Flag Algebra Methods (some applications)

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Flag algebras

Seminal paper:
Razborov, Flag Algebras, *Journal of Symbolic Logic* **72** (2007), 1239–1282.
David P. Robbins Prize by AMS for Razborov in 2013

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Example (Goodman, Razborov)

If density of edges is at least $\rho > 0$, what is the minimum density of triangles?

- designed to attack extremal problems.
- works well if constraints as well as desired value can be computed by checking small subgraphs (or average over small subgraphs)
- the results are in limit (very large graphs)

Applications (early incomplete list)

Author	Year	Application/Result
Razborov	2008	edge density vs. triangle density
Hladký, Kráľ, Norin	2009	Bounds for the Caccetta-Haggvist conjectu
Razborov	2010	On 3-hypergraphs with forbidden 4-vertex co
Hatami, Hladký,Kráľ,Norin, Razborov / Grzesik	2011	Erdős Pentagon problem
Hatami, Hladký, Kráľ, Norin, Razborov	2012	Non-Three-Colourable Common Graphs Exis
Balogh, Hu, L., Liu / Baber	2012	4-cycles in hypercubes
Das, Huang, Ma, Naves, Sudakov	2013	minimum number of k-cliques
Baber, Talbot	2013	A Solution to the 2/3 Conjecture
Falgas-Ravry, Vaughan	2013	Turán density of many 3-graphs
Cummings, Kráľ, Pfender, Sperfeld, Treglown, Young	2013	Monochromatic triangles in 3-edge colored
Kramer, Martin, Young	2013	Boolean lattice
Balogh, Hu, L., Pikhurko, Udvari, Volec	2013	Monotone permutations
Norin, Zwols	2013	New bound on Zarankiewicz's conjecture
Huang, Linial, Naves, Peled, Sudakov	2014	3-local profiles of graphs
Balogh, Hu, L., Pfender, Volec, Young	2014	Rainbow triangles in 3-edge colored graphs
Balogh, Hu, L., Pfender	2014	Induced density of C_5
Goaoc, Hubard, de Verclos, Séréni, Volec	2014	Order type and density of convex subsets
Coregliano, Razborov	2015	Tournaments
Alon, Naves, Sudakov	2015	Phylogenetic trees
...	...	...

Applications to graphs, oriented graphs, hypergraphs, hypercubes, permutations, crossing number of graphs, order types, discrete geometry, Ramsey numbers, phylogenetic trees...

Not everything is a good fit for flag algebras.

- Hamiltonicity
- Connectivity
- Chromatic number
- Lower order terms
- Sparse graphs
  - planar graphs
  - saturation problems
  - $ex(n, C_4)$, $ex(n, K_{t,t})$, ...

The following application are NOT the most impressive ones!

They are just easier to talk about for me.
(And I already had many of the slides)

We ignore things as Caccetta-Häggkvist and Turán for 3-uniform hypergrpahs

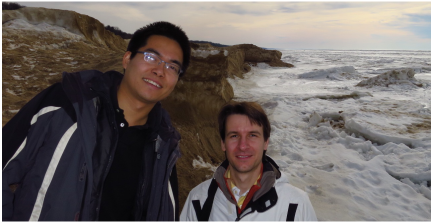
$$\frac{5}{9}\binom{n}{3} \leq ex(n, K_4^{(3)}) \leq 0.5615\binom{n}{3} \text{ by Baber}$$

Flagmatic

J. Balogh

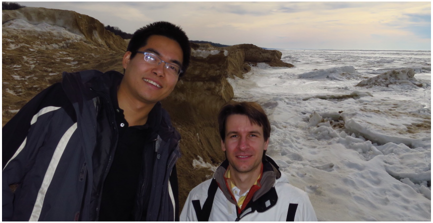
P. Hu

Hypercubes and posets



H. Liu B. L.

Application to sparse structure.



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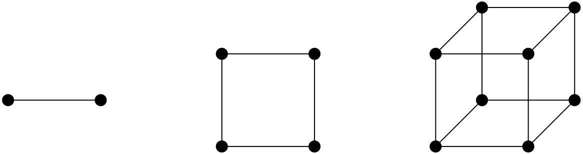
Application to sparse structure.

