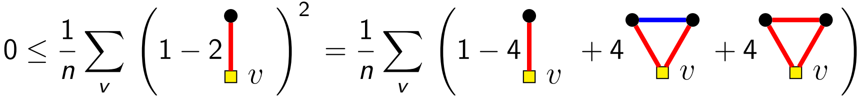
Example - Mantel's Theorem version 1

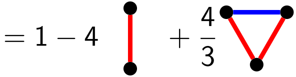
Theorem (Mantel 1907)

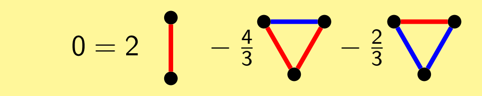
A triangle-free n-vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2}\binom{n}{2}$ *edges.*

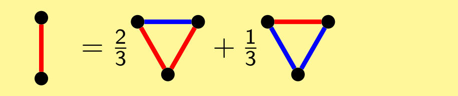
Assume edges are red and non-edges are blue.

Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)







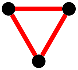


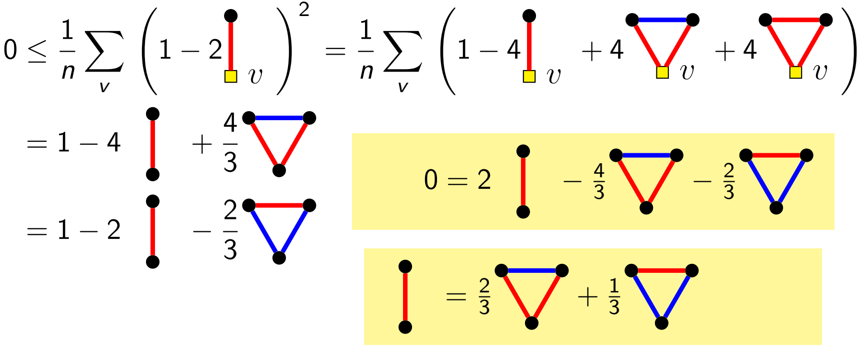
Example - Mantel's Theorem version 1

Theorem (Mantel 1907)

A triangle-free n-vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2}\binom{n}{2}$ *edges.*

Assume edges are red and non-edges are blue.

Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

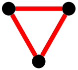


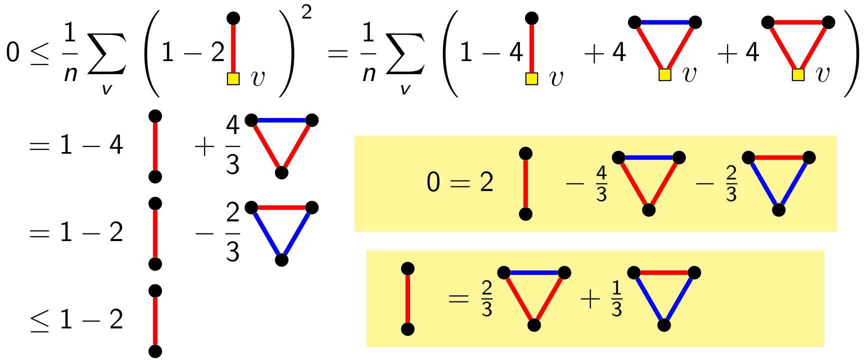
Example - Mantel's Theorem version 1

Theorem (Mantel 1907)

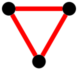
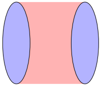
A triangle-free n-vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2}\binom{n}{2}$ *edges.*

Assume edges are red and non-edges are blue.

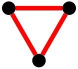
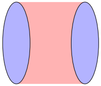
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

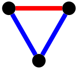


Example - stability for Mantel

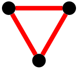
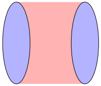
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$  .

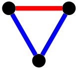
Example - stability for Mantel

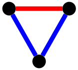
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$ .

$0 \leq 1 - 2$  $- \dfrac{2}{3}$  $+o(1)$

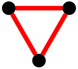
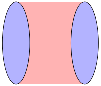
Example - stability for Mantel

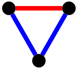
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$ .

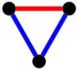
$0 \leq 1 - 2$  $- \frac{2}{3}$  $+o(1)$

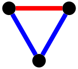
$0 \leq -\frac{2}{3}$ 

Example - stability for Mantel

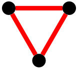
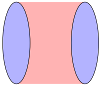
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$ .

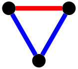
$0 \leq 1 - 2$  $- \frac{2}{3}$  $+o(1)$

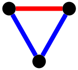
$0 \leq -\frac{2}{3}$ 

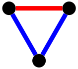
$0 \geq$ 

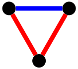
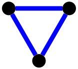
Example - stability for Mantel

Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$ .

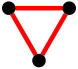
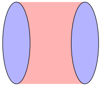
$0 \leq 1 - 2$  $- \frac{2}{3}$  $+o(1)$

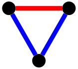
$0 \leq -\frac{2}{3}$ 

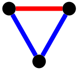
$0 \geq$ 

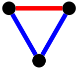
Only  and  appear in G.

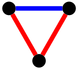
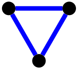
Example - stability for Mantel

Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$  .

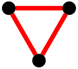
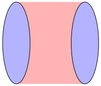
$0 \leq 1 - 2$  $- \dfrac{2}{3}$  $+o(1)$

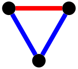
$0 \leq -\dfrac{2}{3}$ 

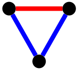
$0 \geq$ 

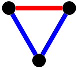
Only  and  appear in G.

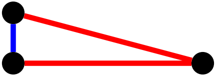
Example - stability for Mantel

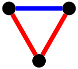
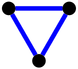
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$ .

$0 \leq 1 - 2$  $- \frac{2}{3}$  $+o(1)$

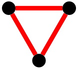
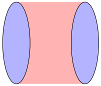
$0 \leq -\frac{2}{3}$ 

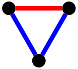
$0 \geq$ 

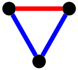


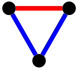
Only  and  appear in G.

