

FLAG ALGEBRA METHODS (HOPEFULLY EASY BASICS)

Bernard Lidický



6th Lake Michigan Workshop on Combinatorics and Graph Theory

Apr 7, 2019

OUTLINE

- Flag Algebras “definitions”
- First try for Mantel’s theorem
- More automatic approach
- Additional constraints
- maybe break
- Define flag algebras
- Graph sequences and homomorphisms
- Turán’s Theorem (in limit)
- Finally Mantel’s Theorem (for real)
- mega break
- Applications

Thanks to Florian Pfender and Jan Volec

FLAG ALGEBRAS

Seminal paper:

Razborov, Flag Algebras, *Journal of Symbolic Logic* **72** (2007), 1239–1282.

David P. Robbins Prize by AMS for Razborov in 2013



FLAG ALGEBRAS

Seminal paper:

Razborov, Flag Algebras, *Journal of Symbolic Logic* **72** (2007), 1239–1282.

David P. Robbins Prize by AMS for Razborov in 2013



EXAMPLE (GOODMAN, RAZBOROV)

If density of edges is at least $\rho > 0$, what is the minimum density of triangles?

- Designed to attack extremal problems.
- Works well if constraints as well as desired value can be computed by checking small subgraphs (or average over small subgraphs).
- The results are for the limit as graphs get very large.

APPLICATIONS (EARLY INCOMPLETE LIST)

AUTHOR	YEAR	APPLICATION/RESULT
RAZBOROV	2008	EDGE DENSITY VS. TRIANGLE DENSITY
HLADKÝ, KRÁL, NORIN	2009	BOUNDS FOR THE CACCETTA-HAGGVIST CONJECTURE
RAZBOROV	2010	ON 3-HYPERGRAPHS WITH FORBIDDEN 4-VERTEX CO
HATAMI, HLADKÝ, KRÁL, NORINE, RAZBOROV / GRZESIK	2011	ERDŐS PENTAGON PROBLEM
HATAMI, HLADKÝ, KRÁL, NORIN, RAZBOROV	2012	NON-THREE-COLOURABLE COMMON GRAPHS EXIST
BALOGH, HU, L., LIU / BABER	2012	4-CYCLES IN HYPERCUBES
DAS, HUANG, MA, NAVES, SUDAKOV	2013	MINIMUM NUMBER OF k -CLIQUES
BABER, TALBOT	2013	A SOLUTION TO THE 2/3 CONJECTURE
FALGAS-RAVRY, VAUGHAN	2013	TURÁN DENSITY OF MANY 3-GRAPHS
CUMMINGS, KRÁL, PFENDER, SPERFELD, TREGLOWN, YOUNG	2013	MONOCHROMATIC TRIANGLES IN 3-EDGE COLORED C
KRAMER, MARTIN, YOUNG	2013	BOOLEAN LATTICE
BALOGH, HU, L., PIKHURKO, UDVARI, VOLEC	2013	MONOTONE PERMUTATIONS
NORIN, ZWOLS	2013	NEW BOUND ON ZARANKIEWICZ'S CONJECTURE
HUANG, LINIAL, NAVES, PELED, SUDAKOV	2014	3-LOCAL PROFILES OF GRAPHS
BALOGH, HU, L., PFENDER, VOLEC, YOUNG	2014	RAINBOW TRIANGLES IN 3-EDGE COLORED GRAPHS
BALOGH, HU, L., PFENDER	2014	INDUCED DENSITY OF C_5
GOAOC, HUBARD, DE VERCLOS, SÉRÉNI, VOLEC	2014	ORDER TYPE AND DENSITY OF CONVEX SUBSETS
COREGLIANO, RAZBOROV	2015	TOURNAMENTS
ALON, NAVES, SUDAKOV	2015	PHYLOGENETIC TREES
...

Applications to graphs, oriented graphs, hypergraphs, hypercubes, permutations, crossing number of graphs, order types, discrete geometry, Ramsey numbers, phylogenetic trees. . .

EXAMPLE EXTREMAL PROBLEM

THEOREM (MANTEL 1907)

Every n -vertex triangle-free graph contains at most $\frac{1}{4}n^2$ edges.



PROBLEM

Maximize a graph parameter ($\#$ of edges) over a class of graphs (triangle-free).

- local condition and global parameter
- threshold
- bound and extremal example

EXAMPLE EXTREMAL PROBLEM

THEOREM (MANTEL 1907)

Every n -vertex triangle-free graph contains at most $\frac{1}{4}n^2$ edges.



PROBLEM

Maximize a graph parameter ($\#$ of edges) over a class of graphs (triangle-free).

- local condition and global parameter
- threshold
- bound and extremal example

We will use colors for **edges** and **non-edges**.

FLAG ALGEBRAS DEFINITIONS

Let G be a 2-edge-colored complete graph on n vertices.



The probability that three random vertices in G span a red triangle, i.e. $\# \text{red triangle} / \binom{n}{3}$.

FLAG ALGEBRAS DEFINITIONS

Let G be a 2-edge-colored complete graph on n vertices.



The probability that three random vertices in G span a red triangle, i.e. $\# \triangle_{\text{red}} / \binom{n}{3}$.



The probability that three random vertices in G span a graph isomorphic to a triangle with one red and two blue edges.

FLAG ALGEBRAS DEFINITIONS

Let G be a 2-edge-colored complete graph on n vertices.



The probability that three random vertices in G span a red triangle, i.e. $\# \triangle_{\text{red}} / \binom{n}{3}$.



The probability that three random vertices in G span a graph isomorphic to a triangle with one red and two blue edges.



The probability that a random vertex other than v is connected to v by a red edge, i.e., the red degree of v divided by $n - 1$.

FLAG ALGEBRAS DEFINITIONS

Let G be a 2-edge-colored complete graph on n vertices.



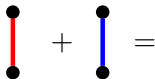
The probability that three random vertices in G span a red triangle, i.e. $\# \triangle_{\text{red}} / \binom{n}{3}$.



The probability that three random vertices in G span a graph isomorphic to a triangle with one red and two blue edges.



The probability that a random vertex other than v is connected to v by a red edge, i.e., the red degree of v divided by $n - 1$.



FLAG ALGEBRAS DEFINITIONS

Let G be a 2-edge-colored complete graph on n vertices.



The probability that three random vertices in G span a red triangle, i.e. $\# \triangle_{\text{red}} / \binom{n}{3}$.



The probability that three random vertices in G span a graph isomorphic to a triangle with one red and two blue edges.



The probability that a random vertex other than v is connected to v by a red edge, i.e., the red degree of v divided by $n - 1$.

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \bullet \end{array} = 1$$

FLAG ALGEBRAS DEFINITIONS



Let G be a 2-edge-colored complete graph on n vertices.

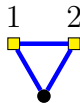


The probability that three random vertices in G span a red triangle, i.e. $\# \triangle_{\text{red}} / \binom{n}{3}$.

The probability that three random vertices in G span a graph isomorphic to a triangle with one red and two blue edges.

The probability that a random vertex other than v is connected to v by a red edge, i.e., the red degree of v divided by $n - 1$.

 $+$  $= 1$

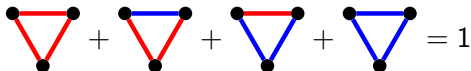


Flag

Type - flag induced by labeled vertices

FLAG ALGEBRAS IDENTITIES

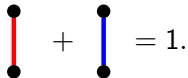
Let G be a 2-edge-colored complete graph on n vertices.



The diagram shows four triangles, each with three vertices and three edges. The first triangle has all three edges colored red. The second triangle has two edges colored red and one edge colored blue. The third triangle has one edge colored red and two edges colored blue. The fourth triangle has all three edges colored blue. The triangles are arranged in a row, separated by plus signs, followed by an equals sign and the number 1.

$$\text{Red Triangle} + \text{Red-Blue Triangle} + \text{Blue-Red Triangle} + \text{Blue Triangle} = 1$$

Same kind as



The diagram shows two vertical edges, each with two vertices. The first edge is colored red and the second edge is colored blue. They are arranged in a row, separated by a plus sign, followed by an equals sign and the number 1.

$$\text{Red Edge} + \text{Blue Edge} = 1.$$

FLAG ALGEBRAS IDENTITIES

Let G be a 2-edge-colored complete graph on n vertices.

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} = \frac{3}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{0}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$$

Expanded version:

$$P \left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right) = P \left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \mid \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right) \cdot P \left(\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right) + P \left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \mid \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right) \cdot P \left(\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right)$$

FLAG ALGEBRAS IDENTITIES

Let G be a 2-edge-colored complete graph on n vertices.

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} = \begin{array}{c} \bullet & \bullet \\ \diagdown & / \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet & \bullet \\ \diagdown & / \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet & \bullet \\ \diagdown & / \\ \bullet \end{array}$$

Expanded version:

$$P\left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array}\right) = P\left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \mid \begin{array}{c} \bullet & \bullet \\ \diagdown & / \\ \bullet \end{array}\right) \cdot P\left(\begin{array}{c} \bullet & \bullet \\ \diagdown & / \\ \bullet \end{array}\right) + P\left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \mid \begin{array}{c} \bullet & \bullet \\ \diagdown & / \\ \bullet \end{array}\right) \cdot P\left(\begin{array}{c} \bullet & \bullet \\ \diagdown & / \\ \bullet \end{array}\right)$$

FLAG ALGEBRAS IDENTITIES

Let G be a 2-edge-colored complete graph on n vertices.


$$\begin{array}{c} \bullet \\ \diagdown \\ \text{red} \\ \square v \end{array} \times \begin{array}{c} \bullet \\ \diagdown \\ \text{red} \\ \square v \end{array} = \begin{array}{c} \bullet \quad ? \quad \bullet \\ \diagdown \quad \diagup \\ \text{red} \quad \text{red} \\ \square v \end{array} + o(1) = \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \text{red} \quad \text{red} \\ \square v \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \text{red} \quad \text{blue} \\ \square v \end{array} + o(1)$$

$o(1)$ as $|V(G)| \rightarrow \infty$ (will be omitted on next slides)

FLAG ALGEBRAS IDENTITIES

Let G be a 2-edge-colored complete graph on n vertices.

$$\begin{array}{c} \bullet \\ \text{red} \\ \square v \end{array} \times \begin{array}{c} \bullet \\ \text{red} \\ \square v \end{array} = \begin{array}{c} \bullet \text{---} ? \text{---} \bullet \\ \text{red} \quad \text{red} \\ \square v \end{array} + o(1) = \begin{array}{c} \bullet \text{---} \text{red} \text{---} \bullet \\ \text{red} \quad \text{red} \\ \square v \end{array} + \begin{array}{c} \bullet \text{---} \text{blue} \text{---} \bullet \\ \text{red} \quad \text{red} \\ \square v \end{array} + o(1)$$

 : The probability of choosing two **different** vertices ...