Hypercube

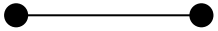
$\mathcal{Q}_n$ is *n*-dimensional hypercube (*n*-cube)



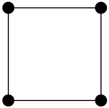
$\mathcal{Q}_1$ $\mathcal{Q}_2$ $\mathcal{Q}_3$

Hypercube

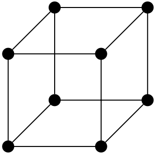
$\mathcal{Q}_n$ is n-dimensional hypercube (n-cube)



$\mathcal{Q}_1$



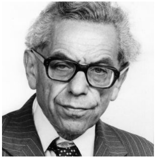
$\mathcal{Q}_2$



$\mathcal{Q}_3$

Problem (Erdős 1984)

What is the maximum number of edges in a subgraph of $\mathcal{Q}_n$ with no $\mathcal{Q}_2$?



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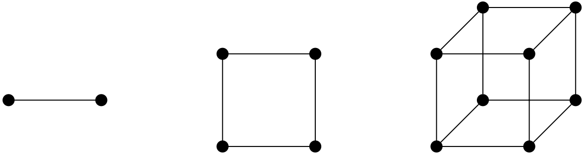

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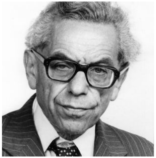

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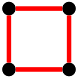
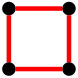
maximize  subject to  $= 0$

Lower bound

Conjecture (Erdős 1984)

In $\mathcal{Q}_n$ where $n \to \infty$:

 $= 0 \Rightarrow$  $\leq \frac{1}{2}.$

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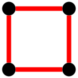
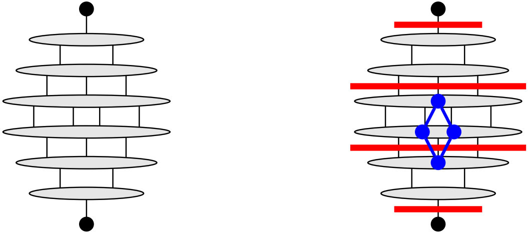
 $= 0 \Rightarrow$  $\leq \frac{1}{2}$.

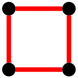
As posets



$\mathcal{Q}_7$ $\mathcal{Q}_7$

By removing every second layer,  $\geq 1/2$.

Results about hypercubes

If  $= 0$ then

Theorem (Chung 1992)

 $\leq 0.62284.$

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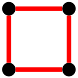
If  $= 0$ then

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Theorem (Thomason and Wagner 2009)

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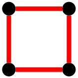
If  $= 0$ then

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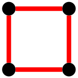
If  $= 0$ then

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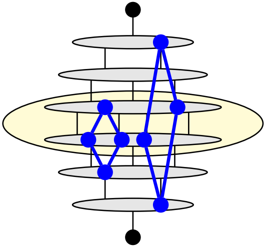
Theorem (Balogh, Hu, L., Liu 2014; Baber)

 ≤ 0.6068.

 ≤ 0.60318 by Baber

Let $\mathcal{B}_n$ denote n-dimensional boolean lattice.
Let F be a subposet of $\mathcal{B}_n$ not containing $\Diamond$.



$\mathcal{B}_7$

Related results - boolean lattice

Let $\mathcal{B}_n$ denote n-dimensional boolean lattice.
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Theorem

$|F| \leq (c + o(1))\binom{n}{\lfloor n/2 \rfloor}$, *where*
$c \leq 2.3$ *[Griggs, Lu 2009]*
$c \leq 2.284$ *[Axenovich, Manske, Martin 2012]*
$c \leq 2.273$ *[Griggs, Li, Lu 2011]*
$c \leq 2.25$ *[Kramer, Martin, Young 2013] FA*
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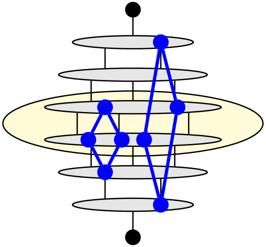
$c \leq 2.20711$ *[Grósz, Methuku, Tompkins 2017]*



$\mathcal{B}_7$

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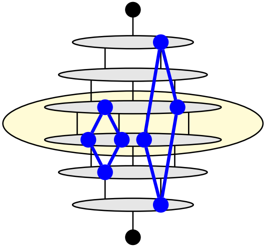

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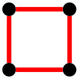
If F is a subposet of only the middle three layers of B_n, then
$c \leq 2.1547$ [Manske, Shen 2013]
$c \leq 2.15121$ [Balogh, Hu, L., Liu 2014] FA

How is it done?

