

FLAG ALGEBRA METHODS (MORE FORMAL APPROACH)

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6th Lake Michigan Workshop on Combinatorics and Graph Theory

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OUTLINE

- Flag Algebras “definitions”
- First try for Mantel’s theorem
- More automatic approach
- Additional constraints
- maybe break
- Define flag algebras
- Graph sequences and homomorphisms
- Turán’s Theorem (in limit)
- Finally Mantel’s Theorem (for real)
- mega break
- Applications

BUILDING AN ALGEBRA

Let \mathcal{F} denote the set of all graphs (up to isomorphism).

Let $\mathbb{R}\mathcal{F}$ be the set of all finite formal linear combinations of graphs.

$$2 \cdot \left(4 \cdot \begin{array}{c} \bullet \\ | \\ \bullet \end{array} - \frac{1}{3} \cdot \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \end{array} + \frac{2}{3} \cdot \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \pi \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right)$$

It comes with natural addition and multiplication by a real number (think of a vector $\in \mathbb{R}\mathcal{F}$.)

How to define multiplication of two elements in $\mathbb{R}\mathcal{F}$?

TOWARDS MULTIPLICATION

Let \mathcal{F} denote the set of all graphs.

Let $F_1, F_2, F \in \mathcal{F}$ such that $|F_1| + |F_2| \leq |F|$.

DEFINITION

Let $X_1, X_2 \subseteq V(F)$ be random disjoint of sizes $|F_1|$ and $|F_2|$.

$$P(F_1, F_2; F)$$

is the probability $F[X_i] \cong F_i$ for both i .

$$P \left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array}, \begin{array}{c} \bullet \\ | \\ \bullet \end{array}; \begin{array}{c} \bullet & \bullet \\ / & \backslash \\ \bullet & \bullet \end{array} \right) = \frac{2}{3} \quad P \left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array}, \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array}; \begin{array}{c} \bullet & \bullet \\ / & \backslash \\ \bullet & \bullet \end{array} \right) = \frac{1}{2}$$

$$P \left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array}, \begin{array}{c} \bullet \\ \bullet \\ \bullet \end{array}; \begin{array}{c} \bullet & \bullet \\ | & | \\ \bullet & \bullet \end{array} \right) = \frac{1}{6} \quad P \left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array}, \begin{array}{c} \bullet \\ | \\ \bullet \end{array}; \begin{array}{c} \bullet & \bullet \\ | & | \\ \bullet & \bullet \end{array} \right) = \frac{1}{6}$$

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MULTIPLICATION

Let \mathcal{F} denote the set of all graphs.

Let $F_1, F_2 \in \mathcal{F}$. We need $F_1 \cdot F_2 \in \mathbb{R}\mathcal{F}$.

Define

$$F_1 \cdot F_2 := \sum_{F \in \mathcal{F}_\ell} P(F_1, F_2; F) F,$$

where $|F_1| + |F_2| = \ell$. Notice there is NO $+o(1)$.

$$\begin{aligned}
 \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \cdot \begin{array}{c} \bullet \\ \bullet \end{array} &= P \left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array}, \begin{array}{c} \bullet \\ \bullet \end{array}; \begin{array}{c} \bullet \\ \bullet \end{array} \right) \cdot \begin{array}{c} \bullet \\ \bullet \end{array} + P \left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array}, \begin{array}{c} \bullet \\ \bullet \end{array}; \begin{array}{c} \bullet \\ \bullet \end{array} \right) \cdot \begin{array}{c} \bullet \\ \bullet \end{array} \\
 &= 0 \cdot \begin{array}{c} \bullet \\ \bullet \end{array} + \frac{1}{6} \begin{array}{c} \bullet \\ \text{---} \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \\ \diagdown \\ \bullet \end{array} + \frac{1}{2} \begin{array}{c} \bullet \\ | \quad \diagdown \\ \bullet \end{array} + \frac{1}{2} \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \dots
 \end{aligned}$$

This extends to $a \cdot b$ for $a, b \in \mathbb{R}\mathcal{F}$.

FACTORIZING

Recall from previous presentation we had identities like

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} = 0 \begin{array}{c} \bullet & \bullet \\ & \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \text{---} \bullet \\ | \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + 1 \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$$

Let \mathcal{K} be linear subspace generated by

$$F - \sum_{F' \in \mathcal{F}_\ell} P(F, F') F' \quad (= 0),$$

where $F \in \mathcal{F}$ and $|F| \leq \ell$.

Algebra \mathcal{A} is $\mathbb{R}\mathcal{F}$ factorized by \mathcal{K} .

$a, b \in \mathbb{R}\mathcal{F}$ are in one equivalence class if $a = b + c$ for some $c \in \mathcal{K}$

Note: Think of $\mathbb{R}\mathcal{F}$ and \mathcal{A} as $\mathbb{Z} \times \mathbb{Z}$ and \mathcal{Q} .

(correctness of definitions proved by Razborov)

ALGEBRA \mathcal{A}

- \mathcal{F} is the set of all graphs.
- \mathcal{F}_ℓ is on ℓ vertices.
- $\mathbb{R}\mathcal{F}$ formal linear combinations
- $\mathcal{K} := \text{span}(F - \sum_{F' \in \mathcal{F}_\ell} P(F, F')F')$
- \mathcal{A} is $\mathbb{R}\mathcal{F}$ factorized by \mathcal{K} .
- addition in \mathcal{A} comes from $\mathbb{R}\mathcal{F}$
- $F_1 \cdot F_2 = \sum_{F \in \mathcal{F}_\ell} P(F_1, F_2; F)F$,
- $F \in \mathcal{F}$ is called a *flag*.
- informally called unlabeled flags

CONVERGENT GRAPH SEQUENCE

Let \mathcal{F} denote the set of all graphs.

