Flag Algebra Methods (more formal approach)

Bernard Lidický

6th Lake Michigan Workshop on Combinatorics and Graph Theory

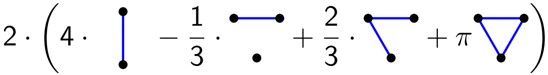
Apr 7, 2019

Outline

- Flag Algebras “definitions”
- First try for Mantel’s theorem
- More automatic approach
- Additional constraints
- maybe break
- Define flag algebras
- Graph sequences and homomorphisms
- Turán’s Theorem (in limit)
- Finally Mantel’s Theorem (for real)
- mega break
- Applications

Building an algebra

Let $\mathcal{F}$ denote the set of all graphs (up to isomorphism).
Let $\mathbb{R}\mathcal{F}$ be the set of all finite formal linear combinations of graphs.



It comes with natural addition and multiplication by a real number (think of a vector $\in \mathbb{R}^{\mathcal{F}}$.)
How to define multiplication of two elements in $\mathbb{R}\mathcal{F}$?

Towards multiplication

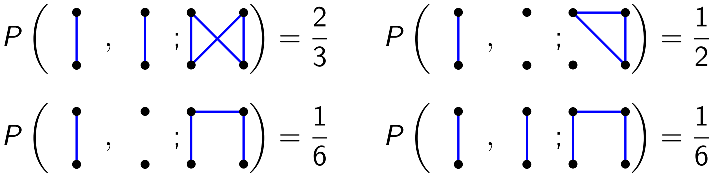
Let $\mathcal{F}$ denote the set of all graphs.
Let $F_1, F_2, F \in \mathcal{F}$ such that $|F_1| + |F_2| \leq |F|$.

Definition

Let $X_1, X_2 \subseteq V(F)$ be random disjoint of sizes $|F_1|$ and $|F_2|$.

$$P(F_1, F_2; F)$$

is the probability $F[X_i] \cong F_i$ for both i.



Towards multiplication

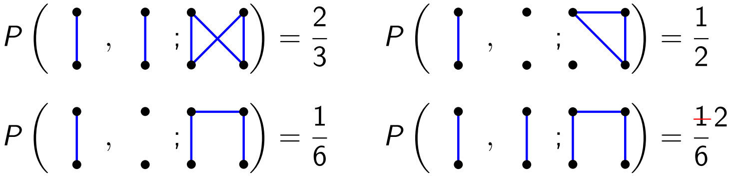
Let $\mathcal{F}$ denote the set of all graphs.
Let $F_1, F_2, F \in \mathcal{F}$ such that $|F_1| + |F_2| \leq |F|$.

Definition

Let $X_1, X_2 \subseteq V(F)$ be random disjoint of sizes $|F_1|$ and $|F_2|$.

$$P(F_1, F_2; F)$$

is the probability $F[X_i] \cong F_i$ for both i.

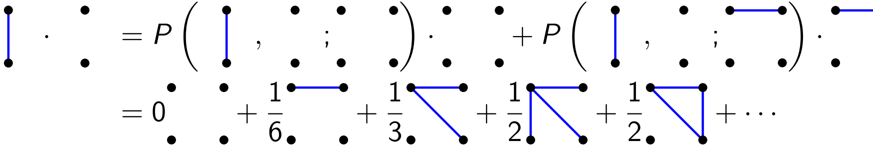


Multiplication

Let $\mathcal{F}$ denote the set of all graphs.
Let $F_1, F_2 \in \mathcal{F}$. We need $F_1 \cdot F_2 \in \mathbb{R}\mathcal{F}$.
Define

$$F_1 \cdot F_2 := \sum_{F \in \mathcal{F}_\ell} P(F_1, F_2; F)F,$$

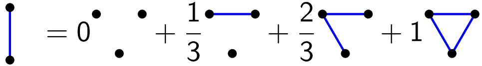
where $|F_1| + |F_2| = \ell$. Notice there is NO $+o(1)$.



This extends to $a \cdot b$ for $a, b \in \mathbb{R}\mathcal{F}$.

Factorizing

Recall from previous presentation we had identities like



Let $\mathcal{K}$ be linear subspace generated by

$$F - \sum_{F' \in \mathcal{F}_\ell} P(F, F')F' \quad (= 0),$$

where $F \in \mathcal{F}$ and $|F| \leq \ell$.
Algebra $\mathcal{A}$ is $\mathbb{R}\mathcal{F}$ factorized by $\mathcal{K}$.
$a, b \in \mathbb{R}\mathcal{F}$ are in one equivalence class if $a = b + c$ for some $c \in \mathcal{K}$
Note: Think of $\mathbb{R}\mathcal{F}$ and $\mathcal{A}$ as $\mathbb{Z} \times \mathbb{Z}$ and $\mathcal{Q}$.
(correctness of definitions proved by Razborov)

Algebra $\mathcal{A}$

- $\mathcal{F}$ is the set of all graphs.
- $\mathcal{F}_\ell$ is on ℓ vertices.
- $\mathbb{R}\mathcal{F}$ formal linear combinations
- $\mathcal{K} := span(F - \sum_{F' \in \mathcal{F}_\ell} P(F, F')F')$
- $\mathcal{A}$ is $\mathbb{R}\mathcal{F}$ factorized by $\mathcal{K}$.
- addition in $\mathcal{A}$ comes from $\mathbb{R}\mathcal{F}$
- $F_1 \cdot F_2 = \sum_{F \in F_\ell} P(F_1, F_2; F)F$,
- $F \in \mathcal{F}$ is called a *flag*.
- informally called unlabeled flags

Convergent Graph Sequence

Let $\mathcal{F}$ denote the set of all graphs.