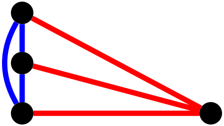
Example - stability for Mantel

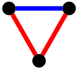
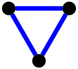
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$ .

$0 \leq 1 - 2$  $- \dfrac{2}{3}$  $+o(1)$

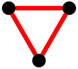
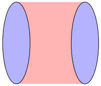
$0 \leq -\dfrac{2}{3}$ 

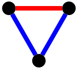
$0 \geq$ 

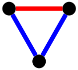


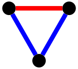
Only  and  appear in G.

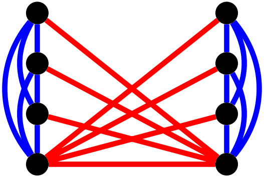
Example - stability for Mantel

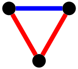
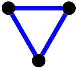
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$ .

$0 \leq 1 - 2$  $- \dfrac{2}{3}$  $+o(1)$

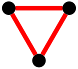
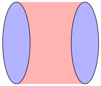
$0 \leq -\dfrac{2}{3}$ 

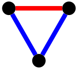
$0 \geq$ 

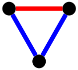


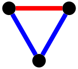
Only  and  appear in G.

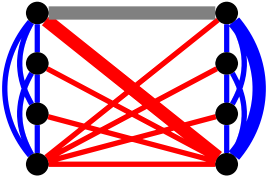
Example - stability for Mantel

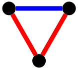
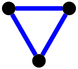
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$ .

$0 \leq 1 - 2$  $- \frac{2}{3}$  $+o(1)$

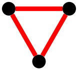
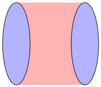
$0 \leq -\frac{2}{3}$ 

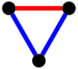
$0 \geq$ 

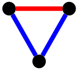


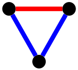
Only  and  appear in G.

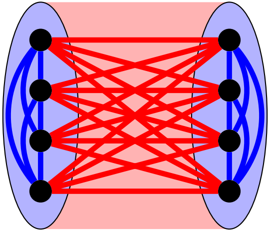
EXAMPLE - STABILITY FOR MANTEL

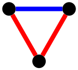
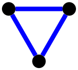
Assume  $= 0$ and  $= \frac{1}{2}$. Goal is $G =$ .

$0 \leq 1 - 2$  $- \frac{2}{3}$  $+o(1)$

$0 \leq -\frac{2}{3}$ 

$0 \geq$ 



Only  and  appear in G.

More automatic approach

- How to use computer to guess the right equation(s) for you.

Flag algebras - example

Theorem (Mantel 1907)

Every triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

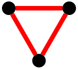
Assume edges are red and non-edges are blue.

Flag algebras - example

Theorem (Mantel 1907)

Every triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

Assume edges are red and non-edges are blue.

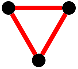
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

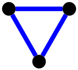
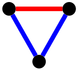
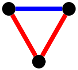
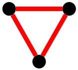
Flag algebras - example

Theorem (Mantel 1907)

Every triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

Assume edges are red and non-edges are blue.

Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

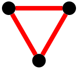
$1 =$  $+$  $+$  $+$ 

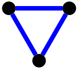
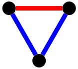
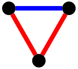
Flag algebras - example

Theorem (Mantel 1907)

Every triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

Assume edges are red and non-edges are blue.

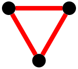
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

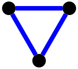
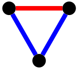
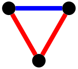
$1 =$  $+$  $+$ 

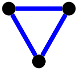
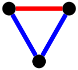
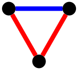
Theorem (Mantel 1907)

Every triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

Assume edges are red and non-edges are blue.

Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

$1 =$  $+$  $+$ 

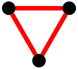
 $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

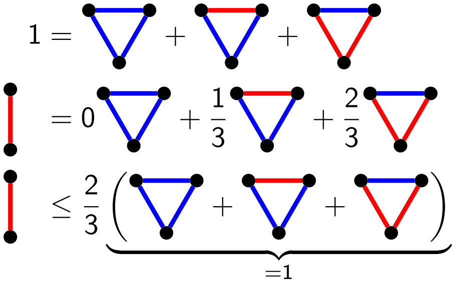
Flag algebras - example

Theorem (Mantel 1907)

Every triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

Assume edges are red and non-edges are blue.

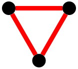
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

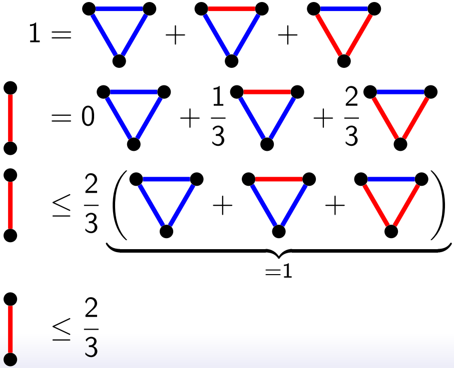


Theorem (Mantel 1907)

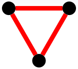
Every triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

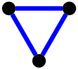
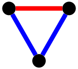
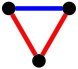
Assume edges are red and non-edges are blue.

Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

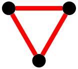


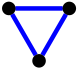
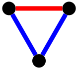
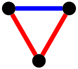
Example - Mantel's theorem version 2

Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

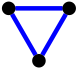
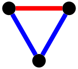
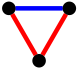
 $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

Example - Mantel's theorem version 2

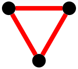
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

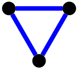
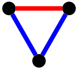
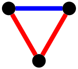
 $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

Idea: find $c_1, c_2, c_3 \in \mathbb{R}$ such that for every graph G

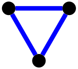
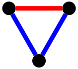
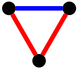
$0 \leq c_1$  $+ c_2$  $+ c_3$  $+ o(1)$.