DEFINITION

A sequence of graphs $(G_n)_{n \in \mathbb{N}}$ is *convergent* if for every finite graph H , $\lim_{n \rightarrow \infty} P(H, G_n)$ exists.

Examples:

$$\lim_{n \rightarrow \infty} P(H, K_n) = \begin{cases} 1 & \text{if } H \cong K_m \\ 0 & \text{otherwise} \end{cases}$$

$$\lim_{n \rightarrow \infty} P(H, K_{n,n}) = \begin{cases} 1/2 & \text{if } H \cong K_2 \\ 3/4 & \text{if } H \cong P_2 \\ \vdots & \end{cases}$$

$$\lim_{n \rightarrow \infty} P(H, P_n) = \begin{cases} 1 & \text{is } |E(H)| = 0 \\ 0 & \text{otherwise} \end{cases}$$

Gives map $\mathcal{F} \rightarrow [0, 1]$ or a point in $[0, 1]^{\mathcal{F}}$.

CONVERGENT GRAPH SEQUENCE

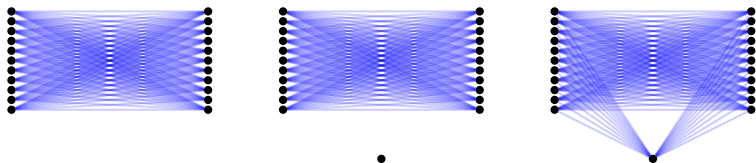
Let \mathcal{F} denote the set of all graphs.

DEFINITION

A sequence of graphs $(G_n)_{n \in \mathbb{N}}$ is *convergent* if for every finite graph H , $\lim_{n \rightarrow \infty} P(H, G_n)$ exists.

For all $H \in \mathcal{F}$

$$\lim_{n \rightarrow \infty} P(H, K_{n,n}) = \lim_{n \rightarrow \infty} P(H, K_{n,n} \cup K_1) = \lim_{n \rightarrow \infty} P(H, K_{n,n} + K_1)$$



In particular for $H \in \left\{ \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \end{array} , \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right\}$.

Small changes in $(G_n)_{n \in \mathbb{N}}$ are not noticeable.

POSITIVE HOMOMORPHISMS

Let $\text{Hom}(\mathcal{A}, \mathbb{R})$ be the set of all homomorphisms from \mathcal{A} to \mathbb{R} .
I.e. for any $\phi \in \text{Hom}(\mathcal{A}, \mathbb{R})$ and $a, b \in \mathcal{A}$:

- $\phi(a + b) = \phi(a) + \phi(b)$
- $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$.

Since $p(H, G) \in [0, 1]$, we consider only $\text{Hom}^+(\mathcal{A}, \mathbb{R})$, i.e.

$\phi(H) \geq 0$ for all $H \in \mathcal{F}$

Notice $\phi(\emptyset) = 1$.

THEOREM (RAZBOROV)

$\text{Hom}^+(\mathcal{A}, \mathbb{R})$ corresponds exactly to the convergent sequences.

THEOREM (FELIX; PODOLSKI; (ONLY FOR GRAPHS))

Let $a \in \mathbb{R}\mathcal{F}$. $\phi(a) = 0$ for all $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$ iff $a \in \mathcal{K}$.

THEOREM (MANTEL)

Every triangle-free graph on n vertices has at most $n^2/4$ edges.

THEOREM (MANTEL FROM PREVIOUS PRESENTATION)

For a large graph if $\phi(\triangle) = 0$ then $\phi(\text{edge}) \leq \frac{1}{2} + o(1)$.

THEOREM (MANTEL WITH HOMOMORPHISMS)

For every $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$ holds that

$$\text{if } \phi(\triangle) = 0 \text{ then } \phi(\text{edge}) \leq \frac{1}{2}.$$

Example from last time

$$0 \leq 3 \cdot \begin{array}{c} \bullet \\ \bullet \end{array} - \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \end{array} - \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \bullet \end{array} + 3 \cdot \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \bullet \end{array} + o(1)$$

We want to find $a \in \mathcal{A}$ such that for EVERY $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$ we have $\phi(a) \geq 0$. We write $a \geq 0$.

THEOREM (HATAMI AND NORIN 11)

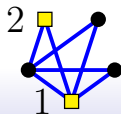
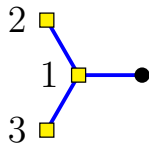
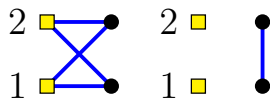
Determining if $a \geq 0$ is not algorithmically decidable.

Norin: “Extremal combinatorics remains an art”.

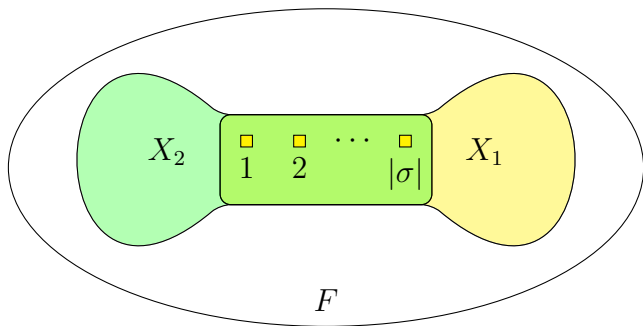
But we can still generate a lot of them!