Definition

A sequence of graphs $(G_n)_{n\in\mathbb{N}}$ is *convergent* if for every finite graph H, $\lim_{n\to\infty} P(H, G_n)$ exists.

Examples:

$$\lim_{n\to\infty} P(H, K_n) = \begin{cases} 1 & \text{if } H \cong K_m \\ 0 & \text{otherwise} \end{cases}$$

$$\lim_{n\to\infty} P(H, K_{n,n}) = \begin{cases} 1/2 & \text{if } H \cong K_2 \\ 3/4 & \text{if } H \cong P_2 \\ \vdots \end{cases}$$

$$\lim_{n\to\infty} P(H, P_n) = \begin{cases} 1 & \text{is } |E(H)| = 0 \\ 0 & otherwise \end{cases}$$

Gives map $\mathcal{F} \to [0, 1]$ or a point in $[0, 1]^{\mathcal{F}}$.

Convergent Graph Sequence

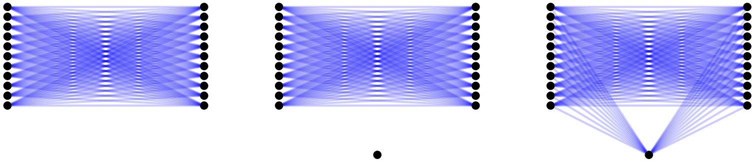
Let $\mathcal{F}$ denote the set of all graphs.

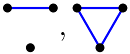
Definition

A sequence of graphs $(G_n)_{n\in\mathbb{N}}$ is *convergent* if for every finite graph H, $\lim_{n\to\infty} P(H, G_n)$ exists.

For all $H \in \mathcal{F}$

$$\lim_{n\to\infty} P(H, K_{n,n}) = \lim_{n\to\infty} P(H, K_{n,n} \cup K_1) = \lim_{n\to\infty} P(H, K_{n,n} + K_1)$$



In particular for $H \in \Big\{$  $\Big\}$.

Small changes in $(G_n)_{n\in\mathbb{N}}$ are not noticeable.

Positive Homomorphisms

Let $Hom(\mathcal{A}, \mathbb{R})$ be the set of all homomorphisms from $\mathcal{A}$ to $\mathbb{R}$.
I.e. for any $\phi \in Hom(\mathcal{A}, \mathbb{R})$ and $a, b \in \mathcal{A}$:

- $\phi(a + b) = \phi(a) + \phi(b)$
- $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$.

Since $p(H, G) \in [0, 1]$, we consider only $Hom^+(\mathcal{A}, \mathbb{R})$, i.e.
$\phi(H) \geq 0$ for all $H \in \mathcal{F}$
Notice $\phi(\emptyset) = 1$.

Theorem (Razborov)

$Hom^+(\mathcal{A}, \mathbb{R})$ corresponds exactly to the convergent sequences.

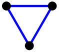
Theorem (Felix; Podolski; (only for graphs))

Let $a \in \mathbb{R}\mathcal{F}$. $\phi(a) = 0$ for all $\phi \in Hom^+(\mathcal{A}, \mathbb{R})$ iff $a \in \mathcal{K}$.

Theorem (Mantel)

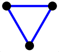
Every triangle-free graph on n vertices has at most $n^2/4$ edges.

Theorem (Mantel from previous presentation)

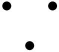
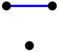
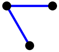
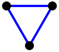
For a large graph if  $= 0$ *then*  $\leq \frac{1}{2} + o(1)$.

Theorem (Mantel with Homomorphisms)

For every $\phi \in Hom^+(\mathcal{A}, \mathbb{R})$ holds that

if $\phi\Big($  $\Big) = 0$ *then* $\phi\Big($  $\Big) \leq \frac{1}{2}$.

Example from last time

$0 \leq 3 \cdot$  $-$  $-$  $+ 3 \cdot$  $+o(1)$

We want to find $a \in \mathcal{A}$ such that for EVERY $\phi \in Hom^{+}(\mathcal{A}, \mathbb{R})$ we have $\phi(a) \geq 0$. We write $a \geq 0$.

Theorem (Hatami and Norin 11)

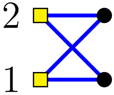
Determining if $a \geq 0$ is not algorithmically decidable.

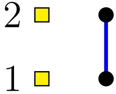
Norin: "Extremal combinatorics remains an art".

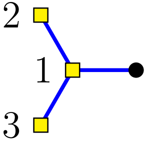
But we can still generate a lot of them!