$o(1)$ as $|V(G)| \rightarrow \infty$ (will be omitted on next slides)

FLAG ALGEBRAS IDENTITIES

Let G be a 2-edge-colored complete graph on n vertices.

$$\begin{array}{c} \bullet \\ \text{red} \\ \square v \end{array} \times \begin{array}{c} \bullet \\ \text{red} \\ \square v \end{array} = \begin{array}{c} \bullet \text{---} ? \text{---} \bullet \\ \text{red} \quad \text{red} \\ \square v \end{array} + o(1) = \begin{array}{c} \bullet \text{---} \text{red} \text{---} \bullet \\ \text{red} \quad \text{red} \\ \square v \end{array} + \begin{array}{c} \bullet \text{---} \text{blue} \text{---} \bullet \\ \text{red} \quad \text{red} \\ \square v \end{array} + o(1)$$

$$\begin{array}{c} \bullet \\ \text{red} \\ \square v \end{array} \times \begin{array}{c} \bullet \\ \text{blue} \\ \square v \end{array} = \frac{1}{2} \begin{array}{c} \bullet \text{---} ? \text{---} \bullet \\ \text{red} \quad \text{blue} \\ \square v \end{array} + o(1) = \frac{1}{2} \begin{array}{c} \bullet \text{---} \text{red} \text{---} \bullet \\ \text{red} \quad \text{blue} \\ \square v \end{array} + \frac{1}{2} \begin{array}{c} \bullet \text{---} \text{blue} \text{---} \bullet \\ \text{red} \quad \text{blue} \\ \square v \end{array} + o(1)$$

 : The probability of choosing two **different** vertices ...


$o(1)$ as $|V(G)| \rightarrow \infty$ (will be omitted on next slides)

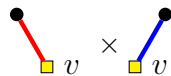
FLAG ALGEBRAS IDENTITIES

Let G be a 2-edge-colored complete graph on n vertices.

$$\begin{array}{c} \bullet \\ \text{red} \\ \square v \end{array} \times \begin{array}{c} \bullet \\ \text{red} \\ \square v \end{array} = \begin{array}{c} \bullet \text{---} \text{?} \text{---} \bullet \\ \text{red} \quad \text{red} \\ \square v \end{array} + o(1) = \begin{array}{c} \bullet \text{---} \text{red} \text{---} \bullet \\ \text{red} \quad \text{red} \\ \square v \end{array} + \begin{array}{c} \bullet \text{---} \text{blue} \text{---} \bullet \\ \text{red} \quad \text{red} \\ \square v \end{array} + o(1)$$

$$\begin{array}{c} \bullet \\ \text{red} \\ \square v \end{array} \times \begin{array}{c} \bullet \\ \text{blue} \\ \square v \end{array} = \frac{1}{2} \begin{array}{c} \bullet \text{---} \text{?} \text{---} \bullet \\ \text{red} \quad \text{blue} \\ \square v \end{array} + o(1) = \frac{1}{2} \begin{array}{c} \bullet \text{---} \text{red} \text{---} \bullet \\ \text{red} \quad \text{blue} \\ \square v \end{array} + \frac{1}{2} \begin{array}{c} \bullet \text{---} \text{blue} \text{---} \bullet \\ \text{red} \quad \text{blue} \\ \square v \end{array} + o(1)$$

 : The probability of choosing two **different** vertices ...

 : The probability that choosing two vertices u_1, u_2 other than v gives red vu_1 and blue vu_2 .

$o(1)$ as $|V(G)| \rightarrow \infty$ (will be omitted on next slides)

FLAG ALGEBRAS IDENTITIES

Let G be a 2-edge-colored complete graph on n vertices.

$$\frac{1}{3} \begin{array}{c} \bullet \text{---} \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c} \bullet \text{---} \text{---} \bullet \\ \diagdown \quad \diagup \\ \square \text{ } v \end{array}$$

FLAG ALGEBRAS IDENTITIES

Let G be a 2-edge-colored complete graph on n vertices.

$$\frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \square v \end{array}$$

$$\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \binom{n}{3} = \sum_{v \in V(G)} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \square v \end{array} \binom{n-1}{2}$$

FLAG ALGEBRAS IDENTITIES

Let G be a 2-edge-colored complete graph on n vertices.

$$\frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \square \text{ } v \end{array}$$

$$\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \square \text{ } v \end{array}$$

$$\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \binom{n}{3} = \sum_{v \in V(G)} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \square \text{ } v \end{array} \binom{n-1}{2}$$

FLAG ALGEBRAS IDENTITIES

Let G be a 2-edge-colored complete graph on n vertices.

$$\frac{1}{3} \text{ (triangle with blue top edge, red other edges) } = \frac{1}{|V(G)|} \sum_{v \in V(G)} \text{ (triangle with blue top edge, red other edges, yellow square at bottom vertex } v \text{)}$$

$$\text{ (triangle with red edges) } = \frac{1}{|V(G)|} \sum_{v \in V(G)} \text{ (triangle with red edges, yellow square at bottom vertex } v \text{)}$$

$$\text{ (triangle with blue top edge, red other edges) } \binom{n}{3} = \sum_{v \in V(G)} \text{ (triangle with blue top edge, red other edges, yellow square at bottom vertex } v \text{) } \binom{n-1}{2}$$

$$\text{ (triangle with red edges) } \binom{n}{3} = \frac{1}{3} \sum_{v \in V(G)} \text{ (triangle with red edges, yellow square at bottom vertex } v \text{) } \binom{n-1}{2}$$

IDENTITIES SUMMARY

$$1 = \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$$

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} = \frac{3}{3} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{0}{3} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$$

$$\begin{array}{c} \bullet \\ | \\ \square v \end{array} \times \begin{array}{c} \bullet \\ | \\ \square v \end{array} = \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \square v \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \square v \end{array}$$

$$\begin{array}{c} \bullet \\ | \\ \square v \end{array} \times \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \square v \end{array} = \frac{1}{2} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \square v \end{array} + \frac{1}{2} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \square v \end{array}$$

$$\frac{1}{3} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \frac{1}{n} \sum_{v \in V(G)} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \square v \end{array} \quad ; \quad \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} = \frac{1}{n} \sum_{v \in V(G)} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \square v \end{array}$$

First try for Mantel's theorem

- How to use the equations to prove something
- Gives bounds as well as helps with extremal examples

EXAMPLE - MANTEL'S THEOREM VERSION 1

THEOREM (MANTEL 1907)

A triangle-free n -vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2} \binom{n}{2}$ edges.



Assume **edges are red** and **non-edges are blue**.

EXAMPLE - MANTEL'S THEOREM VERSION 1

THEOREM (MANTEL 1907)

A triangle-free n -vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2} \binom{n}{2}$ edges.

Assume edges are red and non-edges are blue.



Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

EXAMPLE - MANTEL'S THEOREM VERSION 1

THEOREM (MANTEL 1907)

A triangle-free n -vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2}\binom{n}{2}$ edges.

Assume edges are red and non-edges are blue.

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)



$$0 \leq \left(1 - 2 \begin{array}{c} \bullet \\ | \\ \square v \end{array} \right)^2$$





EXAMPLE - MANTEL'S THEOREM VERSION 1





THEOREM (MANTEL 1907)

A triangle-free n -vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2}\binom{n}{2}$ edges.

Assume edges are red and non-edges are blue.

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$0 \leq \left(1 - 2 \text{  } \right)^2 = \left(1 - 4 \text{  } + 4 \text{  } + 4 \text{  } \right)$$



$$\text{  } \times \text{  } = \text{  } + \text{  }$$





EXAMPLE - MANTEL'S THEOREM VERSION 1

THEOREM (MANTEL 1907)

A triangle-free n -vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2}\binom{n}{2}$ edges.

Assume edges are red and non-edges are blue.

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)



$$0 \leq \frac{1}{n} \sum_v \left(1 - 2 \text{}_v \right)^2 = \frac{1}{n} \sum_v \left(1 - 4 \text{}_v + 4 \text{}_v + 4 \text{}_v \right)$$








EXAMPLE - MANTEL'S THEOREM VERSION 1



THEOREM (MANTEL 1907)



A triangle-free n -vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2} \binom{n}{2}$ edges.

Assume edges are red and non-edges are blue.

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$\begin{aligned}
 0 &\leq \frac{1}{n} \sum_v \left(1 - 2 \text{}_v \right)^2 = \frac{1}{n} \sum_v \left(1 - 4 \text{}_v + 4 \text{}_v + 4 \text{}_v \right) \\
 &= 1 - 4 \text{} + \frac{4}{3} \text{} + 4 \text{}
 \end{aligned}$$

$$\frac{1}{3} \text{} = \frac{1}{n} \sum_{v \in V(G)} \text{}_v$$



$$\text{} = \frac{1}{n} \sum_{v \in V(G)} \text{}_v$$

EXAMPLE - MANTEL'S THEOREM VERSION 1

THEOREM (MANTEL 1907)

A triangle-free n -vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2}\binom{n}{2}$ edges.

Assume edges are red and non-edges are blue.