ALGEBRA \mathcal{A}^σ

- vertices of $\sigma \in F$ are labeled by $1, \dots, |\sigma|$.
- \mathcal{F}^σ is the set of all graphs each containing a fixed induced labeled copy of σ .
- \mathcal{F}_ℓ^σ is on ℓ vertices.
- $\mathbb{R}\mathcal{F}^\sigma$ formal linear combinations
- $\mathcal{K}^\sigma := \text{span}(F - \sum_{F' \in \mathcal{F}_\ell^\sigma} P(F, F')F')$
- \mathcal{A}^σ is $\mathbb{R}\mathcal{F}^\sigma$ factorized by \mathcal{K}^σ .
- addition in \mathcal{A}^σ comes from $\mathbb{R}\mathcal{F}^\sigma$
- $F_1 \cdot F_2 = \sum_{F \in \mathcal{F}_\ell^\sigma} P(F_1, F_2; F)F$,
- $F \in \mathcal{F}^\sigma$ is called a σ -flag.
- σ is called a *type*



$$F_1 \cdot F_2 = \sum_{F \in \mathcal{F}_\ell^\sigma} P(F_1, F_2; F) F$$



Pick randomly $X_1, X_2 \subset V(F)$ such that $X_1 \cap X_2$ are exactly all labeled vertices and $|X_i| = |F_i|$.

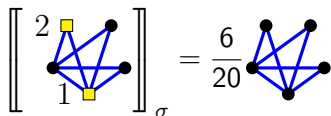
$P(F_1, F_2; F)$ is the probability that $X_1 \cong F_1$ and $X_2 \cong F_2$.

AVERAGING (UNLABELING) OPERATOR

Let $F = (G, \theta)$ be a σ -flag, where $\theta : 1, \dots, |\sigma| \rightarrow V(G)$. Define

$$[[F]]_{\sigma} = q_{\theta}(F) \cdot G,$$

where $q_{\theta}(F)$ is the probability that (G, θ') is isomorphic to F for a random injective $\theta' : 1, \dots, |\sigma| \rightarrow V(G)$.


$$\left[\begin{array}{c} 2 \\ \text{Graph} \\ 1 \end{array} \right]_{\sigma} = \frac{6}{20} \text{Graph}$$

Linear extension gives the *averaging operator*

$$[[\cdot]]_{\sigma} : \mathbb{R}\mathcal{F}^{\sigma} \rightarrow \mathbb{R}\mathcal{F}.$$

It is a linear mapping, *not* a homomorphism.

$$[[F_1 + F_2]]_{\sigma} = [[F_1]]_{\sigma} + [[F_2]]_{\sigma}$$

$$[[F_1 \cdot F_2]]_{\sigma} \text{ might NOT be } [[F_1]]_{\sigma} \cdot [[F_2]]_{\sigma}$$

\mathcal{A}^σ , \mathcal{A} AND Hom^+

For any $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$ and any $a \in \mathcal{A}^\sigma$

$$\phi(\llbracket a \cdot a \rrbracket_\sigma) \geq 0.$$

LEMMA (RAZBOROV)

Cauchy-Schwarz inequality for all $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$

$$\phi(\llbracket a^2 \rrbracket_\sigma \cdot \llbracket b^2 \rrbracket_\sigma) \geq \phi(\llbracket ab \rrbracket_\sigma^2).$$

Special cases:

$$\phi(\llbracket a^2 \rrbracket_\sigma) \geq \frac{\phi(\llbracket a \rrbracket_\sigma^2)}{\phi(\llbracket \sigma \rrbracket_\sigma)} \qquad \phi(\llbracket a^2 \rrbracket_\bullet) \geq \phi(\llbracket a \rrbracket_\bullet^2)$$

Uses special probability distribution on $\text{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$ related to ϕ .

Let σ and $\text{Hom}^+(\mathcal{A}, \mathbb{R})$ be fixed. This gives

Skip this. Hard to follow.

$$G_1, G_2, G_3, \dots$$

To make $\phi^\sigma \in \text{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$, we need a sequence of labeled graphs.

Incorrect: Take a copy of σ in each G_i and get a convergent

$$G_1^\sigma, G_2^\sigma, G_3^\sigma, \dots$$

Correct: Fix G_i . Randomly label a copy of σ and get G_i^σ . This gives $P(F, G_n^\sigma)$ for all $F \in \mathcal{F}^\sigma$.

By a random σ , we get a probability distribution $\mathbf{P}_{G_n}^\sigma$ on the functions $P(\cdot, G_n^\sigma)$.

These $\mathbf{P}_{G_n}^\sigma$ then weakly converge to a (unique) probability distribution \mathbf{P}_ϕ^σ on $\phi^\sigma \in \text{Hom}^+(\mathcal{A}^\sigma, \mathbb{R})$.

A crucial feature: if $a \in \mathcal{A}^\sigma$ and $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$, then

$$\phi([\sigma]_\sigma) \cdot \mathbb{E}_{\mathbf{P}_\phi^\sigma} [\phi^\sigma(a)] = \phi([a]_\sigma).$$

This can be viewed as an analogue of $P(B) \cdot P(A|B) = P(A \wedge B)$.

A crucial feature: if $a \in \mathcal{A}^\sigma$ and $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$, then

$$\phi(\llbracket \sigma \rrbracket_\sigma) \cdot \mathbb{E}_{\mathbf{P}_\phi^\sigma} [\phi^\sigma(a)] = \phi(\llbracket a \rrbracket_\sigma). \quad (1)$$

Bonus: For a given $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$, exists a unique probability distribution \mathbf{P}_ϕ^σ satisfying (1).