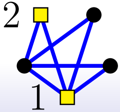
Algebra A^σ

- vertices of $\sigma \in F$ are labeled by $1, \ldots, |\sigma|$.
- $\mathcal{F}^\sigma$ is the set of all graphs each containing a fixed induced labeled copy of σ.
- $\mathcal{F}^\sigma_\ell$ is on ℓ vertices.
- $\mathbb{R}\mathcal{F}^\sigma$ formal linear combinations
- $\mathcal{K}^\sigma := \mathit{span}(F - \sum_{F' \in \mathcal{F}^\sigma_\ell} P(F, F')F')$
- $\mathcal{A}^\sigma$ is $\mathbb{R}\mathcal{F}^\sigma$ factorized by $\mathcal{K}^\sigma$.
- addition in $\mathcal{A}^\sigma$ comes from $\mathbb{R}\mathcal{F}^\sigma$
- $F_1 \cdot F_2 = \sum_{F \in F^\sigma_\ell} P(F_1, F_2; F)F$,
- $F \in \mathcal{F}^\sigma$ is called a σ-*flag*.
- σ is called a *type*

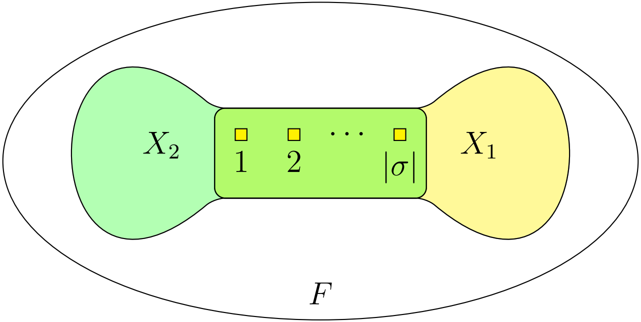








$F_1 \cdot F_2 = \sum_{F \in F_\ell^\sigma} P(F_1, F_2; F)F$



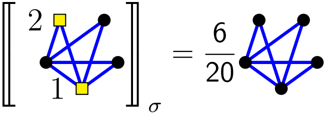
Pick randomly $X_1, X_2 \subset V(F)$ such that $X_1 \cap X_2$ are exactly all labeled vertices and $|X_i| = |F_i|$.
$P(F_1, F_2; F)$ is the probability that $X_1 \cong F_1$ and $X_2 \cong F_2$.

Averaging (unlabeling) operator

Let $F = (G, \theta)$ be a σ-flag, where $\theta : 1, \ldots, |\sigma| \to V(G)$. Define

$$[\![F]\!]_\sigma = q_\theta(F) \cdot G,$$

where $q_\theta(F)$ is the probability that (G, θ') is isomorphic to F for a random injective $\theta' : 1, \ldots, |\sigma| \to V(G)$.



Linear extension gives the *averaging operator*

$$[\![\cdot]\!]_\sigma : \mathbb{R}\mathcal{F}^\sigma \to \mathbb{R}\mathcal{F}.$$

It is a linear mapping, *not* a homomorphism.

$$[\![F_1 + F_2]\!]_\sigma = [\![F_1]\!]_\sigma + [\![F_2]\!]_\sigma$$

$[\![F_1 \cdot F_2]\!]_\sigma$ might NOT be $[\![F_1]\!]_\sigma \cdot [\![F_2]\!]_\sigma$

$\mathcal{A}^\sigma$, $\mathcal{A}$ and *Hom*$^+$

For any $\phi \in Hom^+(\mathcal{A}, \mathbb{R})$ and any $a \in \mathcal{A}^\sigma$

$$\phi([\![a \cdot a]\!]_\sigma) \geq 0.$$

Lemma (Razborov)

Cauchy-Schwarz inequality for all $\phi \in Hom^+(\mathcal{A}, \mathbb{R})$

$$\phi([\![a^2]\!]_\sigma \cdot [\![b^2]\!]_\sigma) \geq \phi([\![ab]\!]_\sigma^2).$$

Special cases:

$$\phi([\![a^2]\!]_\sigma) \geq \frac{\phi([\![a]\!]_\sigma^2)}{\phi([\![\sigma]\!]_\sigma)} \qquad \phi([\![a^2]\!]_{\square}) \geq \phi([\![a]\!]_{\square}^2)$$

Uses special probability distribution on $Hom^+(\mathcal{A}^\sigma, \mathbb{R})$ related to ϕ.

Let σ and $Hom^+(\mathcal{A}, \mathbb{R})$ be fixed. This gives

Skip this. Hard to follow.

$$G_1, G_2, G_3, \ldots$$

To make $\phi^\sigma \in Hom^+(\mathcal{A}^\sigma, \mathbb{R})$, we need a sequence of labeled graphs.
Incorrect: Take a copy of σ in each G_i and get a convergent

$$G_1^\sigma, G_2^\sigma, G_3^\sigma, \ldots$$

Correct: Fix G_i. Randomly label a copy of σ and get G_i^σ. This gives $P(F, G_n^\sigma)$ for all $F \in \mathcal{F}^\sigma$.
By a random σ, we get a probability distribution $\mathbf{P}_{G_n}^\sigma$ on the functions $P(., G_n^\sigma)$.
These $\mathbf{P}_{G_n}^\sigma$ then weakly converge to a (unique) probability distribution $\mathbf{P}_\phi^\sigma$ on $\phi^\sigma \in Hom^+(\mathcal{A}^\sigma, \mathbb{R})$.
A *crucial feature*: if $a \in \mathcal{A}^\sigma$ and $\phi \in Hom^+(\mathcal{A}, \mathbb{R})$, then

$$\phi([\![\sigma]\!]_\sigma) \cdot \mathbb{E}_{\mathbf{P}_\phi^\sigma}[\phi^\sigma(a)] = \phi([\![a]\!]_\sigma).$$

This can be viewed as an analogue of $P(B) \cdot P(A|B) = P(A \wedge B)$.