Example - Mantel's theorem version 2

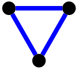
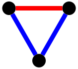
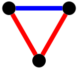
Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

 $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

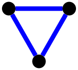
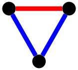
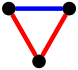
Idea: find $c_1, c_2, c_3 \in \mathbb{R}$ such that for every graph G

$0 \leq c_1$  $+ c_2$  $+ c_3$  $+o(1)$.

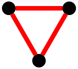
After summing together

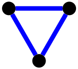
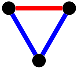
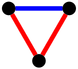
 $\leq c_1$  $+ \left(\frac{1}{3} + c_2\right)$  $+ \left(\frac{2}{3} + c_3\right)$ 

and

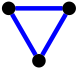
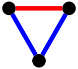
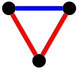
 $\leq \max\left\{0 + c_1, \frac{1}{3} + c_2, \frac{2}{3} + c_3\right\}$ $\underbrace{\Big($  $+$  $+$  $\Big)}_{=1}$

Example - Mantel's theorem version 2

Assume  $= 0$. (We want to conclude  $\leq \frac{1}{2}$.)

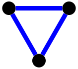
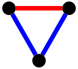
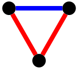
 $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$ 

Idea: find $c_1, c_2, c_3 \in \mathbb{R}$ such that for every graph G

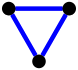
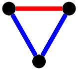
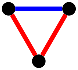
$0 \leq c_1$  $+ c_2$  $+ c_3$  $+ o(1)$

$c_3 < 0$

After summing together

 $\leq c_1$  $+ \left(\frac{1}{3} + c_2\right)$  $+ \left(\frac{2}{3} + c_3\right)$ 

and

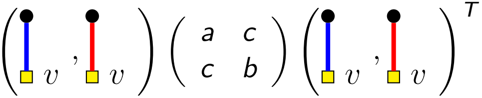
 $\leq \max\left\{0 + c_1, \frac{1}{3} + c_2, \frac{2}{3} + c_3\right\} \underbrace{\Big($  $+$  $+$  $\Big)}_{=1}$

Flag algebras - candidates for c_1, c_2, c_3

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

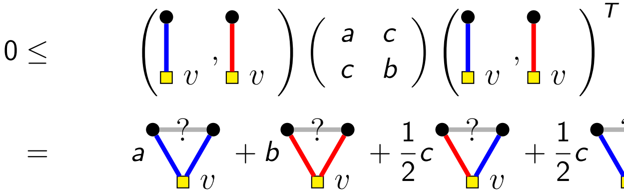
Flag algebras - candidates for c_1, c_2, c_3

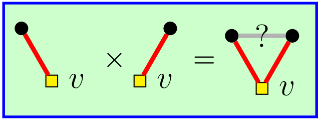
$0 \leq$

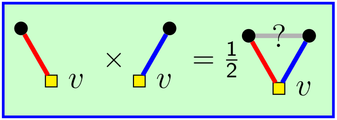


$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

Flag algebras - candidates for c_1, c_2, c_3

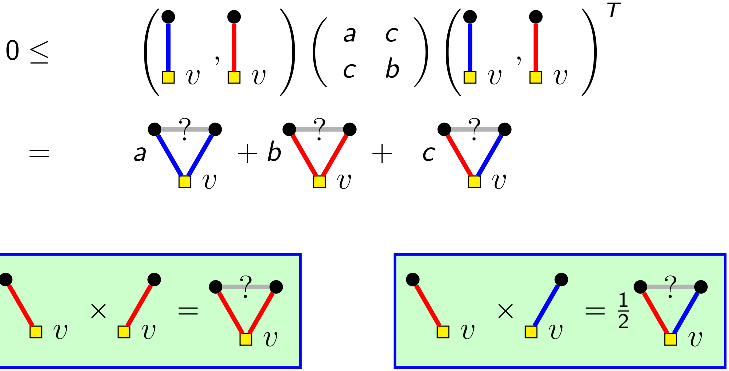




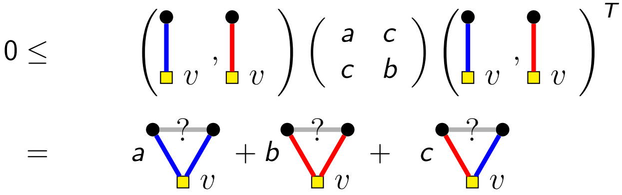


$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

Flag algebras - candidates for c_1, c_2, c_3

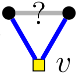
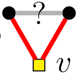
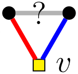


$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$$ (matrix is positive semidefinite)



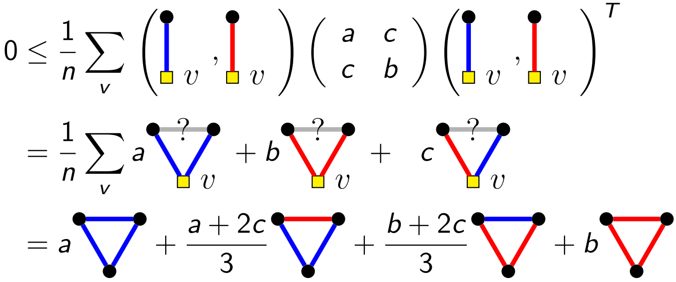
$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

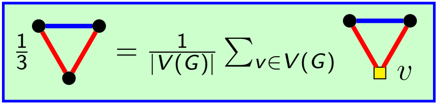
$0 \leq \frac{1}{n} \sum_v$ ( v ,  v) $\begin{pmatrix} a & c \\ c & b \end{pmatrix}$ ( v ,  v)^T

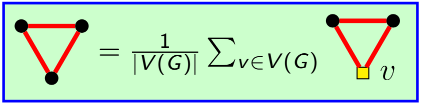
$= \frac{1}{n} \sum_v a$  $+ b$  $+ c$ 

$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$ (matrix is positive semidefinite)

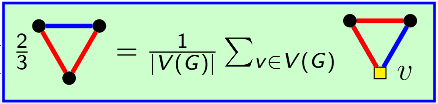
Flag algebras - candidates for c_1, c_2, c_3