(1) is especially useful when $\phi^\sigma(a) \geq 0$ with probability one. It gives $\phi(\llbracket a \rrbracket_\sigma) \geq 0$.

In particular, for any $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$ and any $a \in \mathcal{A}^\sigma$

$$\phi(\llbracket a \cdot a \rrbracket_\sigma) \geq 0.$$

Simplified notation is just $\llbracket a \cdot a \rrbracket_\sigma \geq 0$.

MANTEL'S THEOREM AGAIN

For all $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$

σ is K_1 denoted by 1. Over triangle-free graphs i.e. $\phi(K_3) = 0$

$$\boxed{[a]_1^2 \leq [a^2]_1}$$

$$\phi\left(\left[\begin{array}{c} \bullet \\ | \\ 1 \square \\ \hline \end{array} \right]_1^2\right) \leq \phi\left(\left[\begin{array}{c} \bullet \\ | \\ 1 \square \\ \hline \end{array} \right]_1^2\right)$$

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$$\phi\left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array}^2\right) = \phi\left(\left[\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array}\right]_1^2\right) \leq \phi\left(\left[\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array}\right]_1^2\right) = \phi\left(\left[\begin{array}{c} \bullet & & \bullet \\ & \square & \\ \bullet & & \bullet \end{array}\right]_1\right)$$

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$$\phi\left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array}\right)^2 \leq \phi\left(\frac{1}{3} \begin{array}{c} \bullet & \bullet \\ \diagdown & / \\ \bullet & \bullet \end{array}\right)$$

MANTEL'S THEOREM AGAIN

$$\llbracket a \rrbracket_1^2 \leq \llbracket a^2 \rrbracket_1$$

For all $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$

σ is K_1 denoted by 1. Over triangle-free graphs i.e. $\phi(K_3) = 0$

$$\phi\left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array}^2\right) = \phi\left(\left[\begin{array}{c} \bullet \\ | \\ \square \end{array}\right]_1^2\right) \leq \phi\left(\left[\begin{array}{c} \bullet \\ | \\ \square \end{array}^2\right]_1\right) = \phi\left(\left[\begin{array}{c} \bullet & \bullet \\ \diagdown & / \\ \square \end{array}\right]_1\right)$$

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$$\phi\left(\frac{1}{3} \begin{array}{c} \bullet & \bullet \\ \cdot & \cdot \end{array} + \frac{2}{3} \begin{array}{c} \bullet & \bullet \\ \diagdown & / \\ \bullet \end{array}\right) = \phi\left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array}\right)$$

MANTEL'S THEOREM AGAIN

$$\llbracket a \rrbracket_1^2 \leq \llbracket a^2 \rrbracket_1$$

For all $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$

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$$\phi\left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array}^2\right) = \phi\left(\left[\begin{array}{c} \bullet \\ | \\ \square \end{array}\right]_1^2\right) \leq \phi\left(\left[\begin{array}{c} \bullet \\ | \\ \square \end{array}^2\right]_1\right) = \phi\left(\left[\begin{array}{c} \bullet & \bullet \\ \diagdown & / \\ \square \end{array}\right]_1\right)$$

$$\phi\left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array}\right)^2 \leq \phi\left(\frac{1}{3} \begin{array}{c} \bullet & \bullet \\ \diagdown & / \\ \bullet \end{array}\right)$$

$$\phi\left(\frac{2}{3} \begin{array}{c} \bullet & \bullet \\ \diagdown & / \\ \bullet \end{array}\right) \leq \phi\left(\frac{1}{3} \begin{array}{c} \bullet & \bullet \\ \diagdown & / \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet & \bullet \\ \diagdown & / \\ \bullet \end{array}\right) = \phi\left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array}\right)$$

MANTEL'S THEOREM AGAIN

$$\boxed{[a]_1^2 \leq [a^2]_1}$$

For all $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$

σ is K_1 denoted by 1. Over triangle-free graphs i.e. $\phi(K_3) = 0$

$$\phi\left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array}\right)^2 = \phi\left(\left[\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array}\right]_1^2\right) \leq \phi\left(\left[\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array}\right]_1^2\right) = \phi\left(\left[\begin{array}{c} \bullet & \bullet \\ \diagdown & / \\ \square \\ | \\ \bullet \end{array}\right]_1\right)$$

$$\phi\left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array}\right)^2 \leq \phi\left(\frac{1}{3} \begin{array}{c} \bullet & \bullet \\ \diagdown & / \\ \bullet \end{array}\right)$$

$$\phi\left(\frac{2}{3} \begin{array}{c} \bullet & \bullet \\ \diagdown & / \\ \bullet \end{array}\right) \leq \phi\left(\frac{1}{3} \begin{array}{c} \bullet & \bullet \\ \diagdown & / \\ \bullet \end{array}\right) + \frac{2}{3} \begin{array}{c} \bullet & \bullet \\ \diagdown & / \\ \bullet \end{array} = \phi\left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array}\right)$$

$$\phi\left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array}\right)^2 \leq \phi\left(\frac{1}{3} \begin{array}{c} \bullet & \bullet \\ \diagdown & / \\ \bullet \end{array}\right) \leq \frac{1}{2} \phi\left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array}\right) \quad \text{hence} \quad \phi\left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array}\right) \leq \frac{1}{2}$$

\mathcal{A} AND FORBIDDEN GRAPHS

In previous example, we used

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} = \frac{1}{3} \cdot \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \end{array} + \frac{2}{3} \cdot \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \\ \bullet \end{array}$$

rather than

$$\begin{array}{c} \bullet \\ | \\ \bullet \end{array} = \frac{1}{3} \cdot \begin{array}{c} \bullet \text{---} \bullet \\ \bullet \end{array} + \frac{2}{3} \cdot \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \\ \bullet \end{array} + 1 \cdot \begin{array}{c} \bullet \text{---} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array}$$

This can be done formally by defining \mathcal{A}^∇ over \mathcal{F}^∇ , which are all triangle free graphs. Works for any forbidden graphs.