Skip this. Hard to follow.

A *crucial feature*: if $a \in \mathcal{A}^\sigma$ and $\phi \in Hom^+(\mathcal{A}, \mathbb{R})$, then

$$\phi([\![\sigma]\!]_\sigma) \cdot \mathbb{E}_{\mathbf{P}^\sigma_\phi}[\phi^\sigma(a)] = \phi([\![a]\!]_\sigma). \tag{1}$$

Bonus: For a given $\phi \in Hom^+(\mathcal{A}, \mathbb{R})$, exists a unique probability distribution $\mathbf{P}^\sigma_\phi$ satisfying (1).

(1) is especially useful when $\phi^\sigma(a) \geq 0$ with probability one. It gives $\phi([\![a]\!]_\sigma) \geq 0$.

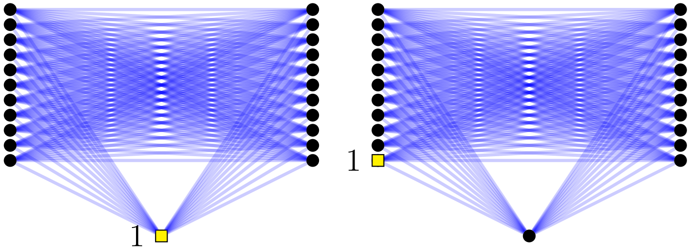
In particular, for any $\phi \in Hom^+(\mathcal{A}, \mathbb{R})$ and any $a \in \mathcal{A}^\sigma$

$$\phi([\![a \cdot a]\!]_\sigma) \geq 0.$$

Simplified notation is just $[\![a \cdot a]\!]_\sigma \geq 0$.

Skip this. Hard to follow.

Consider sequence $(K_{n,n} + K_1)_n$ with $\phi \in Hom^+(\mathcal{A}, \mathbb{R})$ and $\sigma = 1$.



There exists $\phi' \in Hom^+(\mathcal{A}^\sigma, \mathbb{R})$ such that $\phi'\Bigg($  $\Bigg) = 1$.

But

$$\phi\left(\left[\!\!\left[\ \right]\!\!\right]_\sigma\right) = \phi([\![\sigma]\!]_\sigma) \cdot \mathbb{E}_{\mathbf{P}^\sigma_\phi}\left[\phi^\sigma\left(\ \right)\right] = \frac{1}{2}$$

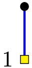
Flag algebras do not see 'rare vertices'.

Mantel's Theorem again

For all $\phi \in Hom^{+}(\mathcal{A}, \mathbb{R})$

$[\![a]\!]_1^2 \le [\![a^2]\!]_1$

σ is K_1 denoted by 1. Over triangle-free graphs i.e. $\phi(K_3) = 0$

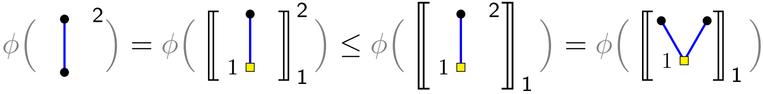
$\phi\Big([\![$  $]\!]_1^2\Big) \le \phi\Big([\![$  $^2]\!]_1\Big)$