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$\begin{aligned}
 0 &\leq \frac{1}{n} \sum_v \left(1 - 2 \cdot \text{img alt="red edge with yellow square at bottom vertex" data-bbox="295 410 325 525"} \right)^2 = \frac{1}{n} \sum_v \left(1 - 4 \cdot \text{img alt="red edge with yellow square at bottom vertex" data-bbox="615 415 655 520"} + 4 \cdot \text{img alt="triangle with blue top edge and red bottom edges, yellow square at bottom vertex" data-bbox="720 415 800 515"} + 4 \cdot \text{img alt="triangle with red edges, yellow square at bottom vertex" data-bbox="855 415 935 515"} \right) \\
 &= 1 - 4 \cdot \text{img alt="red edge" data-bbox="225 550 245 650"} + \frac{4}{3} \cdot \text{img alt="triangle with blue top edge and red bottom edges" data-bbox="325 555 405 645"} + 4 \cdot \text{img alt="triangle with red edges" data-bbox="455 555 535 645"}
 \end{aligned}$$



$$\text{img alt="red edge" data-bbox="485 760 505 860"} = \frac{2}{3} \cdot \text{img alt="triangle with blue top edge and red bottom edges" data-bbox="585 765 665 855"} + \frac{1}{3} \cdot \text{img alt="triangle with blue bottom edge and red top edges" data-bbox="715 765 795 855"} + \text{img alt="triangle with red edges" data-bbox="835 765 915 855"}$$

EXAMPLE - MANTEL'S THEOREM VERSION 1

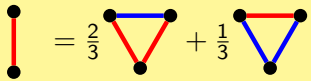
THEOREM (MANTEL 1907)

A triangle-free n -vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2}\binom{n}{2}$ edges.

Assume edges are red and non-edges are blue.

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$\begin{aligned}
 0 &\leq \frac{1}{n} \sum_v \left(1 - 2 \cdot \text{img alt="red edge with yellow square at bottom vertex" data-bbox="295 410 325 525"} \right)^2 = \frac{1}{n} \sum_v \left(1 - 4 \cdot \text{img alt="red edge with yellow square at bottom vertex" data-bbox="610 410 640 525"} + 4 \cdot \text{img alt="triangle with blue top edge and red bottom edges, yellow square at bottom vertex" data-bbox="715 415 795 515"} + 4 \cdot \text{img alt="triangle with red top edge and blue bottom edges, yellow square at bottom vertex" data-bbox="855 415 935 515"} \right) \\
 &= 1 - 4 \cdot \text{img alt="red edge" data-bbox="220 550 240 645"} + \frac{4}{3} \cdot \text{img alt="triangle with blue top edge and red bottom edges" data-bbox="325 555 405 645"}
 \end{aligned}$$





$$\text{img alt="red edge" data-bbox="480 755 500 865"} = \frac{2}{3} \cdot \text{img alt="triangle with blue top edge and red bottom edges" data-bbox="585 765 665 855"} + \frac{1}{3} \cdot \text{img alt="triangle with red top edge and blue bottom edges" data-bbox="715 765 795 855"}$$

EXAMPLE - MANTEL'S THEOREM VERSION 1

THEOREM (MANTEL 1907)

A triangle-free n -vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2} \binom{n}{2}$ edges.

Assume edges are red and non-edges are blue.

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$0 \leq \frac{1}{n} \sum_v \left(1 - 2 \cdot \text{red edge } v \right)^2 = \frac{1}{n} \sum_v \left(1 - 4 \cdot \text{red edge } v + 4 \cdot \text{red triangle } v + 4 \cdot \text{red triangle } v \right)$$

$$= 1 - 4 \cdot \text{red edge} + \frac{4}{3} \cdot \text{red triangle}$$

$$0 = 2 \cdot \text{red edge} - \frac{4}{3} \cdot \text{red triangle} - \frac{2}{3} \cdot \text{red triangle}$$



$$\text{red edge} = \frac{2}{3} \cdot \text{red triangle} + \frac{1}{3} \cdot \text{red triangle}$$

EXAMPLE - MANTEL'S THEOREM VERSION 1

THEOREM (MANTEL 1907)

A triangle-free n -vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2} \binom{n}{2}$ edges.

Assume edges are red and non-edges are blue.

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$0 \leq \frac{1}{n} \sum_v \left(1 - 2 \cdot \text{red edge } v \right)^2 = \frac{1}{n} \sum_v \left(1 - 4 \cdot \text{red edge } v + 4 \cdot \text{red triangle } v + 4 \cdot \text{red triangle } v \right)$$

$$= 1 - 4 \cdot \text{red edge} + \frac{4}{3} \cdot \text{red triangle}$$

$$= 1 - 2 \cdot \text{red edge} - \frac{2}{3} \cdot \text{red triangle}$$

$$0 = 2 \cdot \text{red edge} - \frac{4}{3} \cdot \text{red triangle} - \frac{2}{3} \cdot \text{red triangle}$$



$$\text{red edge} = \frac{2}{3} \cdot \text{red triangle} + \frac{1}{3} \cdot \text{red triangle}$$

EXAMPLE - MANTEL'S THEOREM VERSION 1

THEOREM (MANTEL 1907)

A triangle-free n -vertex graph contains at most $\frac{1}{4}n^2 \approx \frac{1}{2} \binom{n}{2}$ edges.

Assume edges are red and non-edges are blue.

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$0 \leq \frac{1}{n} \sum_v \left(1 - 2 \cdot \text{red edge } v \right)^2 = \frac{1}{n} \sum_v \left(1 - 4 \cdot \text{red edge } v + 4 \cdot \text{red triangle } v + 4 \cdot \text{red triangle } v \right)$$

$$= 1 - 4 \cdot \text{red edge} + \frac{4}{3} \cdot \text{red triangle}$$



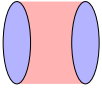
$$= 1 - 2 \cdot \text{red edge} - \frac{2}{3} \cdot \text{red triangle}$$

$$\leq 1 - 2 \cdot \text{red edge}$$



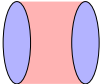
$$0 = 2 \cdot \text{red edge} - \frac{4}{3} \cdot \text{red triangle} - \frac{2}{3} \cdot \text{red triangle}$$



$$\text{red edge} = \frac{2}{3} \cdot \text{red triangle} + \frac{1}{3} \cdot \text{red triangle}$$

EXAMPLE - STABILITY FOR MANTEL



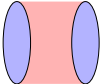
Assume  = 0 and  = $\frac{1}{2}$. Goal is $G =$  .

EXAMPLE - STABILITY FOR MANTEL

Assume  = 0 and  = $\frac{1}{2}$. Goal is $G =$  .

$$0 \leq 1 - 2 \text{  } - \frac{2}{3} \text{  } + o(1)$$



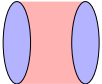
EXAMPLE - STABILITY FOR MANTEL

Assume  = 0 and  = $\frac{1}{2}$. Goal is $G =$  .

$$0 \leq 1 - 2 \cdot \text{edge} - \frac{2}{3} \cdot \text{triangle} + o(1)$$

$$0 \leq -\frac{2}{3} \cdot \text{triangle}$$

EXAMPLE - STABILITY FOR MANTEL



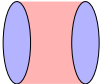
Assume  = 0 and  = $\frac{1}{2}$. Goal is $G =$  .

$$0 \leq 1 - 2 \cdot \text{edge} - \frac{2}{3} \cdot \text{triangle} + o(1)$$

$$0 \leq -\frac{2}{3} \cdot \text{triangle}$$

$$0 \geq \text{triangle}$$



EXAMPLE - STABILITY FOR MANTEL

Assume  = 0 and  = $\frac{1}{2}$. Goal is $G =$  .



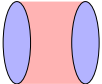
$$0 \leq 1 - 2 \cdot \text{edge} - \frac{2}{3} \cdot \text{triangle} + o(1)$$

$$0 \leq -\frac{2}{3} \cdot \text{triangle}$$

$$0 \geq \text{triangle}$$

Only  and  appear in G .



EXAMPLE - STABILITY FOR MANTEL

Assume  = 0 and  = $\frac{1}{2}$. Goal is $G =$ .

$$0 \leq 1 - 2 \cdot \text{edge} - \frac{2}{3} \cdot \text{triangle} + o(1)$$



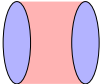
$$0 \leq -\frac{2}{3} \cdot \text{triangle}$$

$$0 \geq \text{triangle}$$

Only  and  appear in G .




EXAMPLE - STABILITY FOR MANTEL

Assume  = 0 and  = $\frac{1}{2}$. Goal is $G =$  .

$$0 \leq 1 - 2 \cdot \text{edge} - \frac{2}{3} \cdot \text{triangle} + o(1)$$



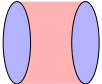
$$0 \leq -\frac{2}{3} \cdot \text{triangle}$$

$$0 \geq \text{triangle}$$

Only  and  appear in G .





EXAMPLE - STABILITY FOR MANTEL

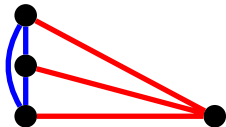
Assume  = 0 and  = $\frac{1}{2}$. Goal is $G =$  .

$$0 \leq 1 - 2 \cdot \text{edge} - \frac{2}{3} \cdot \text{triangle} + o(1)$$



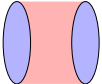
$$0 \leq -\frac{2}{3} \cdot \text{triangle}$$

$$0 \geq \text{triangle}$$

Only  and  appear in G .





EXAMPLE - STABILITY FOR MANTEL

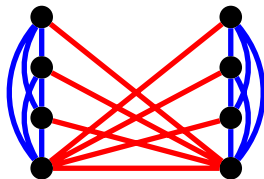
Assume  = 0 and  = $\frac{1}{2}$. Goal is $G =$ .

$$0 \leq 1 - 2 \cdot \text{edge} - \frac{2}{3} \cdot \text{triangle} + o(1)$$



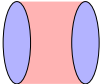
$$0 \leq -\frac{2}{3} \cdot \text{triangle}$$

$$0 \geq \text{triangle}$$

Only  and  appear in G .



EXAMPLE - STABILITY FOR MANTEL

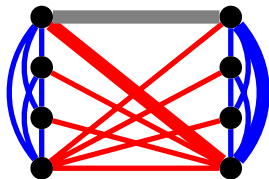
Assume  = 0 and  = $\frac{1}{2}$. Goal is $G =$ .

$$0 \leq 1 - 2 \cdot \text{edge} - \frac{2}{3} \cdot \text{triangle} + o(1)$$



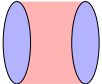
$$0 \leq -\frac{2}{3} \cdot \text{triangle}$$

$$0 \geq \text{triangle}$$

Only  and  appear in G .