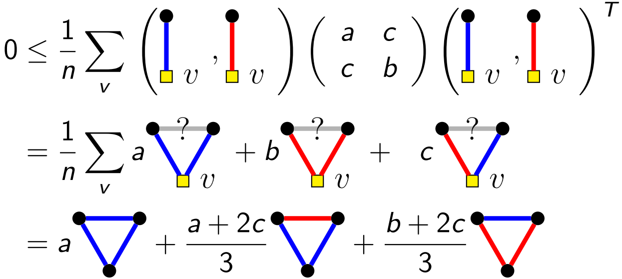


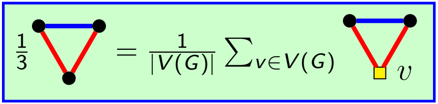


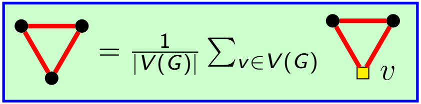
$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$ (



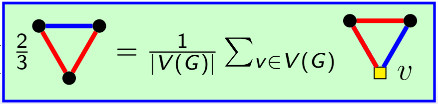
Flag algebras - candidates for c_1, c_2, c_3



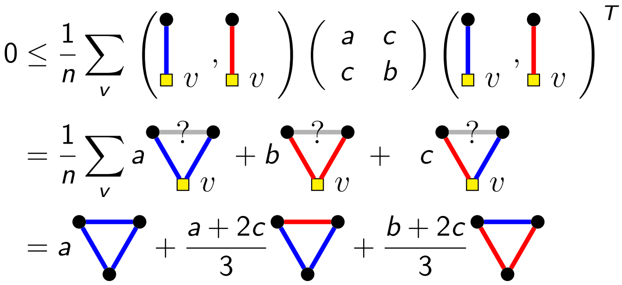




$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \;($$

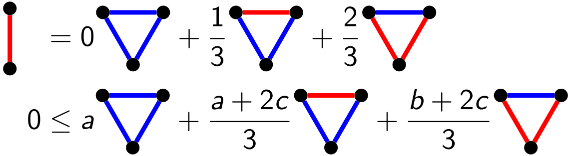


Flag algebras - candidates for c_1, c_2, c_3



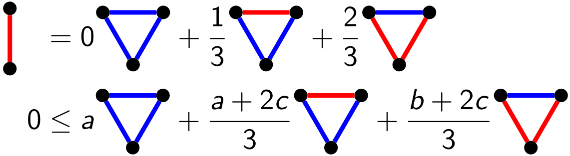
$$c_1 = a,\ c_2 = \frac{a+2c}{3},\ c_3 = \frac{b+2c}{3}$$

$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$ (matrix is positive semidefinite)



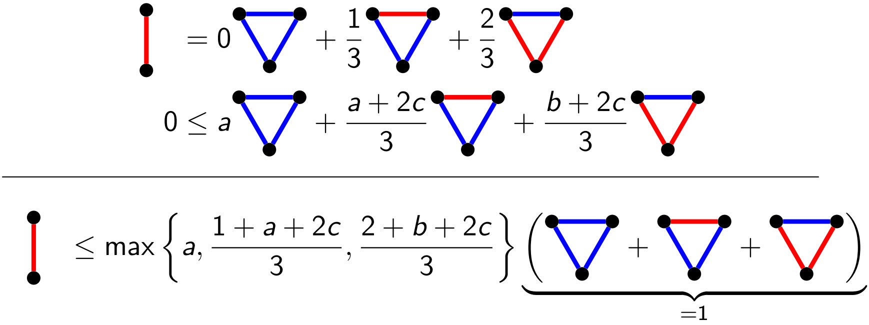
$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0$ (matrix is positive semidefinite)

Flag algebras - using c_1, c_2, c_3



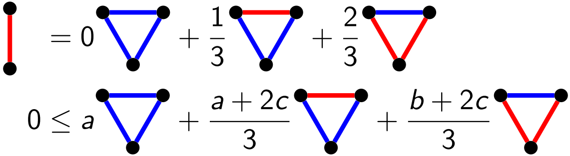
$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

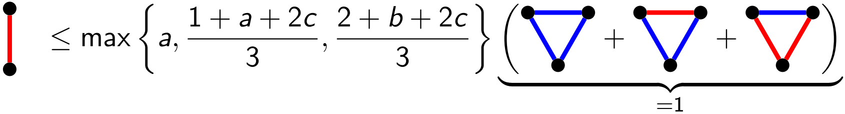
Flag algebras - using c_1, c_2, c_3



$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

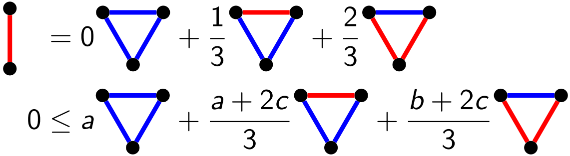
Flag algebras - using c_1, c_2, c_3

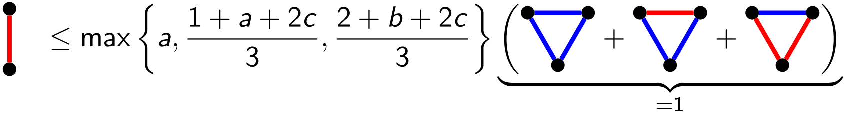




Try

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

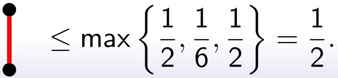




Try

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

It gives



Flag algebras - optimizing a, b, c

 $$\leq \max\left\{a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3}\right\}$$

$$(SDP)\begin{cases}\text{Minimize} & d\\ \text{subject to} & a \leq d\\ & \frac{1+a+2c}{3} \leq d\\ & \frac{2+b+2c}{3} \leq d\\ & \begin{pmatrix} a & c\\ c & b\end{pmatrix} \succcurlyeq 0\end{cases}$$

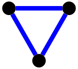
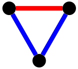
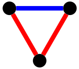
(SDP) can be solved on computers using CSDP or SDPA.
Rounding may be needed for exact results.

How to find extremal constructions?