Mantel's Theorem again

$[\![a]\!]_1^2 \leq [\![a^2]\!]_1$

For all $\phi \in Hom^+(\mathcal{A}, \mathbb{R})$

σ is K_1 denoted by 1. Over triangle-free graphs i.e. $\phi(K_3) = 0$

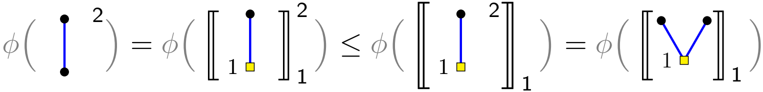


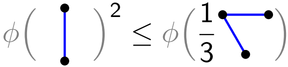
Mantel's Theorem again

$[\![a]\!]_1^2 \leq [\![a^2]\!]_1$

For all $\phi \in Hom^+(\mathcal{A}, \mathbb{R})$

σ is K_1 denoted by 1. Over triangle-free graphs i.e. $\phi(K_3) = 0$



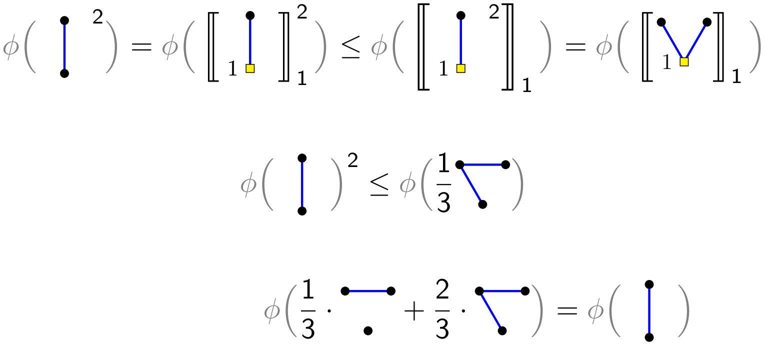


Mantel's Theorem again

For all $\phi \in Hom^{+}(\mathcal{A}, \mathbb{R})$

$[\![a]\!]_1^2 \leq [\![a^2]\!]_1$

σ is K_1 denoted by 1. Over triangle-free graphs i.e. $\phi(K_3) = 0$

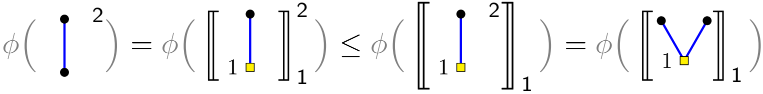


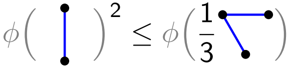
Mantel's Theorem again

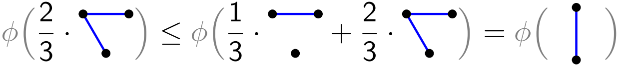
$[\![a]\!]_1^2 \leq [\![a^2]\!]_1$

For all $\phi \in Hom^+(\mathcal{A}, \mathbb{R})$

σ is K_1 denoted by 1. Over triangle-free graphs i.e. $\phi(K_3) = 0$





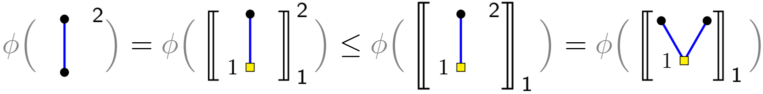


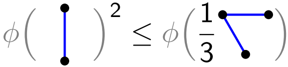
Mantel's Theorem again

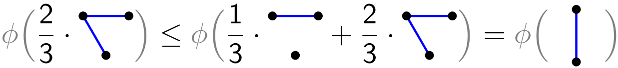
$[\![a]\!]_1^2 \le [\![a^2]\!]_1$

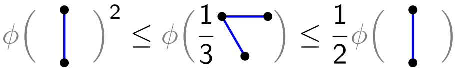
For all $\phi \in Hom^+(\mathcal{A}, \mathbb{R})$

σ is K_1 denoted by 1. Over triangle-free graphs i.e. $\phi(K_3) = 0$

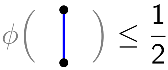






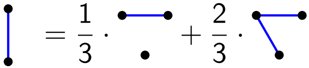


hence

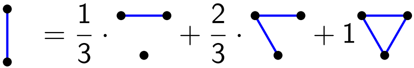


$\mathcal{A}$ AND FORBIDDEN GRAPHS

In previous example, we used



rather than

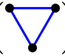


This can be done formally by defining $\mathcal{A}^{\nabla}$ over $\mathcal{F}^{\nabla}$, which are all triangle free graphs. Works for any forbidden graphs.

Forbidding triangles can be done formally by defining $\mathcal{A}^\nabla$ over $\mathcal{F}^\nabla$, which are all triangle free graphs. Works for any forbidden graphs.

Theorem (Mantel - Homomorphisms from $\mathcal{A}$)

For every $\phi \in Hom^+(\mathcal{A}, \mathbb{R})$ holds that

if $\phi\Big($  $\Big) = 0$ *then* $\phi\Big($  $\Big) \leq \frac{1}{2}$.

Theorem (Mantel - Homomorphisms from $\mathcal{A}^\nabla$)

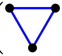
For every $\phi \in Hom^+(\mathcal{A}^\nabla, \mathbb{R})$ holds $\phi\Big($  $\Big) \leq \frac{1}{2}$.

Formulation with $\mathcal{A}^\nabla$ is particularly useful in practical calculations if $|\mathcal{F}_\ell^\nabla| < |\mathcal{F}_\ell|$.

Forbidding triangles can be done forma
which are all triangle free graphs. Worl

THEOREM (MANTEL - HOMOMOR

For every $\phi \in Hom^{+}(\mathcal{A}, \mathbb{R})$ *holds that*

if $\phi\Big($  $\Big) = 0$ *then* ϕ

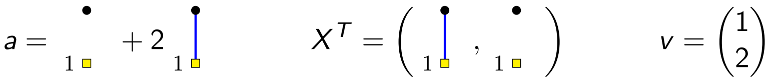
ℓ	$\lvert\mathcal{F}_\ell\rvert$	$\lvert\mathcal{F}_\ell^{\nabla}\rvert$
3	4	3
4	11	7
5	34	14
6	156	38
7	1,044	107
8	12,346	410
9	274,668	1,897
10	12,005,168	12,172
11	1,018,997,864	105,071

THEOREM (MANTEL - HOMOMORPHISMS FROM $\mathcal{A}^{\nabla}$)

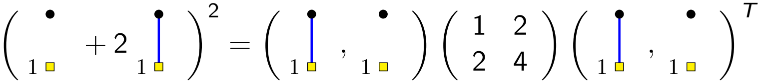
For every $\phi \in Hom^{+}(\mathcal{A}^{\nabla}, \mathbb{R})$ *holds* $\phi\Big($  $\Big) \leq \frac{1}{2}$.