Hypercubes are sparse structures, so

 $= 0$

if one just takes random 4 vertices from $\mathcal{Q}_n$ as $n \to \infty$.

Trick: Pick a vertex as $[0,1]^n$ and then pick randomly d coordinates. This samples a small sub-hypercube.

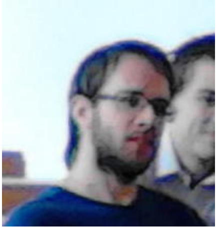
J. Balogh

P. Hu

B. L.

Permutations

O. Pikhurko



B. Udvari

J. Volec

Application with exact result.

Problem

What is the minimum number of monotone subsequences of size k in a permutation of $[n]$?

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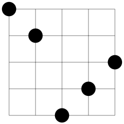
k = 3
n = 5



(5,4,1,2,3)

(5,4,1),(5,4,2),(5,4,3)
(1,2,3)

PROBLEM

What is the minimum number of monotone subsequences of size k in a permutation of $[n]$?

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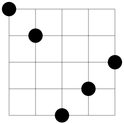

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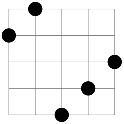


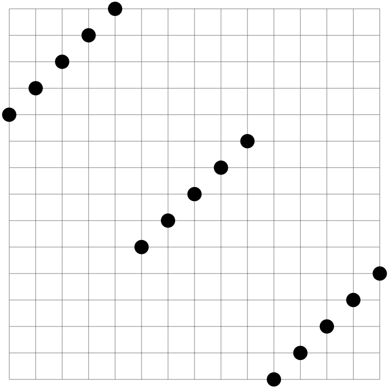
(4,5,1,2,3)

(1,2,3)

Conjecture

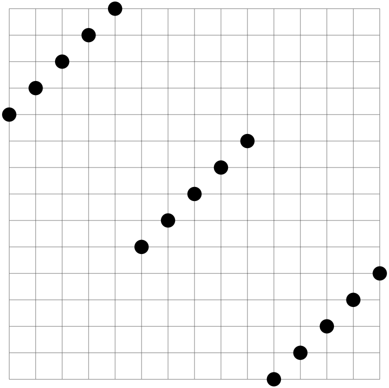
Conjecture (Myers 2002)

The number of monotone subsequences of length k is minimized by a permutation on $[n]$ with $k-1$ increasing runs of as equal lengths as possible.



$k = 4, n = 15$

Extremal case is not unique



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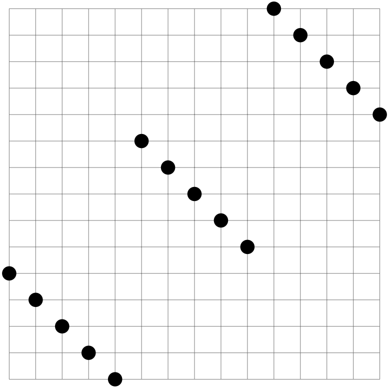

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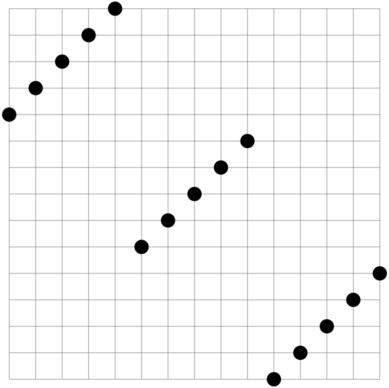

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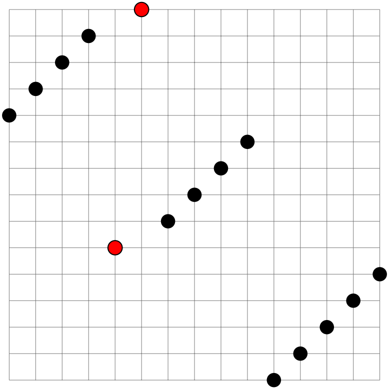

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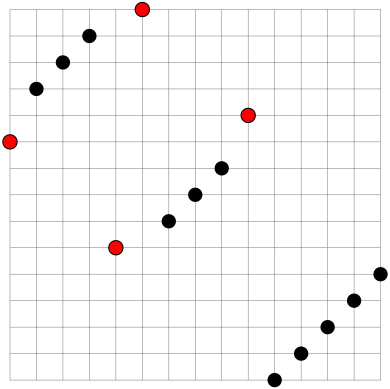