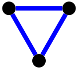
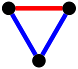
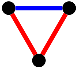
We got

 $\leq \max\left\{\frac{1}{2}, \frac{1}{6}, \frac{1}{2}\right\} = \frac{1}{2}.$

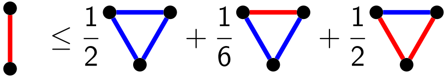
which is

 $\leq \frac{1}{2}$  $+ \frac{1}{6}$  $+ \frac{1}{2}$ 

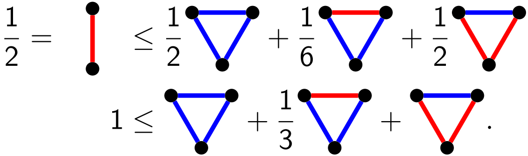
How to find extremal constructions?

 $\leq \frac{1}{2}$  $+ \frac{1}{6}$  $+ \frac{1}{2}$ 

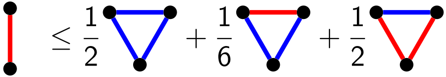
How to find extremal constructions?



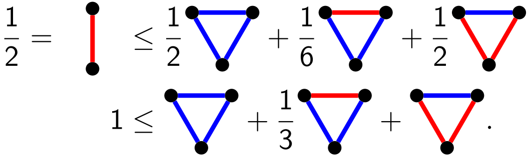
Suppose G is an extremal graph $\left(\; = \frac{1}{2}\right)$. Then



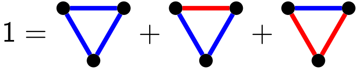
How to find extremal constructions?



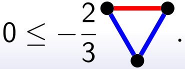
Suppose G is an extremal graph ( $= \frac{1}{2}$). Then



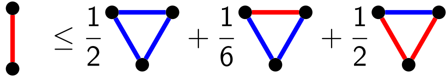
By subtracting



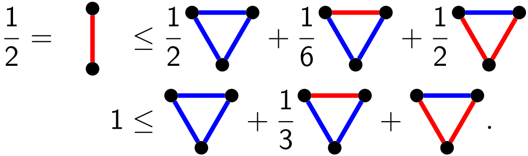
we obtain



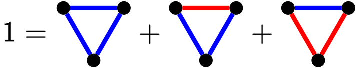
How to find extremal constructions?



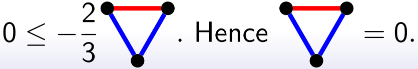
Suppose G is an extremal graph ( $= \frac{1}{2}$). Then



By subtracting



we obtain



 $\leq \max\left\{\frac{1}{2}, \frac{1}{6}, \frac{1}{2}\right\} = \frac{1}{2}$

Tells us that that if ( $= \frac{1}{2}$), then

- graphs with coefficients $< \frac{1}{2}$ do not appear in any extremal example
- all subgraphs of extremal example(s) should have $\frac{1}{2}$
- gives possible subgraphs for extremal examples (if not known)
- having $\frac{1}{2}$ does not mean it appears in any extremal example

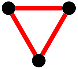
The semidefinite matrix gives a certificate.

Additional constraints

- Adding more constraints
- Considering bigger (but still small) graphs may improve bounds

Small experiment

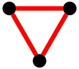
Mantel

- Maximize 
- subject to  $= 0$

Solution is $\frac{1}{2}$.

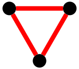
Mantel $\begin{cases} \text{Maximize} \\ \text{subject to} \end{cases}$

Maximize 

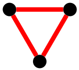
subject to  $= 0$

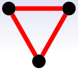
Solution is $\frac{1}{2}$. What if  $= p > \frac{1}{2}$?

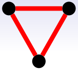
Small experiment

Mantel {
Maximize 
subject to  $= 0$

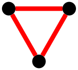
Solution is $\frac{1}{2}$. What if  $= p > \frac{1}{2}$?

{
Minimize 
subject to  $\geq p$

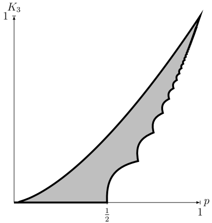
Minimize  subject to  $\geq p$.

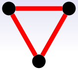
Minimize  subject to  $\geq p$.

Theorem (Razborov '08)

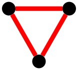
 $\geq \dfrac{(t-1)\left(t-2\sqrt{t(t-p(t+1))}\right)\left(t+\sqrt{t(t-p(t+1))}\right)^2}{t^2(t+1)^2}$

where $t = \lfloor 1/(1-p) \rfloor$. *Tight bound.*



Minimize  subject to  $\geq p$.

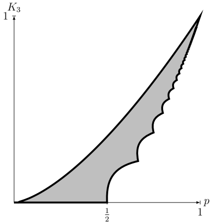
THEOREM (RAZBOROV '08)

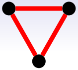
 $\geq \dfrac{(t-1)\left(t-2\sqrt{t(t-p(t+1))}\right)\left(t+\sqrt{t(t-p(t+1))}\right)^2}{t^2(t+1)^2}$

where $t = \lfloor 1/(1-p) \rfloor$. *Tight bound.*

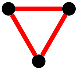
Nontrivial application of FA.

We will try a simple approach for $p = 0.6$



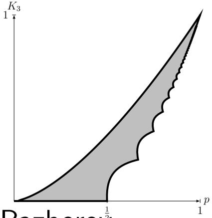
Minimize  subject to  $\geq p$.

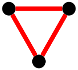
Theorem (Razborov '08)

 $\geq \dfrac{(t-1)\left(t-2\sqrt{t(t-p(t+1))}\right)\left(t+\sqrt{t(t-p(t+1))}\right)^2}{t^2(t+1)^2}$

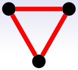
where $t = \lfloor 1/(1-p) \rfloor$. *Tight bound.*

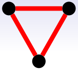
Nontrivial application of FA.
We will try a simple approach for $p = 0.6$
(We not will reproduce the result)

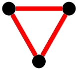
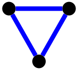
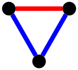
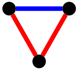
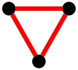


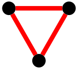
 $\geq 0.14150099\ldots$ for $p = 0.6$ by Razborov

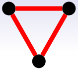
Note: Liu, Pikhurko, Staden: more exact results 2017 (99 pages)

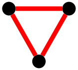
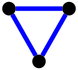
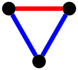
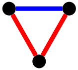
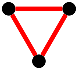
Minimize  subject to  ≥ 0.6.