Formulation with $\mathcal{A}^{\nabla}$ is particularly useful in practical calculations if $|\mathcal{F}_\ell^{\nabla}| < |\mathcal{F}_\ell|$.

Let $a \in \mathcal{A}^\sigma$. Recall $[\![a \cdot a]\!]_\sigma \geq 0$. Let $a = X^T v$, where $X \in (\mathcal{F}^\sigma)^m$ and $v \in \mathbb{R}^m$.



$$a^2 = a \cdot a = X^T v \cdot (X^T v) = X^T v v^T X = X^T M X$$

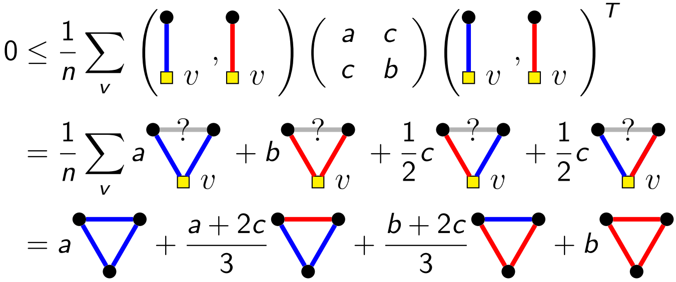


Observation

Let σ be fixed. If $M \succcurlyeq 0$ and X is a vector $(\mathcal{F}^\sigma_\ell)^n$, then $\forall \phi \in Hom^+(\mathcal{A}, \mathbb{R})$

$$\phi\Big([\![X^T M X]\!]_\sigma\Big) \geq 0. \text{ Similar to } \phi\Big([\![\sum_{a \in \mathcal{A}^\sigma} a^2]\!]_\sigma\Big) \geq 0.$$

Last time - Density approach



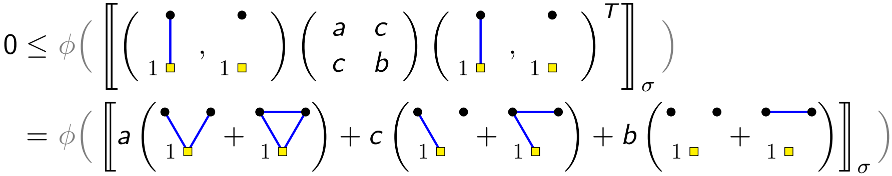
Homomorphism approach

For all $\phi \in Hom^+(\mathcal{A}, \mathbb{R})$

$0 \leq \phi\Big(\Big[\!\!\Big[\Big($  ,  $\Big) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \Big($  ,  $\Big)^T \Big]\!\!\Big]_\sigma \Big)$