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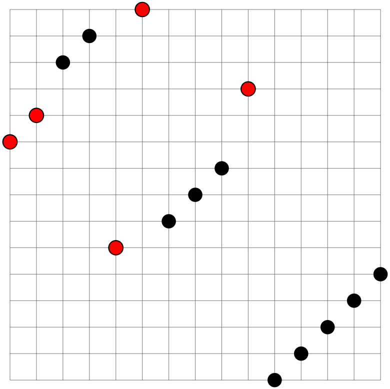

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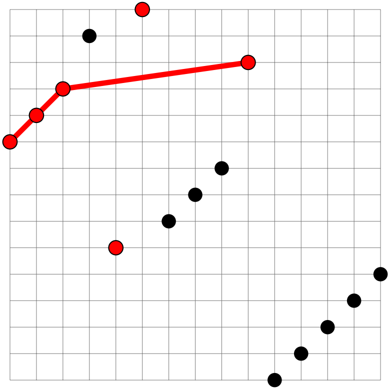

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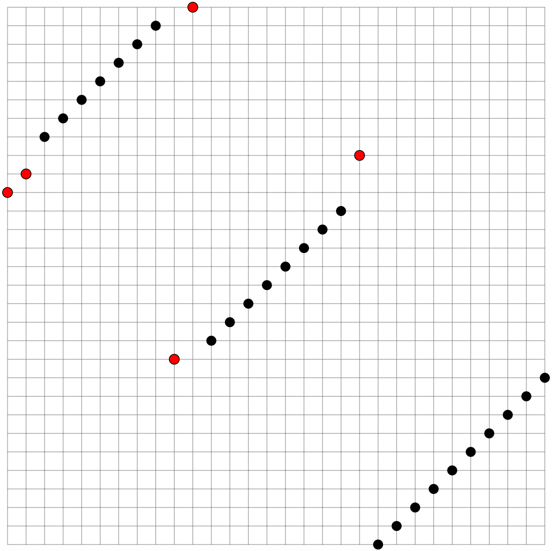


Conjecture (Myers 2002)

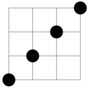
The number of monotone subsequences of length k is minimized by a permutation on $[n]$ with $k-1$ increasing runs of as equal lengths as possible.

Theorem (Samotij, Sudakov '15)

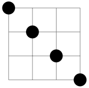
Myers' conjecture is true for sufficiently large k and $n \leq k^2 + ck^{3/2} \log k$, where c is an absolute positive constant.

Theorem (Balogh, Hu, **L.**, Pikhurko, Udvari, Volec '15)

Myers' conjecture is true for $k = 4$ and n sufficiently large.



(1,2,3,4)



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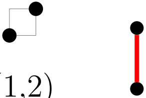
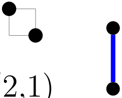
(1,2,3,4) (4,3,2,1)

Use of flag algebras, $k = 5, 6$ also doable, 7 not.

From permutations to permutation graphs



(1,2)



(2,1)

From permutations to permutation graphs

(1,2)

(2,1)

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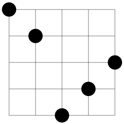
n = 5



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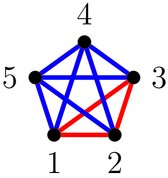
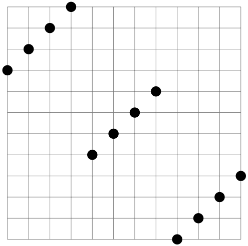
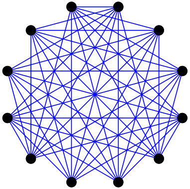
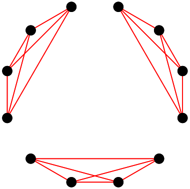
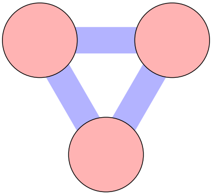


Extremal example ($k = 4$)

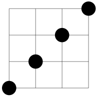




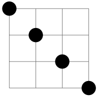




As flag algebras question $(k = 4)$



(1,2,3,4)



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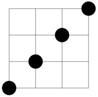

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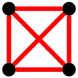
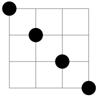
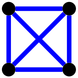
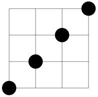