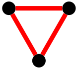
Minimize  subject to  ≥ 0.6.

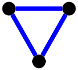
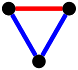
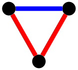
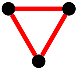
 $= 0$  $+\, 0$  $+\, 0$  $+$ 

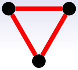
 $\geq \min\{0, 0, 0, 1\}$

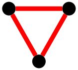
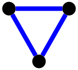
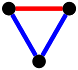
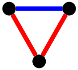
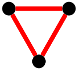
Minimize  subject to  ≥ 0.6.

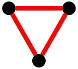
 $= 0$  $+\, 0$  $+\, 0$  $+$ 

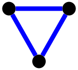
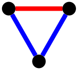
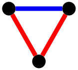
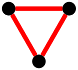
 $\geq \min\{0, 0, 0, 1\}$

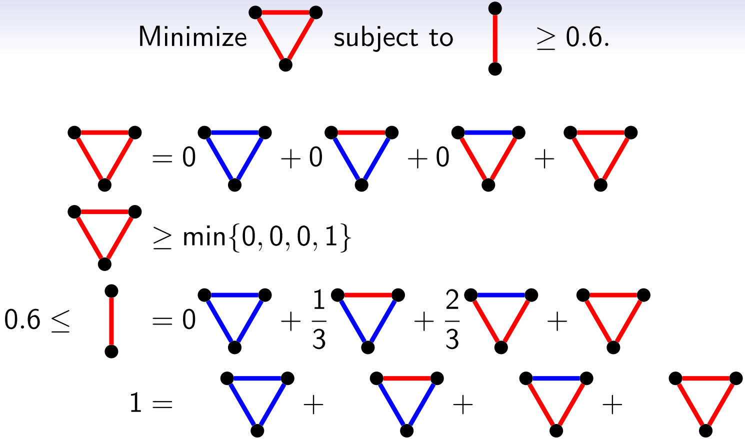
 $= 0$  $+\, \frac{1}{3}$  $+\, \frac{2}{3}$  $+$ 

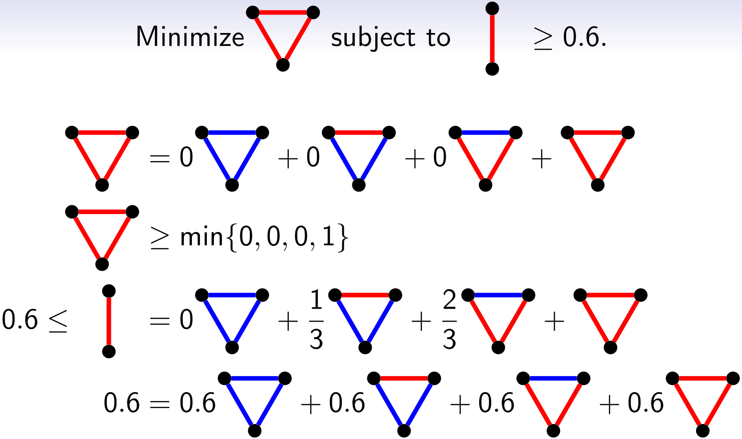
Minimize  subject to  ≥ 0.6.

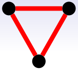
 $= 0$  $+ 0$  $+ 0$  $+$ 

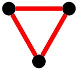
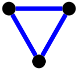
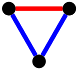
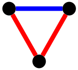
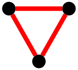
 $\geq \min\{0, 0, 0, 1\}$

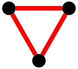
$0.6 \leq$  $= 0$  $+ \frac{1}{3}$  $+ \frac{2}{3}$  $+$ 

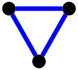
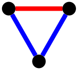
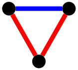
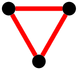

Minimize subject to ≥ 0.6.
= 0 + 0 + 0 +
≥ min{0, 0, 0, 1}
0.6 ≤ = 0 + 1/3 + 2/3 +
1 = + + +

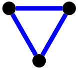
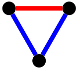
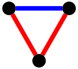
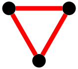

Minimize subject to ≥ 0.6.
= 0 + 0 + 0 +
≥ min{0, 0, 0, 1}
0.6 ≤ = 0 + 1/3 + 2/3 +
0.6 = 0.6 + 0.6 + 0.6 + 0.6

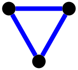
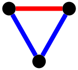
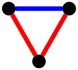
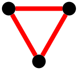
Minimize  subject to  ≥ 0.6.

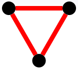
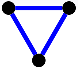
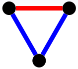
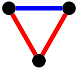
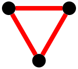
 $= 0$  $+\, 0$  $+\, 0$  $+$ 

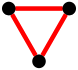
 $\geq \min\{0, 0, 0, 1\}$

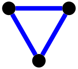
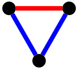
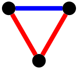
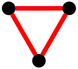
$0.6 \leq$  $= 0$  $+\, \frac{1}{3}$  $+\, \frac{2}{3}$  $+$ 

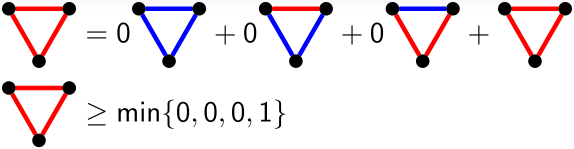
$0.6 = 0.6$  $+\, 0.6$  $+\, 0.6$  $+\, 0.6$ 