EXAMPLE - STABILITY FOR MANTEL

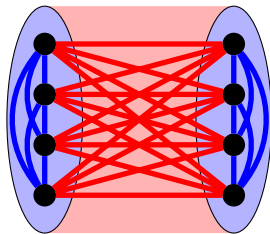
Assume  = 0 and  = $\frac{1}{2}$. Goal is $G =$ .

$$0 \leq 1 - 2 \cdot \text{edge} - \frac{2}{3} \cdot \text{triangle} + o(1)$$

$$0 \leq -\frac{2}{3} \cdot \text{triangle}$$

$$0 \geq \text{triangle}$$

Only  and  appear in G .



More automatic approach

- How to use computer to guess the right equation(s) for you.

FLAG ALGEBRAS - EXAMPLE

THEOREM (MANTEL 1907)

Every triangle-free graph contains at most $\frac{1}{4}n^2$ edges.



Assume **edges are red** and **non-edges are blue**.

FLAG ALGEBRAS - EXAMPLE

THEOREM (MANTEL 1907)

Every triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

Assume **edges are red** and **non-edges are blue**.



Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

FLAG ALGEBRAS - EXAMPLE

THEOREM (MANTEL 1907)

Every triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

Assume edges are red and non-edges are blue.

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)



$$1 = \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \end{array}$$

FLAG ALGEBRAS - EXAMPLE

THEOREM (MANTEL 1907)

Every triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

Assume edges are red and non-edges are blue.

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)



$$1 = \text{triangle with 2 blue edges} + \text{triangle with 1 blue edge} + \text{triangle with 0 blue edges}$$

FLAG ALGEBRAS - EXAMPLE

THEOREM (MANTEL 1907)

Every triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

Assume edges are red and non-edges are blue.

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$1 = \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array}$$



$$\begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} = 0 \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array}$$

FLAG ALGEBRAS - EXAMPLE

THEOREM (MANTEL 1907)

Every triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

Assume edges are red and non-edges are blue.

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)



$$\begin{aligned}
 1 &= \begin{array}{c} \text{triangle with 3 blue edges} \\ \text{triangle with 2 blue edges, 1 red edge} \\ \text{triangle with 1 blue edge, 2 red edges} \end{array} + \begin{array}{c} \text{triangle with 2 blue edges, 1 red edge} \\ \text{triangle with 1 blue edge, 2 red edges} \\ \text{triangle with 3 red edges} \end{array} + \begin{array}{c} \text{triangle with 1 blue edge, 2 red edges} \\ \text{triangle with 2 red edges, 1 blue edge} \\ \text{triangle with 3 red edges} \end{array} \\
 \begin{array}{c} \text{edge with 2 red edges} \\ \text{edge with 1 red edge, 1 blue edge} \\ \text{edge with 2 blue edges} \end{array} &= 0 \begin{array}{c} \text{triangle with 3 blue edges} \\ \text{triangle with 2 blue edges, 1 red edge} \\ \text{triangle with 1 blue edge, 2 red edges} \end{array} + \frac{1}{3} \begin{array}{c} \text{triangle with 2 blue edges, 1 red edge} \\ \text{triangle with 1 blue edge, 2 red edges} \\ \text{triangle with 3 red edges} \end{array} + \frac{2}{3} \begin{array}{c} \text{triangle with 1 blue edge, 2 red edges} \\ \text{triangle with 2 red edges, 1 blue edge} \\ \text{triangle with 3 red edges} \end{array} \\
 \begin{array}{c} \text{edge with 2 red edges} \\ \text{edge with 1 red edge, 1 blue edge} \\ \text{edge with 2 blue edges} \end{array} &\leq \frac{2}{3} \underbrace{\left(\begin{array}{c} \text{triangle with 3 blue edges} \\ \text{triangle with 2 blue edges, 1 red edge} \\ \text{triangle with 1 blue edge, 2 red edges} \end{array} + \begin{array}{c} \text{triangle with 2 blue edges, 1 red edge} \\ \text{triangle with 1 blue edge, 2 red edges} \\ \text{triangle with 3 red edges} \end{array} + \begin{array}{c} \text{triangle with 1 blue edge, 2 red edges} \\ \text{triangle with 2 red edges, 1 blue edge} \\ \text{triangle with 3 red edges} \end{array} \right)}_{=1}
 \end{aligned}$$

FLAG ALGEBRAS - EXAMPLE

THEOREM (MANTEL 1907)

Every triangle-free graph contains at most $\frac{1}{4}n^2$ edges.

Assume edges are red and non-edges are blue.

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)



$$1 = \begin{array}{c} \text{triangle with 2 blue edges, 1 red edge} \\ + \\ \text{triangle with 1 blue edge, 2 red edges} \\ + \\ \text{triangle with 0 blue edges, 3 red edges} \end{array}$$





$$\begin{array}{c} \text{red edge} \\ = 0 \\ \text{triangle with 2 blue edges, 1 red edge} \\ + \frac{1}{3} \text{triangle with 1 blue edge, 2 red edges} \\ + \frac{2}{3} \text{triangle with 0 blue edges, 3 red edges} \end{array}$$

$$\begin{array}{c} \text{red edge} \\ \leq \frac{2}{3} \left(\begin{array}{c} \text{triangle with 2 blue edges, 1 red edge} \\ + \\ \text{triangle with 1 blue edge, 2 red edges} \\ + \\ \text{triangle with 0 blue edges, 3 red edges} \end{array} \right) \\ \underbrace{\hspace{10em}}_{=1} \end{array}$$



$$\begin{array}{c} \text{red edge} \\ \leq \frac{2}{3} \end{array}$$

EXAMPLE - MANTEL'S THEOREM VERSION 2

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$\text{Edge} = 0 \cdot \text{Triangle} + \frac{1}{3} \cdot \text{Triangle} + \frac{2}{3} \cdot \text{Triangle}$$
 = 0  + $\frac{1}{3}$  + $\frac{2}{3}$ 

EXAMPLE - MANTEL'S THEOREM VERSION 2



Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$\text{img alt="A single edge colored red." data-bbox="285 245 305 340"} = 0 \cdot \text{img alt="A triangle with all three edges colored blue." data-bbox="385 245 465 340"} + \frac{1}{3} \cdot \text{img alt="A triangle with the top edge colored red and the other two edges colored blue." data-bbox="520 245 600 340"} + \frac{2}{3} \cdot \text{img alt="A triangle with the bottom edge colored red and the other two edges colored blue." data-bbox="655 245 735 340"/>$$

Idea: find $c_1, c_2, c_3 \in \mathbb{R}$ such that for every graph G

$$0 \leq c_1 \cdot \text{img alt="A triangle with all three edges colored blue." data-bbox="315 435 395 525"} + c_2 \cdot \text{img alt="A triangle with the top edge colored red and the other two edges colored blue." data-bbox="455 435 535 525"} + c_3 \cdot \text{img alt="A triangle with the bottom edge colored red and the other two edges colored blue." data-bbox="600 435 680 525"} + o(1).$$

EXAMPLE - MANTEL'S THEOREM VERSION 2

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$\text{red edge} = 0 \cdot \text{triangle (all blue)} + \frac{1}{3} \cdot \text{triangle (2 red, 1 blue)} + \frac{2}{3} \cdot \text{triangle (1 red, 2 blue)}$$

Idea: find $c_1, c_2, c_3 \in \mathbb{R}$ such that for every graph G

$$0 \leq c_1 \cdot \text{triangle (all blue)} + c_2 \cdot \text{triangle (2 red, 1 blue)} + c_3 \cdot \text{triangle (1 red, 2 blue)} + o(1).$$



After summing together

$$\text{red edge} \leq c_1 \cdot \text{triangle (all blue)} + \left(\frac{1}{3} + c_2\right) \cdot \text{triangle (2 red, 1 blue)} + \left(\frac{2}{3} + c_3\right) \cdot \text{triangle (1 red, 2 blue)}$$

and

$$\text{red edge} \leq \max \left\{ 0 + c_1, \frac{1}{3} + c_2, \frac{2}{3} + c_3 \right\} \underbrace{\left(\text{triangle (all blue)} + \text{triangle (2 red, 1 blue)} + \text{triangle (1 red, 2 blue)} \right)}_{=1}$$

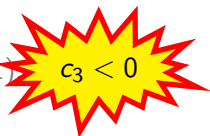
EXAMPLE - MANTEL'S THEOREM VERSION 2

Assume  = 0. (We want to conclude  $\leq \frac{1}{2}$.)

$$\text{edge} = 0 \cdot \text{triangle}_{\text{blue}} + \frac{1}{3} \cdot \text{triangle}_{\text{red}} + \frac{2}{3} \cdot \text{triangle}_{\text{blue-red}}$$

Idea: find $c_1, c_2, c_3 \in \mathbb{R}$ such that for every graph G

$$0 \leq c_1 \cdot \text{triangle}_{\text{blue}} + c_2 \cdot \text{triangle}_{\text{red}} + c_3 \cdot \text{triangle}_{\text{blue-red}} + o(1)$$



After summing together

$$\text{edge} \leq c_1 \cdot \text{triangle}_{\text{blue}} + \left(\frac{1}{3} + c_2\right) \cdot \text{triangle}_{\text{red}} + \left(\frac{2}{3} + c_3\right) \cdot \text{triangle}_{\text{blue-red}}$$

and

$$\text{edge} \leq \max \left\{ 0 + c_1, \frac{1}{3} + c_2, \frac{2}{3} + c_3 \right\} \underbrace{\left(\text{triangle}_{\text{blue}} + \text{triangle}_{\text{red}} + \text{triangle}_{\text{blue-red}} \right)}_{=1}$$

FLAG ALGEBRAS - CANDIDATES FOR c_1, c_2, c_3

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

FLAG ALGEBRAS - CANDIDATES FOR $\mathfrak{C}_1, \mathfrak{C}_2, \mathfrak{C}_3$

$$0 \leq \left(\begin{array}{c} \bullet \\ \color{blue}{|} \\ \color{yellow}{\square} \end{array} v, \begin{array}{c} \bullet \\ \color{red}{|} \\ \color{yellow}{\square} \end{array} v \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ \color{blue}{|} \\ \color{yellow}{\square} \end{array} v, \begin{array}{c} \bullet \\ \color{red}{|} \\ \color{yellow}{\square} \end{array} v \right)^T$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