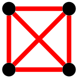
(4,3,2,1)

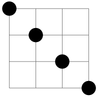


As flag algebras question ($k = 4$)



(1,2,3,4)





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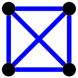
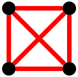
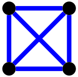


minimize  $+$ 

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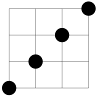

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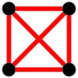

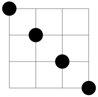

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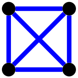
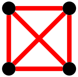
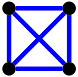
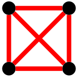
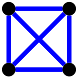


minimize  $+$ 

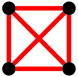
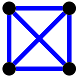
Theorem (Balogh, Hu, **L.**, Pikhurko, Udvari, Volec '15)

 $+$  $\geq \frac{1}{27}$

for every permutation graph.

Only for permutation graphs

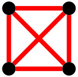
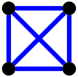
Theorem (Balogh, Hu, **L.**, Pikhurko, Udvari, Volec '15)

$\min \Big($  $+$  $\Big) = \frac{1}{27}$

over permutation graphs (and extremal permutations described using Myers' results - stability arguments).

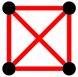
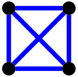
Only for permutation graphs

Theorem (Balogh, Hu, **L.**, Pikhurko, Udvari, Volec '15)

min ( + ) $= \frac{1}{27}$

over permutation graphs (and extremal permutations described using Myers' results - stability arguments).

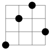
Theorem (Sperfeld '12; Thomason '89)

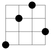
$\frac{1}{35} <$ min ( + ) $< \frac{1}{33}$

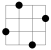
over **all** *sufficiently large 2-edge-colored complete graphs.*

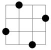
Other permutations - maximizing 1342 and 2413

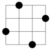
$d(X)$ denotes the maximum packing density of a permutation X.

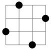
$0.19657 \leq d($$) \leq 2/9 = 0.22222\ldots$ AAHHS

$d($$) \leq 0.1988373$ BHLPUV

$51/511 = 0.0998\ldots \leq d($$) \leq 2/9 = 0.22222$ AAHHS

$0.1024732 \leq d($$)$ P

$0.10472\ldots \leq d($$)$ PS

$d($$) \leq 0.1047805$ BHLPUV

AAHHS . . . Albert, Atkinson, Handley, Holton, Stromquist 2002

P. . . Presutti 2008

PS. . . Presutti, Stromquist 2010

BHLPUV. . . FA 2015

Many other patterns investigated by Sliačan and Stromquist 2018.

How is it done?

For some permutations, one can use permutation graphs.

For other permutations, flag algebras can be developed on linearly ordered vertices.

Crossing numbers

S. Norin

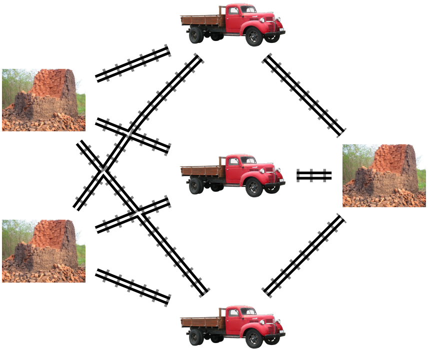
Application to graph drawing.

Crossing number

Turán 1945: In a forced labor camp, prisoners transfer carts of bricks from kilns to shipping yards.

When two tracks cross, cart is likely to derail.

How to connect every kiln and shipping yard that minimizes the number of crossings?



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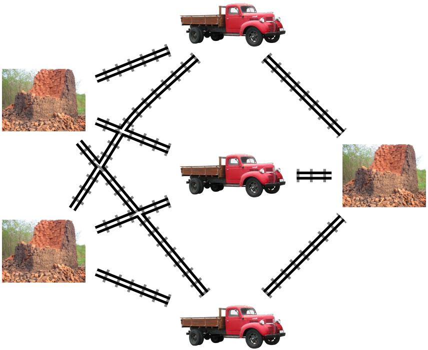


$K_{m,n}$ is a complete bipartite graph with sizes m and n, $K_{3,3}$ is above.

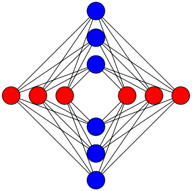
For a graph G, $cr(G)$ is the *crossing number*.