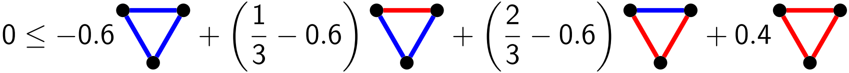
$0 \leq -0.6$  $+ \left(\frac{1}{3} - 0.6\right)$  $+ \left(\frac{2}{3} - 0.6\right)$  $+\, 0.4$ 

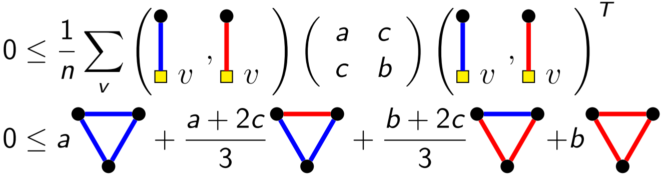
 $= 0$  $+\, 0$  $+\, 0$  $+$ 

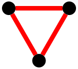
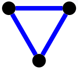
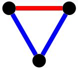
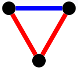
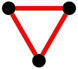
 $\geq \min\{0, 0, 0, 1\}$

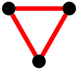
$0 \leq -0.6$  $+ \left(\frac{1}{3} - 0.6\right)$  $+ \left(\frac{2}{3} - 0.6\right)$  $+\, 0.4$ 

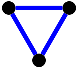
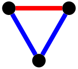
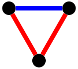
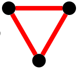

= 0 + 0 + 0 +
≥ min{0, 0, 0, 1}

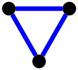
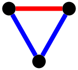
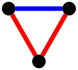
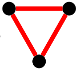

0 ≤ −0.6 + (1/3 − 0.6) + (2/3 − 0.6) + 0.4


0 ≤ 1/n ∑_v (v , v) (a c ; c b) (v , v)^T
0 ≤ a + (a + 2c)/3 + (b + 2c)/3 +b

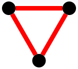
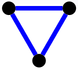
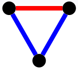
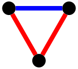
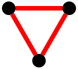
 $= 0$  $+ 0$  $+ 0$  $+$ 

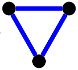
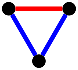
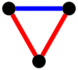
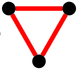
 $\geq \min\{0, 0, 0, 1\}$

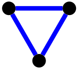
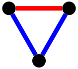
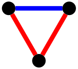
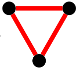
$0 \geq -a$  $- \dfrac{a+2c}{3}$  $- \dfrac{b+2c}{3}$  $-b$ 

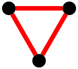
$0 \geq d \Bigg(0.6$  $+ \left(0.6 - \dfrac{1}{3}\right)$  $+ \left(0.6 - \dfrac{2}{3}\right)$  $- 0.4$  $\Bigg)$

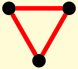
$$\begin{pmatrix} a & c & 0 \\ c & b & 0 \\ 0 & 0 & d \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

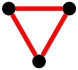
 $= 0$  $+ 0$  $+ 0$  $+$ 

$0 \geq -a$  $- \frac{a+2c}{3}$  $- \frac{b+2c}{3}$  $-b$ 

$0 \geq d \Big(0.6$  $+ \left(0.6 - \frac{1}{3}\right)$  $+ \left(0.6 - \frac{2}{3}\right)$  $- 0.4$  $\Big)$

 $\geq \min \left\{ 0.6d - a, \left(0.6 - \frac{1}{3}\right) d - \frac{a+2c}{3}, \left(0.6 - \frac{2}{3}\right) d - \frac{b+2c}{3}, 1 - 0.4d - b \right\}$

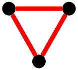
 $\geq 0.14150099\ldots$



$$\geq \min\left\{0.6d - a, \left(0.6 - \frac{1}{3}\right)d - \frac{a+2c}{3}, \left(0.6 - \frac{2}{3}\right)d - \frac{b+2c}{3}, 1 - 0.4d - b\right\}$$

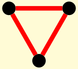
Numerical solution from CSDP:

$$\begin{aligned} a &= 6 \times 0.12000065088497\,79385 & \qquad a &= 0.72 \\ b &= 6 \times 0.05333290843810910981 & b &= 0.32 \\ c &= 6 \times -0.07999989818128358521 & c &= -0.48 \\ d &= 1.40000645402718\,5265 & d &= 1.4 \end{aligned}$$



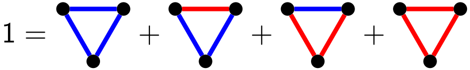
$$\geq \min\left\{0.12, 0.45\overline{3}, 0.12, 0.12\right\} = 0.12$$

How to improve 0.12?

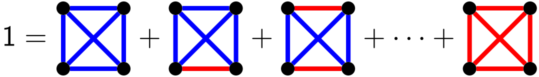


$\geq 0.14150099\ldots$

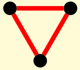
Sample bigger graphs. Instead of



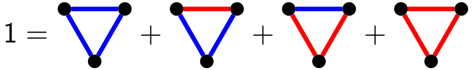
use



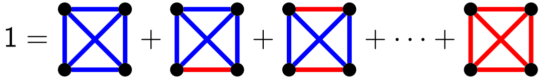
How to improve 0.12?



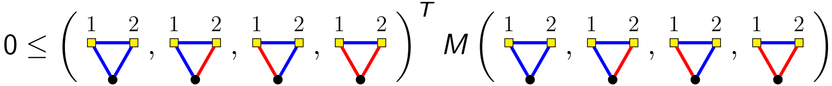
Sample bigger graphs. Instead of

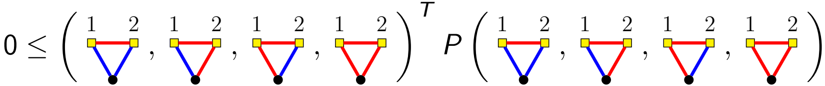


use

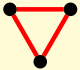


and include also $M, P \succcurlyeq 0$



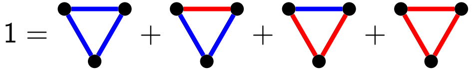


How to improve 0.12?