Forbidding triangles can be done formally by defining \mathcal{A}^∇ over \mathcal{F}^∇ , which are all triangle free graphs. Works for any forbidden graphs.

THEOREM (MANTEL - HOMOMORPHISMS FROM \mathcal{A})

For every $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$ holds that

$$\text{if } \phi \left(\begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \right) = 0 \text{ then } \phi \left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right) \leq \frac{1}{2}.$$

THEOREM (MANTEL - HOMOMORPHISMS FROM \mathcal{A}^∇)

For every $\phi \in \text{Hom}^+(\mathcal{A}^\nabla, \mathbb{R})$ holds $\phi \left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right) \leq \frac{1}{2}$.

Formulation with \mathcal{A}^∇ is particularly useful in practical calculations if $|\mathcal{F}_\ell^\nabla| < |\mathcal{F}_\ell|$.

Forbidding triangles can be done formally by considering graphs which are all triangle free graphs. Work

THEOREM (MANTEL - HOMOMORPHISMS FROM \mathcal{A})

For every $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$ holds that

$$\text{if } \phi \left(\begin{array}{c} \bullet & & \bullet \\ & \diagdown & / \\ & \bullet & \\ & / & \diagdown \\ \bullet & & \bullet \end{array} \right) = 0 \text{ then } \phi$$

ℓ	$ \mathcal{F}_\ell $	$ \mathcal{F}_\ell^\nabla $
3	4	3
4	11	7
5	34	14
6	156	38
7	1,044	107
8	12,346	410
9	274,668	1,897
10	12,005,168	12,172
11	1,018,997,864	105,071

THEOREM (MANTEL - HOMOMORPHISMS FROM \mathcal{A}^∇)

For every $\phi \in \text{Hom}^+(\mathcal{A}^\nabla, \mathbb{R})$ holds $\phi \left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right) \leq \frac{1}{2}$.

Formulation with \mathcal{A}^∇ is particularly useful in practical calculations if $|\mathcal{F}_\ell^\nabla| < |\mathcal{F}_\ell|$.

Let $a \in \mathcal{A}^\sigma$. Recall $\llbracket a \cdot a \rrbracket_\sigma \geq 0$. Let $a = X^T v$, where $X \in (\mathcal{F}^\sigma)^m$ and $v \in \mathbb{R}^m$.

$$a = \begin{array}{c} \bullet \\ 1 \blacksquare \end{array} + 2 \begin{array}{c} \bullet \\ 1 \blacksquare \end{array} \quad X^T = \left(\begin{array}{c} \bullet \\ 1 \blacksquare \end{array}, \begin{array}{c} \bullet \\ 1 \blacksquare \end{array} \right) \quad v = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$a^2 = a \cdot a = X^T v \cdot (X^T v) = X^T v v^T X = X^T M X$$

$$\left(\begin{array}{c} \bullet \\ 1 \blacksquare \end{array} + 2 \begin{array}{c} \bullet \\ 1 \blacksquare \end{array} \right)^2 = \left(\begin{array}{c} \bullet \\ 1 \blacksquare \end{array}, \begin{array}{c} \bullet \\ 1 \blacksquare \end{array} \right) \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \left(\begin{array}{c} \bullet \\ 1 \blacksquare \end{array}, \begin{array}{c} \bullet \\ 1 \blacksquare \end{array} \right)^T$$

OBSERVATION

Let σ be fixed. If $M \succcurlyeq 0$ and X is a vector $(\mathcal{F}_\ell^\sigma)^n$, then

$\forall \phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$

$$\phi\left(\llbracket X^T M X \rrbracket_\sigma\right) \geq 0. \text{ Similar to } \phi\left(\llbracket \sum_{a \in \mathcal{A}^\sigma} a^2 \rrbracket_\sigma\right) \geq 0.$$

LAST TIME - DENSITY APPROACH

$$\begin{aligned}
 0 &\leq \frac{1}{n} \sum_v \left(\begin{array}{c} \bullet \\ | \\ \square_v \end{array}, \begin{array}{c} \bullet \\ | \\ \square_v \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ | \\ \square_v \end{array}, \begin{array}{c} \bullet \\ | \\ \square_v \end{array} \right)^T \\
 &= \frac{1}{n} \sum_v a \begin{array}{c} \bullet \quad ? \quad \bullet \\ \diagdown \quad \diagup \\ \square_v \end{array} + b \begin{array}{c} \bullet \quad ? \quad \bullet \\ \diagup \quad \diagdown \\ \square_v \end{array} + \frac{1}{2}c \begin{array}{c} \bullet \quad ? \quad \bullet \\ \diagdown \quad \diagup \\ \square_v \end{array} + \frac{1}{2}c \begin{array}{c} \bullet \quad ? \quad \bullet \\ \diagup \quad \diagdown \\ \square_v \end{array} \\
 &= a \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \square_v \end{array} + \frac{a+2c}{3} \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \square_v \end{array} + \frac{b+2c}{3} \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \square_v \end{array} + b \begin{array}{c} \bullet \quad \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \square_v \end{array}
 \end{aligned}$$