FLAG ALGEBRAS - CANDIDATES FOR $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$

$$\begin{aligned}
 0 &\leq \left(\begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \right)^T \\
 &= a \begin{array}{c} \bullet \quad ? \quad \bullet \\ \diagdown \quad \diagup \\ \square v \end{array} + b \begin{array}{c} \bullet \quad ? \quad \bullet \\ \diagdown \quad \diagup \\ \square v \end{array} + \frac{1}{2}c \begin{array}{c} \bullet \quad ? \quad \bullet \\ \diagdown \quad \diagup \\ \square v \end{array} + \frac{1}{2}c \begin{array}{c} \bullet \quad ? \quad \bullet \\ \diagdown \quad \diagup \\ \square v \end{array}
 \end{aligned}$$

$$\begin{array}{c} \bullet \\ \diagdown \\ \square v \end{array} \times \begin{array}{c} \bullet \\ \diagdown \\ \square v \end{array} = \begin{array}{c} \bullet \quad ? \quad \bullet \\ \diagdown \quad \diagup \\ \square v \end{array}$$

$$\begin{array}{c} \bullet \\ \diagdown \\ \square v \end{array} \times \begin{array}{c} \bullet \\ \diagdown \\ \square v \end{array} = \frac{1}{2} \begin{array}{c} \bullet \quad ? \quad \bullet \\ \diagdown \quad \diagup \\ \square v \end{array}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

FLAG ALGEBRAS - CANDIDATES FOR $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$

$$\begin{aligned}
 0 &\leq \left(\begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \right)^T \\
 &= a \begin{array}{c} \bullet \\ \text{?} \\ \diagdown \quad \diagup \\ \square v \end{array} + b \begin{array}{c} \bullet \\ \text{?} \\ \diagdown \quad \diagup \\ \square v \end{array} + c \begin{array}{c} \bullet \\ \text{?} \\ \diagdown \quad \diagup \\ \square v \end{array}
 \end{aligned}$$

$$\begin{array}{c} \bullet \\ | \\ \square v \end{array} \times \begin{array}{c} \bullet \\ | \\ \square v \end{array} = \begin{array}{c} \bullet \quad \bullet \\ \text{?} \\ \diagdown \quad \diagup \\ \square v \end{array}$$

$$\begin{array}{c} \bullet \\ | \\ \square v \end{array} \times \begin{array}{c} \bullet \\ | \\ \square v \end{array} = \frac{1}{2} \begin{array}{c} \bullet \quad \bullet \\ \text{?} \\ \diagdown \quad \diagup \\ \square v \end{array}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

FLAG ALGEBRAS - CANDIDATES FOR c_1, c_2, c_3

$$\begin{aligned}
 0 &\leq \left(\begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ | \\ \square v \end{array}, \begin{array}{c} \bullet \\ | \\ \square v \end{array} \right)^T \\
 &= a \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \square v \end{array} + b \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \square v \end{array} + c \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \square v \end{array}
 \end{aligned}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

FLAG ALGEBRAS - CANDIDATES FOR c_1, c_2, c_3

$$\begin{aligned}
 0 &\leq \frac{1}{n} \sum_v \left(\begin{array}{c} \bullet \\ | \\ \square_v \end{array}, \begin{array}{c} \bullet \\ | \\ \square_v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ | \\ \square_v \end{array}, \begin{array}{c} \bullet \\ | \\ \square_v \end{array} \right)^T \\
 &= \frac{1}{n} \sum_v a \begin{array}{c} \bullet \quad ? \quad \bullet \\ \diagdown \quad \diagup \\ \square_v \end{array} + b \begin{array}{c} \bullet \quad ? \quad \bullet \\ \diagup \quad \diagdown \\ \square_v \end{array} + c \begin{array}{c} \bullet \quad ? \quad \bullet \\ \diagdown \quad \diagup \\ \square_v \end{array}
 \end{aligned}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

FLAG ALGEBRAS - CANDIDATES FOR $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$

$$\begin{aligned}
 0 &\leq \frac{1}{n} \sum_v \left(\begin{array}{c} \bullet \\ | \\ \square_v \end{array}, \begin{array}{c} \bullet \\ | \\ \square_v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ | \\ \square_v \end{array}, \begin{array}{c} \bullet \\ | \\ \square_v \end{array} \right)^T \\
 &= \frac{1}{n} \sum_v a \begin{array}{c} \bullet \\ ? \\ \bullet \\ | \\ \square_v \end{array} + b \begin{array}{c} \bullet \\ ? \\ \bullet \\ | \\ \square_v \end{array} + c \begin{array}{c} \bullet \\ ? \\ \bullet \\ | \\ \square_v \end{array} \\
 &= a \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ | \\ \square_v \end{array} + \frac{a+2c}{3} \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ | \\ \square_v \end{array} + \frac{b+2c}{3} \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ | \\ \square_v \end{array} + b \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ | \\ \square_v \end{array}
 \end{aligned}$$

$$\frac{1}{3} \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ | \\ \square_v \end{array} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ | \\ \square_v \end{array}$$

$$\begin{array}{c} \bullet \\ \bullet \\ \bullet \\ | \\ \square_v \end{array} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ | \\ \square_v \end{array}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \left(\frac{2}{3} \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ | \\ \square_v \end{array} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c} \bullet \\ \bullet \\ \bullet \\ | \\ \square_v \end{array} \right)$$

FLAG ALGEBRAS - CANDIDATES FOR $\mathcal{C}_1, \mathcal{C}_2, \mathcal{C}_3$

$$\begin{aligned}
 0 &\leq \frac{1}{n} \sum_v \left(\begin{array}{c} \bullet \\ | \\ \square_v \end{array}, \begin{array}{c} \bullet \\ | \\ \square_v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ | \\ \square_v \end{array}, \begin{array}{c} \bullet \\ | \\ \square_v \end{array} \right)^T \\
 &= \frac{1}{n} \sum_v a \begin{array}{c} \bullet \quad ? \quad \bullet \\ \diagdown \quad \diagup \\ \square_v \end{array} + b \begin{array}{c} \bullet \quad ? \quad \bullet \\ \diagup \quad \diagdown \\ \square_v \end{array} + c \begin{array}{c} \bullet \quad ? \quad \bullet \\ \diagdown \quad \diagup \\ \square_v \end{array} \\
 &= a \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \square_v \end{array} + \frac{a+2c}{3} \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \square_v \end{array} + \frac{b+2c}{3} \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \square_v \end{array}
 \end{aligned}$$

$$\frac{1}{3} \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \square_v \end{array} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \square_v \end{array}$$

$$\begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \square_v \end{array} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \square_v \end{array}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \left(\frac{2}{3} \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \square_v \end{array} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \square_v \end{array} \right)$$

FLAG ALGEBRAS - CANDIDATES FOR c_1, c_2, c_3

$$\begin{aligned}
 0 &\leq \frac{1}{n} \sum_v \left(\begin{array}{c} \bullet \\ | \\ \square_v \end{array}, \begin{array}{c} \bullet \\ | \\ \square_v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ | \\ \square_v \end{array}, \begin{array}{c} \bullet \\ | \\ \square_v \end{array} \right)^T \\
 &= \frac{1}{n} \sum_v a \begin{array}{c} \bullet \quad ? \\ \diagdown \quad \diagup \\ \square_v \end{array} + b \begin{array}{c} \bullet \quad ? \\ \diagup \quad \diagdown \\ \square_v \end{array} + c \begin{array}{c} \bullet \quad ? \\ \diagdown \quad \diagup \\ \square_v \end{array} \\
 &= a \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{a+2c}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{b+2c}{3} \begin{array}{c} \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \\
 c_1 &= a, \quad c_2 = \frac{a+2c}{3}, \quad c_3 = \frac{b+2c}{3}
 \end{aligned}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

FLAG ALGEBRAS - USING c_1, c_2, c_3

$$\begin{aligned} \begin{array}{c} \bullet \\ | \\ \bullet \end{array} &= 0 \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \\ 0 \leq a &\begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{a+2c}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{b+2c}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \end{aligned}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

FLAG ALGEBRAS - USING c_1, c_2, c_3

$$\begin{aligned} \text{Red edge} &= 0 \cdot \text{Blue triangle} + \frac{1}{3} \cdot \text{Red top triangle} + \frac{2}{3} \cdot \text{Red bottom triangle} \\ 0 \leq a &\leq a \cdot \text{Blue triangle} + \frac{a+2c}{3} \cdot \text{Red top triangle} + \frac{b+2c}{3} \cdot \text{Red bottom triangle} \end{aligned}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

FLAG ALGEBRAS - USING c_1, c_2, c_3

$$\begin{aligned}
 \text{red edge} &= 0 \cdot \text{blue triangle} + \frac{1}{3} \cdot \text{triangle with top edge red} + \frac{2}{3} \cdot \text{triangle with bottom edge red} \\
 0 \leq a &+ \frac{a+2c}{3} \cdot \text{triangle with top edge red} + \frac{b+2c}{3} \cdot \text{triangle with bottom edge red}
 \end{aligned}$$

$$\text{red edge} \leq \max \left\{ a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3} \right\} \underbrace{\left(\text{blue triangle} + \text{triangle with top edge red} + \text{triangle with bottom edge red} \right)}_{=1}$$

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

FLAG ALGEBRAS - USING c_1, c_2, c_3

$$\begin{aligned}
 \text{Diagram} &= 0 \cdot \text{Diagram}_1 + \frac{1}{3} \cdot \text{Diagram}_2 + \frac{2}{3} \cdot \text{Diagram}_3 \\
 0 &\leq a \cdot \text{Diagram}_1 + \frac{a+2c}{3} \cdot \text{Diagram}_2 + \frac{b+2c}{3} \cdot \text{Diagram}_3
 \end{aligned}$$





$$\text{Diagram} \leq \max \left\{ a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3} \right\} \underbrace{\left(\text{Diagram}_1 + \text{Diagram}_2 + \text{Diagram}_3 \right)}_{=1}$$

Try





$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

FLAG ALGEBRAS - USING c_1, c_2, c_3

$$\begin{aligned}
 \text{Diagram} &= 0 \cdot \text{Diagram}_1 + \frac{1}{3} \cdot \text{Diagram}_2 + \frac{2}{3} \cdot \text{Diagram}_3 \\
 0 \leq a &+ \frac{a+2c}{3} \cdot \text{Diagram}_2 + \frac{b+2c}{3} \cdot \text{Diagram}_3
 \end{aligned}$$

$$\text{Diagram} \leq \max \left\{ a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3} \right\} \underbrace{\left(\text{Diagram}_1 + \text{Diagram}_2 + \text{Diagram}_3 \right)}_{=1}$$







Try

$$\begin{pmatrix} a & c \\ c & b \end{pmatrix} = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}.$$

It gives

$$\text{Diagram} \leq \max \left\{ \frac{1}{2}, \frac{1}{6}, \frac{1}{2} \right\} = \frac{1}{2}.$$



FLAG ALGEBRAS - OPTIMIZING a, b, c

$$\bullet \leq \max \left\{ a, \frac{1+a+2c}{3}, \frac{2+b+2c}{3} \right\}$$

$$(SDP) \left\{ \begin{array}{l} \text{Minimize } d \\ \text{subject to } a \leq d \\ \frac{1+a+2c}{3} \leq d \\ \frac{2+b+2c}{3} \leq d \\ \begin{pmatrix} a & c \\ c & b \end{pmatrix} \succeq 0 \end{array} \right.$$

(*SDP*) can be solved on computers using CSDP or SDPA.
Rounding may be needed for exact results.