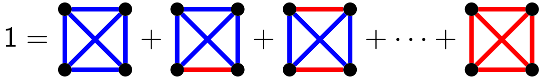


$\geq 0.14150099\ldots$

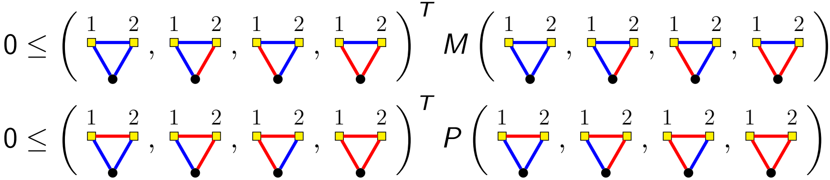
Sample bigger graphs. Instead of



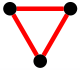
use



and include also $M, P \succcurlyeq 0$

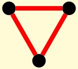


This gives



$\geq 0.127815\ldots$

How to improve 0.12781...?

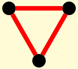
 $\geq 0.14150099\ldots$

Sample even bigger graphs.

Use K_5 instead of K_4

Include even more types and flags.

How to improve 0.12781...?

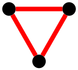
 $\geq 0.14150099\ldots$

Sample even bigger graphs.

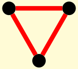
Use K_5 instead of K_4

Include even more types and flags.

This gives

 $\geq 0.1333333 = 2/15.$

How to improve 0.12781...?

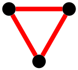
 $\geq 0.14150099\ldots$

Sample even bigger graphs.

Use K_5 instead of K_4

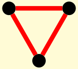
Include even more types and flags.

This gives

 $\geq 0.1333333 = 2/15.$

Try even bigger!

How to improve 0.12781...?

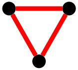
 $\geq 0.14150099\ldots$

Sample even bigger graphs.

Use K_5 instead of K_4

Include even more types and flags.

This gives

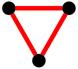
 $\geq 0.1333333 = 2/15.$

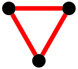
Try even bigger!

vertices	# graphs	time	bound
3	4	instant	0.12
4	11	instant	0.127815...
5	34	instant	0.13333...
6	156	seconds	0.13333...
7	1044	minutes	0.13333...
8	12346	day(s)	0.13333...
9	274668	not computable*	?

$\star$ needs hundreds of GB of RAM, maybe easy in 10 years?

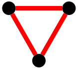
Getting 0.14150099...

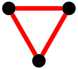
 $\geq 0.14150099\ldots$ for $p = 0.6$ by Razborov

Minimize 

subject to  $-\, p \geq 0$

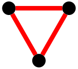
Getting 0.14150099 . . .

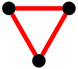
 $\geq 0.14150099\ldots$ for $p = 0.6$ by Razborov

Minimize 

subject to $H \cdot \Big($  $- p\Big) \geq 0$ for any graph H

Getting 0.14150099 . . .

 $\geq 0.14150099\ldots$ for $p = 0.6$ by Razborov

Minimize 

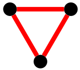
subject to $H \cdot \Big($  $- p\Big) \geq 0$ for any graph H

vertices	# graphs	time	bound	new bound
3	4	instant	0.12	0.12
4	11	instant	0.12781. . .	0.131746. . .
5	34	instant	0.13333. . .	0.14046241. . .
6	156	seconds	0.13333. . .	0.14150099. . .
7	1044	minutes	0.13333. . .	0.14150099. . .
8	12346	day(s)	0.13333. . .	0.14150099. . .

These are just numerical bounds! Not exact.

Goodman's bound

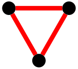
Recall we got for $p = 0.6$



$$\geq \min\left\{0.6d - a, \left(0.6 - \frac{1}{3}\right) d - \frac{a+2c}{3}, \left(0.6 - \frac{2}{3}\right) d - \frac{b+2c}{3}, 1 - (1-0.6)d - b\right\}$$

Goodman's bound

Recall we got for $p = 0.6$

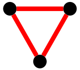


$$\geq \min\left\{0.6d - a, \left(0.6 - \frac{1}{3}\right) d - \frac{a+2c}{3}, \left(0.6 - \frac{2}{3}\right) d - \frac{b+2c}{3}, 1 - (1-0.6)d - b\right\}$$

Same thing holds when 0.6 is replaced by a parameter p.

Goodman's bound

Recall we got for $p = 0.6$

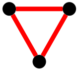


$$\geq \min\left\{0.6d - a, \left(0.6 - \frac{1}{3}\right) d - \frac{a+2c}{3}, \left(0.6 - \frac{2}{3}\right) d - \frac{b+2c}{3}, 1 - (1-0.6)d - b\right\}$$

Same thing holds when 0.6 is replaced by a parameter p.

$$a = 2p^2 \qquad b = 2p^2 - 4p + 2 \qquad c = p(2p-2) \qquad d = 4p - 1$$

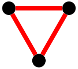
gives Goodman's bound:



$$\geq 2p^2 - p$$

Goodman's bound

Recall we got for $p = 0.6$

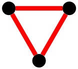


$$\geq \min\left\{0.6d - a, \left(0.6 - \frac{1}{3}\right) d - \frac{a+2c}{3}, \left(0.6 - \frac{2}{3}\right) d - \frac{b+2c}{3}, 1 - (1-0.6)d - b\right\}$$

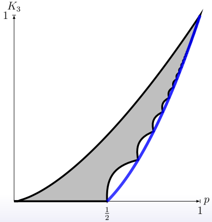
Same thing holds when 0.6 is replaced by a parameter p.

$$a = 2p^2 \quad b = 2p^2 - 4p + 2 \quad c = p(2p-2) \quad d = 4p - 1$$

gives Goodman's bound:



$$\geq 2p^2 - p$$



This is tight for $p \in \{\frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots\}$.

See you again in few minutes!

- Flag Algebras "definitions"
- First try for Mantel's theorem
- More automatic approach
- Additional constraints
- maybe break
- Define flag algebras
- Graph sequences and homomorphisms
- Turán's Theorem (in limit)
- Finally Mantel's Theorem (for real)
- mega break
- Applications