HOMOMORPHISM APPROACH

For all $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$

$$0 \leq \phi \left(\left[\left(\begin{array}{c|c} \bullet & \bullet \\ \hline 1_{\square} & 1_{\square} \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \begin{pmatrix} \bullet & \bullet \\ \hline 1_{\square} & 1_{\square} \end{pmatrix}^T \right]_{\sigma} \right)$$

HOMOMORPHISM APPROACH

For all $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$

$$\begin{aligned}
 0 &\leq \phi \left(\left[\left(\begin{array}{c} \bullet \\ | \\ \text{1} \square \end{array}, \begin{array}{c} \bullet \\ | \\ \text{1} \square \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ | \\ \text{1} \square \end{array}, \begin{array}{c} \bullet \\ | \\ \text{1} \square \end{array} \right)^T \right]_{\sigma} \right) \\
 &= \phi \left(\left[a \left(\begin{array}{c} \bullet \\ / \backslash \\ \text{1} \square \end{array} + \begin{array}{c} \bullet \backslash / \\ \text{1} \square \end{array} \right) + c \left(\begin{array}{c} \bullet \\ | \\ \text{1} \square \end{array} \bullet + \begin{array}{c} \bullet \bullet \\ | \quad | \\ \text{1} \square \end{array} \right) + b \left(\begin{array}{c} \bullet \bullet \\ | \quad | \\ \text{1} \square \end{array} + \begin{array}{c} \bullet \bullet \\ \text{---} \\ \text{1} \square \end{array} \right) \right]_{\sigma} \right)
 \end{aligned}$$

HOMOMORPHISM APPROACH

For all $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$

$$\begin{aligned}
 0 &\leq \phi \left(\left[\left(\begin{array}{c} \bullet \\ | \\ \text{1} \square \end{array}, \begin{array}{c} \bullet \\ | \\ \text{1} \square \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ | \\ \text{1} \square \end{array}, \begin{array}{c} \bullet \\ | \\ \text{1} \square \end{array} \right)^T \right]_{\sigma} \right) \\
 &= \phi \left(\left[a \left(\begin{array}{c} \bullet \\ / \backslash \\ \text{1} \square \end{array} + \begin{array}{c} \bullet \backslash / \\ \text{1} \square \end{array} \right) + c \left(\begin{array}{c} \bullet \\ | \\ \text{1} \square \end{array} \bullet + \begin{array}{c} \bullet \text{---} \bullet \\ | \quad | \\ \text{1} \square \end{array} \right) + b \left(\begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ \text{1} \square \end{array} + \begin{array}{c} \bullet \text{---} \bullet \\ | \quad | \\ \text{1} \square \end{array} \right) \right]_{\sigma} \right) \\
 &= \phi \left(a \left(\left[\begin{array}{c} \bullet \\ / \backslash \\ \text{1} \square \end{array} \right]_{\sigma} + \left[\begin{array}{c} \bullet \backslash / \\ \text{1} \square \end{array} \right]_{\sigma} \right) + c \left(\left[\begin{array}{c} \bullet \\ | \\ \text{1} \square \end{array} \right]_{\sigma} \bullet + \left[\begin{array}{c} \bullet \text{---} \bullet \\ | \quad | \\ \text{1} \square \end{array} \right]_{\sigma} \right) + b \left(\left[\begin{array}{c} \bullet \quad \bullet \\ | \quad | \\ \text{1} \square \end{array} \right]_{\sigma} + \left[\begin{array}{c} \bullet \text{---} \bullet \\ | \quad | \\ \text{1} \square \end{array} \right]_{\sigma} \right) \right)
 \end{aligned}$$

HOMOMORPHISM APPROACH

For all $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$

$$\begin{aligned}
 0 &\leq \phi \left(\left[\left(\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array}, \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array}, \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right)^T \right]_{\sigma} \right) \\
 &= \phi \left(\left[a \left(\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right) + c \left(\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right) + b \left(\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right) \right]_{\sigma} \right) \\
 &= \phi \left(a \left(\left[\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right]_{\sigma} + \left[\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right]_{\sigma} \right) + c \left(\left[\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right]_{\sigma} + \left[\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right]_{\sigma} \right) + b \left(\left[\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right]_{\sigma} + \left[\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right]_{\sigma} \right) \right) \\
 &= \phi \left(a \left(\frac{1}{3} \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right) + c \left(\frac{2}{3} \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right) + b \left(\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right) \right)
 \end{aligned}$$

HOMOMORPHISM APPROACH

For all $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$


$$\begin{aligned}
 0 &\leq \phi \left(\left[\left(\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array}, \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right) \begin{pmatrix} a & c \\ c & b \end{pmatrix} \left(\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array}, \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right)^T \right]_{\sigma} \right) \\
 &= \phi \left(\left[a \left(\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right) + c \left(\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right) + b \left(\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right) \right]_{\sigma} \right) \\
 &= \phi \left(a \left(\left[\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right]_{\sigma} + \left[\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right]_{\sigma} \right) + c \left(\left[\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right]_{\sigma} + \left[\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right]_{\sigma} \right) + b \left(\left[\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right]_{\sigma} + \left[\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right]_{\sigma} \right) \right) \\
 &= \phi \left(a \left(\frac{1}{3} \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} + \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right) + c \left(\frac{2}{3} \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} + \frac{2}{3} \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right) + b \left(\begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} + \frac{1}{3} \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right) \right) \\
 &= \phi \left(a \cdot \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} + \frac{a+2c}{3} \cdot \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} + \frac{b+2c}{3} \cdot \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} + b \cdot \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \begin{array}{c} \bullet \\ | \\ \square \\ | \\ \bullet \end{array} \right)
 \end{aligned}$$

TURÁN THEOREM

We do not write $\phi(\dots)$ any more but it is there!