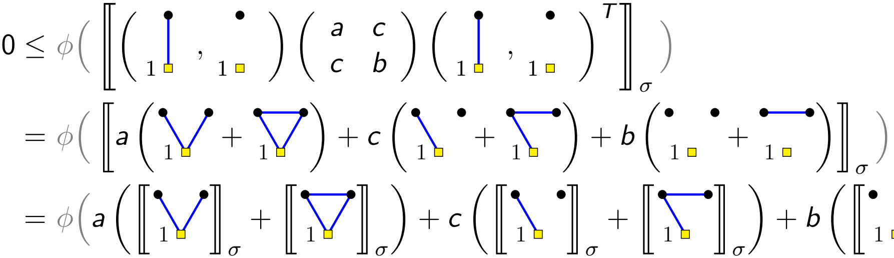
Homomorphism approach

For all $\phi \in Hom^{+}(\mathcal{A}, \mathbb{R})$



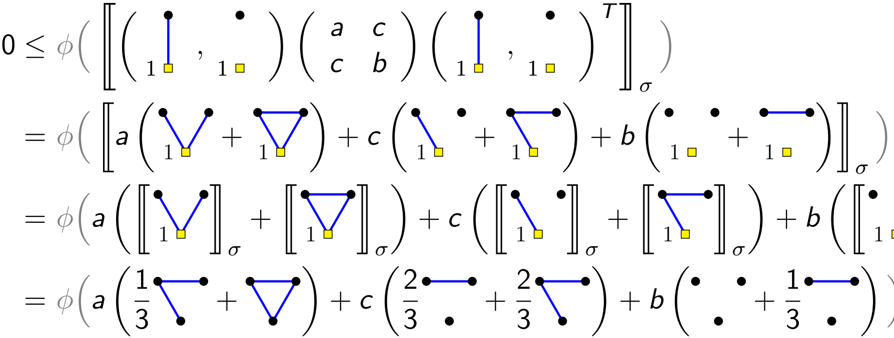
Homomorphism approach

For all $\phi \in Hom^{+}(\mathcal{A}, \mathbb{R})$



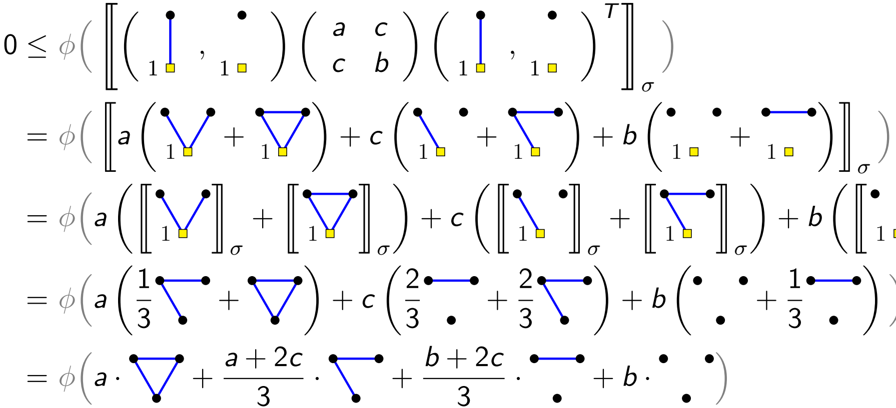
Homomorphism approach

For all $\phi \in Hom^{+}(\mathcal{A}, \mathbb{R})$



Homomorphism approach

For all $\phi \in Hom^{+}(\mathcal{A}, \mathbb{R})$



Turán Theorem

We do not write $\phi(\dots)$ any more but it is there!

Turán question

What is the maximum number of edges in K_t-free n-vertex graphs?

We seek answer as  $\leq \frac{t-2}{t-1}$.

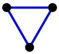
Computer can do at most $t = 8$.

We do by hand induction on t.

Main tool

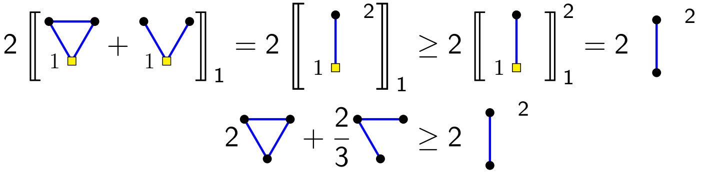
$$[\![a^2]\!]_\sigma \geq \frac{[\![a]\!]_\sigma^2}{[\![\sigma]\!]_\sigma}.$$

Our first goal is to prove

 ≥ 2  $\cdot \Big($  $- \frac{1}{2}\Big)$

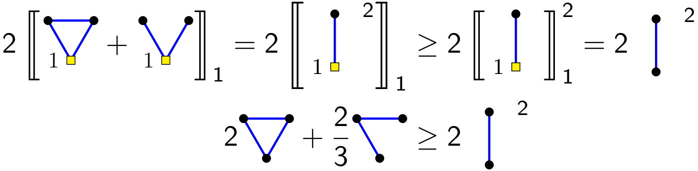
Fill the rest of the following

$[\![a^2]\!]_1 \geq [\![a]\!]_1^2$

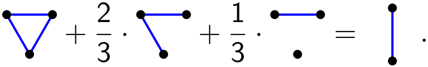


Fill the rest of the following

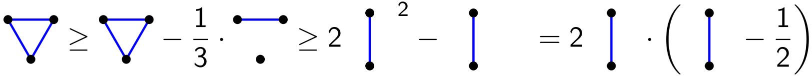
$$[\![a^2]\!]_1 \geq [\![a]\!]_1^2$$



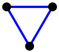
Recall



By subtracting, we obtain



We got the base of induction.

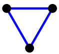
 ≥ 2  $\cdot \Big($  $- \frac{1}{2} \Big)$ (2)

The real goal is

$K_{t+1} \geq t \cdot K_t \cdot \Big($  $- \frac{t-1}{t} \Big)$. (3)

Inequality (2) is just (3) with $t = 2$.
Why is (3) *proving* the Turáns theorem?

We got the base of induction.

 ≥ 2  $\cdot \Big($  $- \frac{1}{2}\Big)$ (2)

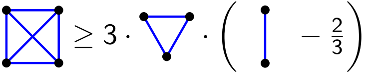
The real goal is

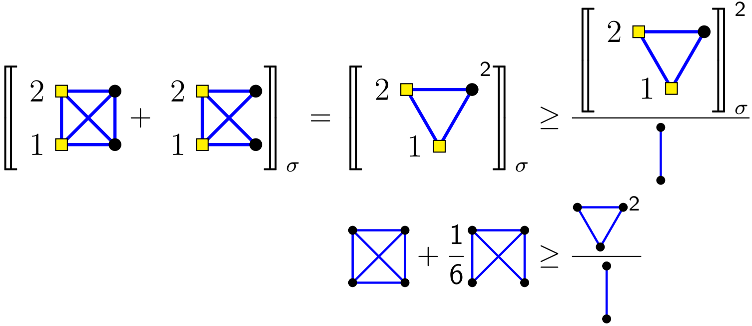
$K_{t+1} \geq t \cdot K_t \cdot \Big($  $- \frac{t-1}{t}\Big).$ (3)