HOW TO FIND EXTREMAL CONSTRUCTIONS?

We got

$$\text{red edge} \leq \max \left\{ \frac{1}{2}, \frac{1}{6}, \frac{1}{2} \right\} = \frac{1}{2}.$$

which is

$$\text{red edge} \leq \frac{1}{2} \text{ (blue triangle)} + \frac{1}{6} \text{ (red triangle)} + \frac{1}{2} \text{ (red triangle)}$$

HOW TO FIND EXTREMAL CONSTRUCTIONS?

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \leq \frac{1}{2} \begin{array}{c} \bullet & \bullet \\ \text{---} & \text{---} \\ \backslash & / \\ \bullet \end{array} + \frac{1}{6} \begin{array}{c} \bullet & \bullet \\ \text{---} & \text{---} \\ \backslash & / \\ \bullet \end{array} + \frac{1}{2} \begin{array}{c} \bullet & \bullet \\ \text{---} & \text{---} \\ \backslash & / \\ \bullet \end{array}$$

HOW TO FIND EXTREMAL CONSTRUCTIONS?

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \leq \frac{1}{2} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \frac{1}{6} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \frac{1}{2} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$$

Suppose G is an extremal graph $\left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} = \frac{1}{2} \right)$. Then

$$\begin{array}{c} \frac{1}{2} = \\ \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \end{array} \leq \frac{1}{2} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \frac{1}{6} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \frac{1}{2} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$$

$$1 \leq \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} .$$

HOW TO FIND EXTREMAL CONSTRUCTIONS?

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \leq \frac{1}{2} \begin{array}{c} \bullet & \bullet \\ / \quad \backslash \\ \bullet \end{array} + \frac{1}{6} \begin{array}{c} \bullet & \bullet \\ / \quad \backslash \\ \bullet \end{array} + \frac{1}{2} \begin{array}{c} \bullet & \bullet \\ / \quad \backslash \\ \bullet \end{array}$$

Suppose G is an extremal graph $\left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} = \frac{1}{2} \right)$. Then

$$\begin{array}{c} \frac{1}{2} \\ = \\ \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \end{array} \leq \frac{1}{2} \begin{array}{c} \bullet & \bullet \\ / \quad \backslash \\ \bullet \end{array} + \frac{1}{6} \begin{array}{c} \bullet & \bullet \\ / \quad \backslash \\ \bullet \end{array} + \frac{1}{2} \begin{array}{c} \bullet & \bullet \\ / \quad \backslash \\ \bullet \end{array}$$

$$1 \leq \begin{array}{c} \bullet & \bullet \\ / \quad \backslash \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet & \bullet \\ / \quad \backslash \\ \bullet \end{array} + \begin{array}{c} \bullet & \bullet \\ / \quad \backslash \\ \bullet \end{array}.$$

By subtracting

$$1 = \begin{array}{c} \bullet & \bullet \\ / \quad \backslash \\ \bullet \end{array} + \begin{array}{c} \bullet & \bullet \\ / \quad \backslash \\ \bullet \end{array} + \begin{array}{c} \bullet & \bullet \\ / \quad \backslash \\ \bullet \end{array}$$

we obtain

$$0 \leq -\frac{2}{3} \begin{array}{c} \bullet & \bullet \\ / \quad \backslash \\ \bullet \end{array}.$$

HOW TO FIND EXTREMAL CONSTRUCTIONS?

$$\text{red edge} \leq \frac{1}{2} \text{blue triangle} + \frac{1}{6} \text{red triangle} + \frac{1}{2} \text{red-blue triangle}$$

Suppose G is an extremal graph $\left(\text{red edge} = \frac{1}{2} \right)$. Then

$$\begin{aligned} \frac{1}{2} \text{red edge} &\leq \frac{1}{2} \text{blue triangle} + \frac{1}{6} \text{red triangle} + \frac{1}{2} \text{red-blue triangle} \\ 1 &\leq \text{blue triangle} + \frac{1}{3} \text{red triangle} + \text{red-blue triangle} \end{aligned}$$

By subtracting

$$1 = \text{blue triangle} + \text{red triangle} + \text{red-blue triangle}$$

we obtain

$$0 \leq -\frac{2}{3} \text{red triangle} \text{ . Hence } \text{red triangle} = 0.$$

$$\rho \leq \max \left\{ \frac{1}{2}, \frac{1}{6}, \frac{1}{2} \right\} = \frac{1}{2}$$

Tells us that that if $\left(\rho = \frac{1}{2} \right)$, then

- graphs with coefficients $< \frac{1}{2}$ do not appear in any extremal example
- all subgraphs of extremal example(s) should have $\frac{1}{2}$
- gives possible subgraphs for extremal examples (if not known)
- having $\frac{1}{2}$ does not mean it appears in any extremal example

The semidefinite matrix gives a certificate.

Additional constraints

- Adding more constraints
- Considering bigger (but still small) graphs may improve bounds

SMALL EXPERIMENT

$$\text{Mantel} \begin{cases} \text{Maximize} \\ \text{subject to} \end{cases} \begin{array}{c} \bullet \\ | \\ \bullet \\ \triangle \\ \bullet \end{array} = 0$$

Solution is $\frac{1}{2}$.

SMALL EXPERIMENT

$$\text{Mantel} \left\{ \begin{array}{l} \text{Maximize} \\ \text{subject to} \end{array} \right. \begin{array}{l} \text{Diagram 1} \\ \text{Diagram 2} = 0 \end{array}$$



Solution is $\frac{1}{2}$. What if $\text{Diagram 1} = p > \frac{1}{2}$?



SMALL EXPERIMENT

$$\text{Mantel} \begin{cases} \text{Maximize} \\ \text{subject to} \end{cases} \begin{array}{c} \bullet \\ | \\ \bullet \\ \triangle \\ \bullet \end{array} = 0$$

Solution is $\frac{1}{2}$. What if $\begin{array}{c} \bullet \\ | \\ \bullet \end{array} = p > \frac{1}{2}$?

$$\begin{cases} \text{Minimize} \\ \text{subject to} \end{cases} \begin{array}{c} \triangle \\ \bullet \\ | \\ \bullet \end{array} \geq p$$

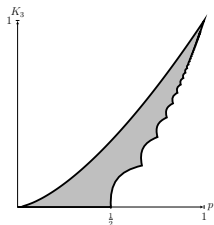
Minimize  subject to  $\geq p$.



Minimize  subject to  $\geq p$.

THEOREM (RAZBOROV '08)

$$\text{triangle} \geq \frac{(t-1) \left(t - 2\sqrt{t(t-p(t+1))} \right) \left(t + \sqrt{t(t-p(t+1))} \right)^2}{t^2(t+1)^2}$$

where $t = \lfloor 1/(1-p) \rfloor$. *Tight bound.*



Minimize  subject to  $\geq p$.

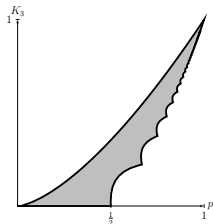
THEOREM (RAZBOROV '08)



$$\text{triangle} \geq \frac{(t-1) \left(t - 2\sqrt{t(t-p(t+1))} \right) \left(t + \sqrt{t(t-p(t+1))} \right)^2}{t^2(t+1)^2}$$

where $t = \lfloor 1/(1-p) \rfloor$. *Tight bound.*

Nontrivial application of FA.

We will try a simple approach for $p = 0.6$



Minimize  subject to  $\geq p$.

THEOREM (RAZBOROV '08)

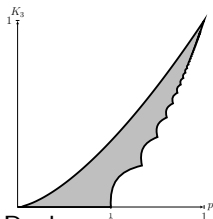
$$\text{red triangle} \geq \frac{(t-1) \left(t - 2\sqrt{t(t-p(t+1))} \right) \left(t + \sqrt{t(t-p(t+1))} \right)^2}{t^2(t+1)^2}$$

where $t = \lfloor 1/(1-p) \rfloor$. *Tight bound.*

Nontrivial application of FA.



We will try a simple approach for $p = 0.6$



(We not will reproduce the result)



$$\text{red triangle} \geq 0.14150099\dots \text{ for } p = 0.6 \text{ by Razborov}$$



Note: Liu, Pikhurko, Staden: more exact results 2017 (99 pages)

Minimize  subject to  ≥ 0.6 .