TURÁN QUESTION

What is the maximum number of edges in K_t -free n -vertex graphs?

We seek answer as  $\leq \frac{t-2}{t-1}$.

Computer can do at most $t = 8$.

We do by hand induction on t .

Main tool

$$[[a^2]]_\sigma \geq \frac{[[a]]_\sigma^2}{[[\sigma]]_\sigma}.$$

Our first goal is to prove

$$\img alt="Diagram of a triangle with three vertices" data-bbox="325 775 380 850"/> $\geq 2 \cdot \img alt="Diagram of a vertical edge with two vertices" data-bbox="455 770 475 855"/> \cdot \left(\img alt="Diagram of a vertical edge with two vertices" data-bbox="555 770 575 855"/> - \frac{1}{2} \right)$$$

Fill the rest of the following

$$\boxed{[a^2]_1 \geq [a]_1^2}$$

$$2 \left[\left[\begin{array}{c} \text{triangle} \\ 1 \end{array} \right] + \left[\begin{array}{c} \text{V-shape} \\ 1 \end{array} \right] \right]_1 = 2 \left[\left[\begin{array}{c} \text{edge} \\ 1 \end{array} \right]^2 \right]_1 \geq 2 \left[\left[\begin{array}{c} \text{edge} \\ 1 \end{array} \right] \right]_1^2 = 2 \left[\begin{array}{c} \text{edge} \\ 1 \end{array} \right]^2$$

$$2 \left[\begin{array}{c} \text{triangle} \\ 1 \end{array} \right] + \frac{2}{3} \left[\begin{array}{c} \text{V-shape} \\ 1 \end{array} \right] \geq 2 \left[\begin{array}{c} \text{edge} \\ 1 \end{array} \right]^2$$

Fill the rest of the following

$$\boxed{[a^2]_1 \geq [a]_1^2}$$

$$2 \left[\left[\begin{array}{c} \triangle \\ 1 \end{array} \right] + \left[\begin{array}{c} \vee \\ 1 \end{array} \right] \right]_1 = 2 \left[\left[\begin{array}{c} \bullet \\ | \\ 1 \end{array} \right]^2 \right]_1 \geq 2 \left[\left[\begin{array}{c} \bullet \\ | \\ 1 \end{array} \right] \right]_1^2 = 2 \left[\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right]^2$$

$$2 \left[\begin{array}{c} \triangle \\ \bullet \end{array} \right] + \frac{2}{3} \left[\begin{array}{c} \vee \\ \bullet \end{array} \right] \geq 2 \left[\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right]^2$$

Recall

$$\left[\begin{array}{c} \triangle \\ \bullet \end{array} \right] + \frac{2}{3} \left[\begin{array}{c} \vee \\ \bullet \end{array} \right] + \frac{1}{3} \left[\begin{array}{c} \bullet \\ \bullet \end{array} \right] = \left[\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right]$$

By subtracting, we obtain

$$\left[\begin{array}{c} \triangle \\ \bullet \end{array} \right] \geq \left[\begin{array}{c} \triangle \\ \bullet \end{array} \right] - \frac{1}{3} \left[\begin{array}{c} \bullet \\ \bullet \end{array} \right] \geq 2 \left[\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right]^2 - \left[\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right] = 2 \left[\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right] \cdot \left(\left[\begin{array}{c} \bullet \\ | \\ \bullet \end{array} \right] - \frac{1}{2} \right)$$

We got the base of induction.

$$K_3 \geq 2 K_2 \cdot \left(K_2 - \frac{1}{2} \right) \quad (2)$$

The real goal is

$$K_{t+1} \geq t \cdot K_t \cdot \left(K_t - \frac{t-1}{t} \right). \quad (3)$$

Inequality (2) is just (3) with $t = 2$.

Why is (3) *proving* the Turán's theorem?

We got the base of induction.

$$\begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \geq 2 \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \cdot \left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} - \frac{1}{2} \right) \quad (2)$$

The real goal is

$$K_{t+1} \geq t \cdot K_t \cdot \left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} - \frac{t-1}{t} \right). \quad (3)$$

Inequality (2) is just (3) with $t = 2$.

Why is (3) *proving* the Turán's theorem?



Let $(G_n)_{n \in \mathbb{N}}$ is K_{t+1} free sequence with $\phi \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$.

Then $\phi(K_{t+1}) = 0$. This gives in (3) that

$$0 \geq t \cdot \phi(K_t) \cdot \left(\phi(K_2) - \frac{t-1}{t} \right).$$

If $\phi(K_t) > 0$ then $\phi(K_2) \leq \frac{t-1}{t}$.

If $\phi(K_t) = 0$ induction on t gives $\phi(K_2) \leq \frac{t-2}{t-1}$.