Inequality (2) is just (3) with $t = 2$.
Why is (3) *proving* the Turáns theorem?
Let $(G_n)_{n\in\mathbb{N}}$ is K_{t+1} free sequence with $\phi \in Hom^+(\mathcal{A}, \mathbb{R})$.
Then $\phi(K_{t+1}) = 0$. This gives in (3) that

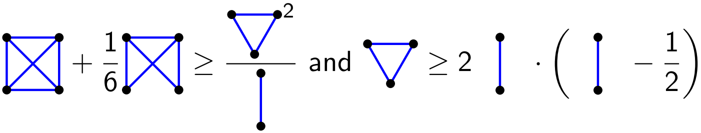
$$0 \geq t \cdot \phi(K_t) \cdot \left(\phi\left(K_2\right) - \frac{t-1}{t}\right).$$

If $\phi(K_t) > 0$ then $\phi(K_2) \leq \frac{t-1}{t}$.
If $\phi(K_t) = 0$ induction on t gives $\phi(K_2) \leq \frac{t-2}{t-1}$.

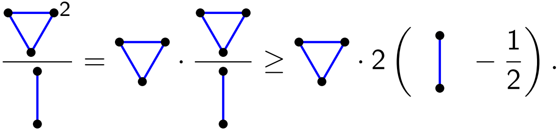
Induction step for $t = 4$. Goal is 



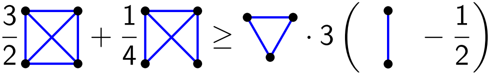
$$[\![a^2]\!]_\sigma \geq \frac{[\![a]\!]_\sigma^2}{[\![\sigma]\!]_\sigma}$$

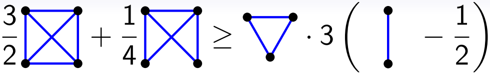


When we use induction, which means (2) and we get

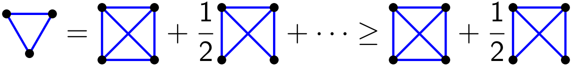


Hence we get after scaling by $\frac{3}{2}$

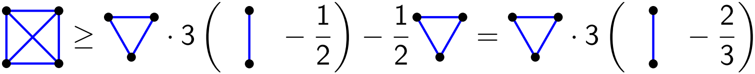




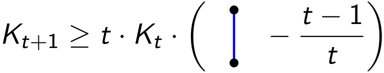
Add $\frac{1}{2}$ times the following



and get



In a similar way we get also



Actual proof of Mantel's theorem

Theorem (Mantel for all n)

If G is a triangle free graph on n vertices, then $|E(G)| \leq n^2/4$.

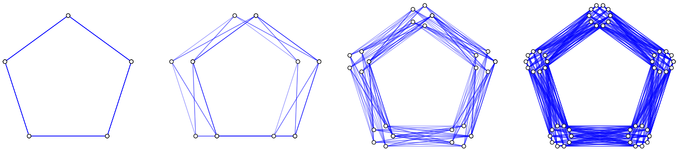
For contradiction let H be a triangle-free graph on n_0 vertices, with $|E(H)| = cn_0^2$, where $c > \frac{1}{4}$.

Actual proof of Mantel's theorem

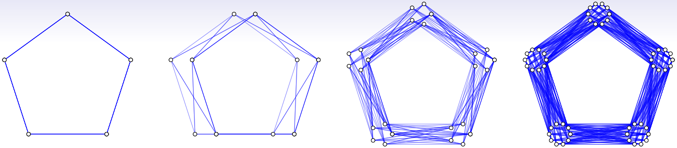
Theorem (Mantel for all n)

If G is a triangle free graph on n vertices, then $|E(G)| \leq n^2/4$.

For contradiction let H be a triangle-free graph on n_0 vertices, with $|E(H)| = cn_0^2$, where $c > \frac{1}{4}$.
Let $B(H,k)$ be a blow-up of H by k vertices.



$(B(H,k))_{k\in\mathbb{N}}$ is convergent, gives $\phi_H \in Hom^+(\mathcal{A},\mathbb{R})$.



$(B(H,k))_{k\in\mathbb{N}}$ is convergent, gives $\phi_H \in Hom^+(\mathcal{A},\mathbb{R})$.

Observation

$B(H,k)$ is a triangle-free graph.

Hence $\phi_H(K_3) = 0$
$|E(B(H,k))| = cn^2k^2$. Hence

$$\phi_H(K_2) = \lim_{k\to\infty} \frac{|E(B(H,k)|}{\binom{|V((B(H,k)|}{2}} = \lim_{k\to\infty} \frac{cn_0^2k^2}{\binom{n_0k}{2}} = 2c > \frac{1}{2}$$

Before, we proved that $\phi(K_2) \leq \frac{1}{2}$ whenever $\phi(K_3) = 0$, contradiction.

Turán's theorem can be proved with the same trick.

See you in the afternoon! (maybe)

- Flag Algebras “definitions”
- First try for Mantel’s theorem
- More automatic approach
- Additional constraints
- maybe break
- Define flag algebras
- Graph sequences and homomorphisms
- Turán’s Theorem (in limit)
- Finally Mantel’s Theorem (for real)
- mega break
- Applications