Minimize  subject to  ≥ 0.6 .

$$\begin{array}{c} \text{red triangle} \\ \text{red triangle} \end{array} = 0 \begin{array}{c} \text{blue triangle} \\ \text{blue triangle} \end{array} + 0 \begin{array}{c} \text{blue triangle} \\ \text{blue triangle} \end{array} + 0 \begin{array}{c} \text{red triangle} \\ \text{red triangle} \end{array} + \begin{array}{c} \text{red triangle} \\ \text{red triangle} \end{array}$$



$$\begin{array}{c} \text{red triangle} \\ \text{red triangle} \end{array} \geq \min\{0, 0, 0, 1\}$$

Minimize  subject to  ≥ 0.6 .

$$\begin{array}{c}
 \text{triangle with red edges} \\
 \text{triangle with blue edges} \\
 \text{triangle with red edges} \\
 \text{triangle with blue edges} \\
 \text{triangle with red edges} \\
 \text{triangle with red edges}
 \end{array}
 = 0 + 0 + 0 + 0 + 1$$

$$\begin{array}{c}
 \text{triangle with red edges} \\
 \text{triangle with red edges} \\
 \text{edge with red line}
 \end{array}
 \geq \min\{0, 0, 0, 1\}$$



$$\begin{array}{c}
 \text{triangle with red edges} \\
 \text{triangle with blue edges} \\
 \text{triangle with red edges} \\
 \text{triangle with blue edges} \\
 \text{triangle with red edges} \\
 \text{triangle with red edges}
 \end{array}
 = 0 + \frac{1}{3} + \frac{2}{3} + 0 + 0 + 1$$

Minimize  subject to  ≥ 0.6 .

$$\begin{array}{c}
 \text{triangle with red edges} \\
 \text{triangle with blue edges} \\
 \text{triangle with red edges} \\
 \text{triangle with blue edges} \\
 \text{triangle with red edges}
 \end{array}
 = 0 + 0 + 0 + 0 + 1$$

$$\begin{array}{c}
 \text{triangle with red edges} \\
 \text{triangle with blue edges} \\
 \text{triangle with red edges} \\
 \text{triangle with blue edges} \\
 \text{triangle with red edges}
 \end{array}
 \geq \min\{0, 0, 0, 0, 1\}$$

$$0.6 \leq \begin{array}{c} \text{edge with red line} \end{array} = 0 \begin{array}{c} \text{triangle with blue edges} \end{array} + \frac{1}{3} \begin{array}{c} \text{triangle with red edges} \end{array} + \frac{2}{3} \begin{array}{c} \text{triangle with blue edges} \end{array} + \begin{array}{c} \text{triangle with red edges} \end{array}$$



Minimize  subject to  ≥ 0.6 .

$$\begin{array}{c} \text{triangle (red)} \\ \text{triangle (blue)} \\ \text{triangle (blue)} \\ \text{triangle (red)} \\ \text{triangle (red)} \end{array} = 0 + 0 + 0 + 0 + 1$$

$$\begin{array}{c} \text{triangle (red)} \\ \text{edge (red)} \end{array} \geq \min\{0, 0, 0, 1\}$$

$$0.6 \leq \begin{array}{c} \text{edge (red)} \\ \text{triangle (blue)} \\ \frac{1}{3} \text{triangle (red)} \\ \frac{2}{3} \text{triangle (red)} \\ \text{triangle (red)} \end{array} = 0 + \frac{1}{3} + \frac{2}{3} + 0 + 1$$

$$1 = \begin{array}{c} \text{triangle (blue)} \\ \text{triangle (blue)} \\ \text{triangle (red)} \\ \text{triangle (red)} \end{array} + \begin{array}{c} \text{triangle (blue)} \\ \text{triangle (red)} \\ \text{triangle (red)} \\ \text{triangle (red)} \end{array} + \begin{array}{c} \text{triangle (blue)} \\ \text{triangle (red)} \\ \text{triangle (red)} \\ \text{triangle (red)} \end{array} + \begin{array}{c} \text{triangle (red)} \\ \text{triangle (red)} \\ \text{triangle (red)} \\ \text{triangle (red)} \end{array}$$



Minimize  subject to  ≥ 0.6 .

$$\begin{array}{c} \text{triangle (red)} \\ \text{triangle (blue)} \end{array} = 0 \begin{array}{c} \text{triangle (blue)} \\ \text{triangle (blue)} \end{array} + 0 \begin{array}{c} \text{triangle (red)} \\ \text{triangle (blue)} \end{array} + 0 \begin{array}{c} \text{triangle (blue)} \\ \text{triangle (red)} \end{array} + \begin{array}{c} \text{triangle (red)} \\ \text{triangle (red)} \end{array}$$

$$\begin{array}{c} \text{triangle (red)} \\ \text{triangle (red)} \end{array} \geq \min\{0, 0, 0, 1\}$$

$$0.6 \leq \begin{array}{c} \text{edge} \end{array} = 0 \begin{array}{c} \text{triangle (blue)} \\ \text{triangle (blue)} \end{array} + \frac{1}{3} \begin{array}{c} \text{triangle (red)} \\ \text{triangle (blue)} \end{array} + \frac{2}{3} \begin{array}{c} \text{triangle (blue)} \\ \text{triangle (red)} \end{array} + \begin{array}{c} \text{triangle (red)} \\ \text{triangle (red)} \end{array}$$

$$0.6 = 0.6 \begin{array}{c} \text{triangle (blue)} \\ \text{triangle (blue)} \end{array} + 0.6 \begin{array}{c} \text{triangle (red)} \\ \text{triangle (blue)} \end{array} + 0.6 \begin{array}{c} \text{triangle (blue)} \\ \text{triangle (red)} \end{array} + 0.6 \begin{array}{c} \text{triangle (red)} \\ \text{triangle (red)} \end{array}$$

Minimize  subject to  ≥ 0.6 .

$$\text{triangle (red)} = 0 \cdot \text{triangle (blue)} + 0 \cdot \text{triangle (blue)} + 0 \cdot \text{triangle (red)} + 1 \cdot \text{triangle (red)}$$

$$\text{triangle (red)} \geq \min\{0, 0, 0, 1\}$$

$$0.6 \leq \text{edge (red)} = 0 \cdot \text{triangle (blue)} + \frac{1}{3} \cdot \text{triangle (blue)} + \frac{2}{3} \cdot \text{triangle (red)} + 1 \cdot \text{triangle (red)}$$

$$0.6 = 0.6 \cdot \text{triangle (blue)} + 0.6 \cdot \text{triangle (blue)} + 0.6 \cdot \text{triangle (red)} + 0.6 \cdot \text{triangle (red)}$$

$$0 \leq -0.6 \cdot \text{triangle (blue)} + \left(\frac{1}{3} - 0.6\right) \cdot \text{triangle (blue)} + \left(\frac{2}{3} - 0.6\right) \cdot \text{triangle (red)} + 0.4 \cdot \text{triangle (red)}$$

$$\begin{array}{c}
 \text{Red Triangle} \\
 \text{Blue Triangle} \\
 \text{Red Triangle} \\
 \text{Blue Triangle} \\
 \text{Red Triangle} \\
 \text{Red Triangle}
 \end{array}
 = 0 + 0 + 0 + 0 + 0 + 1$$

$$\text{Red Triangle} \geq \min\{0, 0, 0, 1\}$$

$$0 \leq -0.6 \text{ (Blue Triangle)} + \left(\frac{1}{3} - 0.6\right) \text{ (Blue Triangle)} + \left(\frac{2}{3} - 0.6\right) \text{ (Red Triangle)} + 0.4 \text{ (Red Triangle)}$$

$$\begin{aligned}
 & \text{Red triangle} = 0 \cdot \text{Blue triangle} + 0 \cdot \text{Mixed triangle} + 0 \cdot \text{Red triangle} + \text{Red triangle} \\
 & \text{Red triangle} \geq \min\{0, 0, 0, 1\}
 \end{aligned}$$

$$0 \leq -0.6 \cdot \text{Blue triangle} + \left(\frac{1}{3} - 0.6\right) \cdot \text{Mixed triangle} + \left(\frac{2}{3} - 0.6\right) \cdot \text{Red triangle} + 0.4 \cdot \text{Red triangle}$$

$$\begin{aligned}
 0 & \leq \frac{1}{n} \sum_v \left(\begin{array}{c} \bullet \\ \text{blue} \\ \square v \end{array}, \begin{array}{c} \bullet \\ \text{red} \\ \square v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ \text{blue} \\ \square v \end{array}, \begin{array}{c} \bullet \\ \text{red} \\ \square v \end{array} \right)^T \\
 0 & \leq a \cdot \text{Blue triangle} + \frac{a+2c}{3} \cdot \text{Mixed triangle} + \frac{b+2c}{3} \cdot \text{Red triangle} + b \cdot \text{Red triangle}
 \end{aligned}$$

$$\begin{array}{c}
 \text{Red triangle} \\
 \text{Blue triangle} \\
 \text{Red triangle} \\
 \text{Blue triangle} \\
 \text{Red triangle}
 \end{array}
 = 0 \text{ Blue triangle} + 0 \text{ Red triangle} + 0 \text{ Blue triangle} + \text{Red triangle}$$

$$\text{Red triangle} \geq \min\{0, 0, 0, 1\}$$

$$0 \geq -a \text{ Blue triangle} - \frac{a+2c}{3} \text{ Red triangle} - \frac{b+2c}{3} \text{ Blue triangle} - b \text{ Red triangle}$$

$$0 \geq d \left(0.6 \text{ Blue triangle} + \left(0.6 - \frac{1}{3}\right) \text{ Red triangle} + \left(0.6 - \frac{2}{3}\right) \text{ Blue triangle} - 0.4 \text{ Red triangle} \right)$$


$$\begin{pmatrix} a & c & 0 \\ c & b & 0 \\ 0 & 0 & d \end{pmatrix} \succcurlyeq 0 \text{ (matrix is positive semidefinite)}$$

$$\begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} = 0 \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} + 0 \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} + 0 \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array}$$


$$0 \geq -a \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} - \frac{a+2c}{3} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} - \frac{b+2c}{3} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} - b \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array}$$

$$0 \geq d \left(0.6 \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} + \left(0.6 - \frac{1}{3} \right) \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} + \left(0.6 - \frac{2}{3} \right) \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} - 0.4 \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \right)$$

$$\begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} \geq \min \left\{ 0.6d - a, \left(0.6 - \frac{1}{3} \right) d - \frac{a+2c}{3}, \left(0.6 - \frac{2}{3} \right) d - \frac{b+2c}{3}, 1 - 0.4d - b \right\}$$



$$\geq 0.14150099 \dots$$



$$\geq \min \left\{ 0.6d - a, \left(0.6 - \frac{1}{3}\right) d - \frac{a + 2c}{3}, \left(0.6 - \frac{2}{3}\right) d - \frac{b + 2c}{3}, 1 - 0.4d - b \right\}$$

Numerical solution from CSDP:

$$a = 6 \times 0.1200006508849779385$$

$$a = 0.72$$

$$b = 6 \times 0.05333290843810910981$$


$$b = 0.32$$

$$c = 6 \times -0.07999989818128358521$$

$$c = -0.48$$


$$d = 1.400006454027185265$$

$$d = 1.4$$



$$\geq \min \{0.12, 0.45\bar{3}, 0.12, 0.12\} = 0.12$$

HOW TO IMPROVE 0.12?