Induction step for $t = 4$. Goal is  $\geq 3 \cdot$  $\cdot \left(\begin{array}{c} | \\ | \\ | \end{array} - \frac{2}{3} \right)$

$$\left[\begin{array}{c} 2 \\ 1 \end{array} \begin{array}{c} \text{square with diagonals} \\ \text{square with diagonals} \end{array} \right]_{\sigma} = \left[\begin{array}{c} 2 \\ 1 \end{array} \begin{array}{c} \text{triangle} \\ \text{triangle} \end{array} \right]_{\sigma} \geq \frac{\left[\begin{array}{c} 2 \\ 1 \end{array} \begin{array}{c} \text{triangle} \\ \text{triangle} \end{array} \right]_{\sigma}^2}{\begin{array}{c} | \\ | \\ | \end{array}}$$

$$\begin{array}{c} \text{square with diagonals} \\ + \frac{1}{6} \text{square with diagonals} \end{array} \geq \frac{\begin{array}{c} \text{triangle} \\ \text{triangle} \end{array}^2}{\begin{array}{c} | \\ | \\ | \end{array}}$$

$$\boxed{[a^2]_{\sigma} \geq \frac{[a]_{\sigma}^2}{[\sigma]_{\sigma}}}$$

$$\begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{1}{6} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \geq \frac{\begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array}^2}{\begin{array}{c} \bullet \\ | \\ \bullet \end{array}} \text{ and } \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \geq 2 \begin{array}{c} \bullet \\ | \\ \bullet \end{array} \cdot \left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} - \frac{1}{2} \right)$$

When we use induction, which means (2) and we get

$$\frac{\begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array}^2}{\begin{array}{c} \bullet \\ | \\ \bullet \end{array}} = \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \cdot \frac{\begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array}}{\begin{array}{c} \bullet \\ | \\ \bullet \end{array}} \geq \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \cdot 2 \left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} - \frac{1}{2} \right).$$

Hence we get after scaling by $\frac{3}{2}$

$$\frac{3}{2} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} + \frac{1}{4} \begin{array}{c} \bullet \\ \diagup \quad \diagdown \\ \bullet \quad \bullet \\ \diagdown \quad \diagup \\ \bullet \end{array} \geq \begin{array}{c} \bullet \\ \diagdown \quad \diagup \\ \bullet \quad \bullet \\ \diagup \quad \diagdown \\ \bullet \end{array} \cdot 3 \left(\begin{array}{c} \bullet \\ | \\ \bullet \end{array} - \frac{1}{2} \right)$$

$$\frac{3}{2} \text{K}_4 + \frac{1}{4} \text{K}_4 \geq \text{K}_3 \cdot 3 \left(\text{K}_2 - \frac{1}{2} \right)$$

Add $\frac{1}{2}$ times the following

$$\text{K}_3 = \text{K}_4 + \frac{1}{2} \text{K}_4 + \dots \geq \text{K}_4 + \frac{1}{2} \text{K}_4$$

and get

$$\text{K}_4 \geq \text{K}_3 \cdot 3 \left(\text{K}_2 - \frac{1}{2} \right) - \frac{1}{2} \text{K}_3 = \text{K}_3 \cdot 3 \left(\text{K}_2 - \frac{2}{3} \right)$$

In a similar way we get also

$$K_{t+1} \geq t \cdot K_t \cdot \left(\text{K}_2 - \frac{t-1}{t} \right)$$

ACTUAL PROOF OF MANTEL'S THEOREM

THEOREM (MANTEL FOR ALL n)

If G is a triangle free graph on n vertices, then $|E(G)| \leq n^2/4$.

For contradiction let H be a triangle-free graph on n_0 vertices, with $|E(H)| = cn_0^2$, where $c > \frac{1}{4}$.

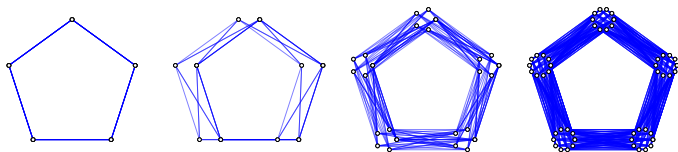
ACTUAL PROOF OF MANTEL'S THEOREM

THEOREM (MANTEL FOR ALL n)

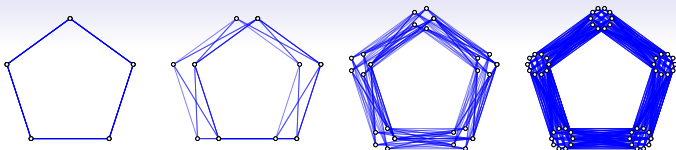
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For contradiction let H be a triangle-free graph on n_0 vertices, with $|E(H)| = cn_0^2$, where $c > \frac{1}{4}$.

Let $B(H, k)$ be a blow-up of H by k vertices.



$(B(H, k))_{k \in \mathbb{N}}$ is convergent, gives $\phi_H \in \text{Hom}^+(\mathcal{A}, \mathbb{R})$.



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OBSERVATION

$B(H, k)$ is a triangle-free graph.

Hence $\phi_H(K_3) = 0$

$|E(B(H, k))| = cn^2 k^2$. Hence

$$\phi_H(K_2) = \lim_{k \rightarrow \infty} \frac{|E(B(H, k))|}{\binom{|V(B(H, k))|}{2}} = \lim_{k \rightarrow \infty} \frac{cn_0^2 k^2}{\binom{n_0 k}{2}} = 2c > \frac{1}{2}$$

Before, we proved that $\phi(K_2) \leq \frac{1}{2}$ whenever $\phi(K_3) = 0$, contradiction.

Turán's theorem can be proved with the same trick.

SEE YOU IN THE AFTERNOON! (MAYBE)

- Flag Algebras “definitions”
- First try for Mantel’s theorem
- More automatic approach
- Additional constraints
- maybe break
- Define flag algebras
- Graph sequences and homomorphisms
- Turán’s Theorem (in limit)
- Finally Mantel’s Theorem (for real)
- mega break
- Applications