 $\geq 0.14150099\dots$

Sample bigger graphs. Instead of

$$1 = \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array}$$

use

$$1 = \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array} + \dots + \begin{array}{c} \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \end{array}$$

HOW TO IMPROVE 0.12?

$$\begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \geq 0.14150099 \dots$$

Sample bigger graphs. Instead of

$$1 = \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$$

use


$$1 = \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \dots + \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$$

and include also $M, P \succcurlyeq 0$

$$0 \leq \begin{pmatrix} \begin{array}{c} 1 & 2 \\ \square & \square \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}, \begin{array}{c} 1 & 2 \\ \square & \square \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}, \begin{array}{c} 1 & 2 \\ \square & \square \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}, \begin{array}{c} 1 & 2 \\ \square & \square \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \end{pmatrix}^T M \begin{pmatrix} \begin{array}{c} 1 & 2 \\ \square & \square \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}, \begin{array}{c} 1 & 2 \\ \square & \square \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}, \begin{array}{c} 1 & 2 \\ \square & \square \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}, \begin{array}{c} 1 & 2 \\ \square & \square \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \end{pmatrix}$$

$$0 \leq \begin{pmatrix} \begin{array}{c} 1 & 2 \\ \square & \square \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}, \begin{array}{c} 1 & 2 \\ \square & \square \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}, \begin{array}{c} 1 & 2 \\ \square & \square \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}, \begin{array}{c} 1 & 2 \\ \square & \square \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \end{pmatrix}^T P \begin{pmatrix} \begin{array}{c} 1 & 2 \\ \square & \square \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}, \begin{array}{c} 1 & 2 \\ \square & \square \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}, \begin{array}{c} 1 & 2 \\ \square & \square \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}, \begin{array}{c} 1 & 2 \\ \square & \square \\ \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \end{pmatrix}$$

HOW TO IMPROVE 0.12?


 $\geq 0.14150099\dots$

Sample bigger graphs. Instead of

$$1 = \begin{array}{c} \text{triangle} \\ \text{blue} \end{array} + \begin{array}{c} \text{triangle} \\ \text{blue/red} \end{array} + \begin{array}{c} \text{triangle} \\ \text{red/blue} \end{array} + \begin{array}{c} \text{triangle} \\ \text{red} \end{array}$$

use


$$1 = \begin{array}{c} \text{square} \\ \text{blue} \end{array} + \begin{array}{c} \text{square} \\ \text{blue/red} \end{array} + \begin{array}{c} \text{square} \\ \text{red/blue} \end{array} + \dots + \begin{array}{c} \text{square} \\ \text{red} \end{array}$$

and include also $M, P \succcurlyeq 0$


$$0 \leq \left(\begin{array}{c} 1 \quad 2 \\ \text{triangle} \\ \text{blue/red} \end{array}, \begin{array}{c} 1 \quad 2 \\ \text{triangle} \\ \text{red/blue} \end{array}, \begin{array}{c} 1 \quad 2 \\ \text{triangle} \\ \text{red} \end{array}, \begin{array}{c} 1 \quad 2 \\ \text{triangle} \\ \text{blue} \end{array} \right)^T M \left(\begin{array}{c} 1 \quad 2 \\ \text{triangle} \\ \text{blue/red} \end{array}, \begin{array}{c} 1 \quad 2 \\ \text{triangle} \\ \text{red/blue} \end{array}, \begin{array}{c} 1 \quad 2 \\ \text{triangle} \\ \text{red} \end{array}, \begin{array}{c} 1 \quad 2 \\ \text{triangle} \\ \text{blue} \end{array} \right)$$

$$0 \leq \left(\begin{array}{c} 1 \quad 2 \\ \text{triangle} \\ \text{blue/red} \end{array}, \begin{array}{c} 1 \quad 2 \\ \text{triangle} \\ \text{red/blue} \end{array}, \begin{array}{c} 1 \quad 2 \\ \text{triangle} \\ \text{red} \end{array}, \begin{array}{c} 1 \quad 2 \\ \text{triangle} \\ \text{blue} \end{array} \right)^T P \left(\begin{array}{c} 1 \quad 2 \\ \text{triangle} \\ \text{blue/red} \end{array}, \begin{array}{c} 1 \quad 2 \\ \text{triangle} \\ \text{red/blue} \end{array}, \begin{array}{c} 1 \quad 2 \\ \text{triangle} \\ \text{red} \end{array}, \begin{array}{c} 1 \quad 2 \\ \text{triangle} \\ \text{blue} \end{array} \right)$$

This gives


 $\geq 0.127815\dots$

HOW TO IMPROVE 0.12781...?



$$\geq 0.14150099\dots$$

Sample even bigger graphs.

Use K_5 instead of K_4

Include even more types and flags.

HOW TO IMPROVE 0.12781...?



$$\geq 0.14150099\dots$$

Sample even bigger graphs.


Use K_5 instead of K_4

Include even more types and flags.

This gives


$$\geq 0.1333333 = 2/15.$$

HOW TO IMPROVE 0.12781...?



$$\geq 0.14150099\dots$$

Sample even bigger graphs.

Use K_5 instead of K_4

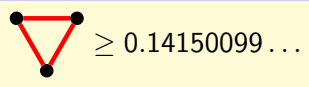
Include even more types and flags.

This gives


$$\geq 0.1333333 = 2/15.$$

Try even bigger!

HOW TO IMPROVE 0.12781...?

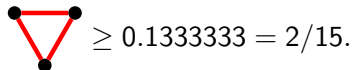


Sample even bigger graphs.

Use K_5 instead of K_4

Include even more types and flags.

This gives




Try even bigger!

vertices	# graphs	time	bound
3	4	instant	0.12
4	11	instant	0.127815...
5	34	instant	0.13333...
6	156	seconds	0.13333...
7	1044	minutes	0.13333...
8	12346	day(s)	0.13333...
9	274668	not computable*	?


* needs hundreds of GB of RAM, maybe easy in 10 years?

GETTING 0.14150099...


$$\geq 0.14150099\dots \text{ for } p = 0.6 \text{ by Razborov}$$


$$\left\{ \begin{array}{l} \text{Minimize} \\ \text{subject to} \end{array} \right. \quad \begin{array}{l} \text{triangle} \\ \text{edge} \end{array} \quad - p \geq 0$$

GETTING 0.14150099...


$$\geq 0.14150099\dots \text{ for } p = 0.6 \text{ by Razborov}$$

$$\left\{ \begin{array}{l} \text{Minimize} \\ \text{subject to} \end{array} \right. \quad \begin{array}{l} \text{triangle} \\ H \cdot \left(\begin{array}{c} \text{edge} \\ -p \end{array} \right) \geq 0 \text{ for any graph } H \end{array}$$

GETTING 0.14150099...



$$\geq 0.14150099\dots \text{ for } p = 0.6 \text{ by Razborov}$$


$$\left\{ \begin{array}{l} \text{Minimize} \\ \text{subject to} \end{array} \right. \quad \begin{array}{c} \text{triangle} \\ H \cdot \left(\text{edge} - p \right) \geq 0 \text{ for any graph } H \end{array}$$

vertices	# graphs	time	bound	new bound
3	4	instant	0.12	0.12
4	11	instant	0.12781...	0.131746...
5	34	instant	0.13333...	0.14046241...
6	156	seconds	0.13333...	0.14150099...
7	1044	minutes	0.13333...	0.14150099...
8	12346	day(s)	0.13333...	0.14150099...

These are just **numerical** bounds! Not exact.


GOODMAN'S BOUND

Recall we got for $p = 0.6$


$$\geq \min \left\{ 0.6d - a, \left(0.6 - \frac{1}{3}\right) d - \frac{a + 2c}{3}, \left(0.6 - \frac{2}{3}\right) d - \frac{b + 2c}{3}, 1 - (1 - 0.6)d - b \right\}$$

GOODMAN'S BOUND


Recall we got for $p = 0.6$


$$\geq \min \left\{ 0.6d - a, \left(0.6 - \frac{1}{3} \right) d - \frac{a + 2c}{3}, \right. \\ \left. \left(0.6 - \frac{2}{3} \right) d - \frac{b + 2c}{3}, 1 - (1 - 0.6)d - b \right\}$$

Same thing holds when 0.6 is replaced by a parameter p .

GOODMAN'S BOUND


Recall we got for $p = 0.6$


$$\geq \min \left\{ 0.6d - a, \left(0.6 - \frac{1}{3}\right) d - \frac{a + 2c}{3}, \left(0.6 - \frac{2}{3}\right) d - \frac{b + 2c}{3}, 1 - (1 - 0.6)d - b \right\}$$

Same thing holds when 0.6 is replaced by a parameter p .

$$a = 2p^2 \quad b = 2p^2 - 4p + 2 \quad c = p(2p - 2) \quad d = 4p - 1$$

gives Goodman's bound:


$$\geq 2p^2 - p$$

GOODMAN'S BOUND

Recall we got for $p = 0.6$

$$\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \geq \min \left\{ 0.6d - a, \left(0.6 - \frac{1}{3}\right) d - \frac{a + 2c}{3}, \left(0.6 - \frac{2}{3}\right) d - \frac{b + 2c}{3}, 1 - (1 - 0.6)d - b \right\}$$

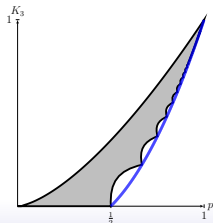
Same thing holds when 0.6 is replaced by a parameter p .

$$a = 2p^2 \quad b = 2p^2 - 4p + 2 \quad c = p(2p - 2) \quad d = 4p - 1$$

gives Goodman's bound:

$$\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \geq 2p^2 - p$$

This is tight for $p \in \left\{ \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \dots \right\}$.



SEE YOU AGAIN IN FEW MINUTES!

- Flag Algebras “definitions”
- First try for Mantel’s theorem
- More automatic approach
- Additional constraints
- maybe break
- Define flag algebras
- Graph sequences and homomorphisms
- Turán’s Theorem (in limit)
- Finally Mantel’s Theorem (for real)
- mega break
- Applications