Conjecture (Zarankiewicz 1954)

$$cr(K_{m,n}) = \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{(n-1)}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{(m-1)}{2} \right\rfloor.$$

Theorem (Norin, Zwols 2013+)

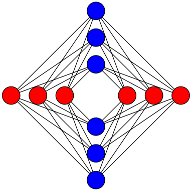
$$cr(K_{m,n}) \geq 0.9 \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{(n-1)}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{(m-1)}{2} \right\rfloor$$



$K_{6,6}$

for large m and n.
(Zarankiewicz's conjecture is 90% true)
80% Kleitman 1970
83% De Klerk, Maharry, Pasechnik, Richter, Salazar 2006
85.9% De Klerk, Pasechnik, Schrijver 2007

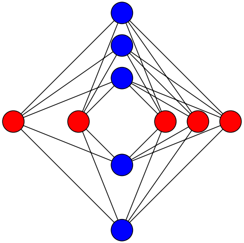
Rectilinear crossing number is with straight line drawing.



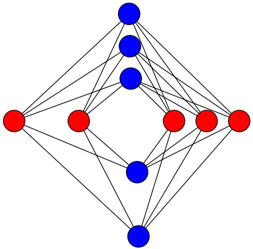
Theorem (Balogh, L., Norin, Pfender, Salazar)

Rectilinear version of the Zarankiewicz is 97.3% true for large m and n.

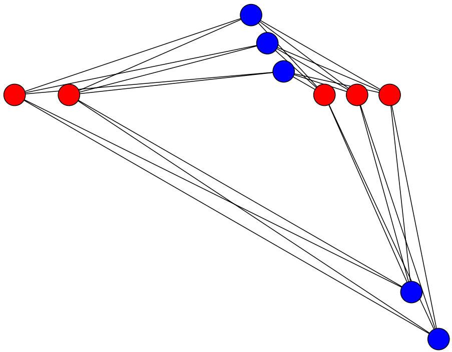
$K_{5,5,5}$



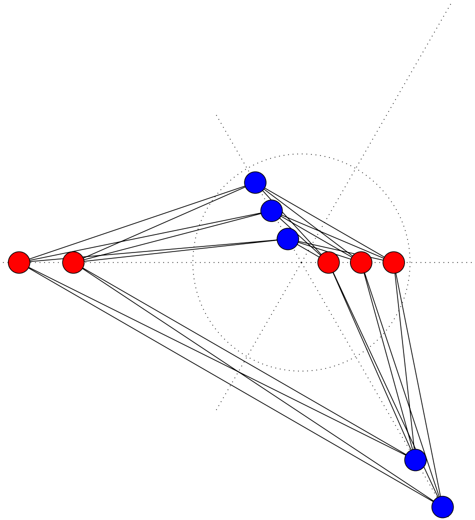
$K_{5,5,5}$



$K_{5,5,5}$

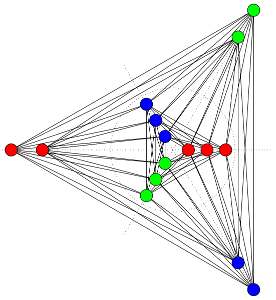


$K_{5,5,5}$



For a graph G, $\overline{cr}(G)$ is the rectilinear crossing number.

Conjecture

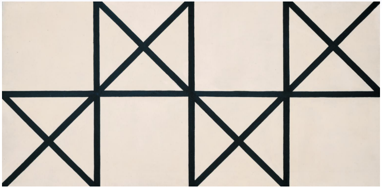
$\overline{cr}(K_{n_1,n_2,n_3})$ is minimized by  $K_{5,5,5}$

Theorem (Gethner, Hogben, L., Pfender, Ruiz, Young, '17)

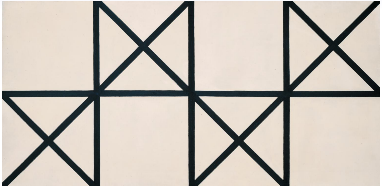
$\overline{cr}(K_{n_1,n_2,n_3})$ conjecture is 97.3% true for large n_1, n_2, and n_3.

Problem

What about partite graphs with more parts?



Anthony Hill
Orthogonal / Diagonal
Composition
1954



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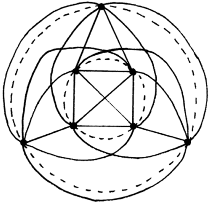
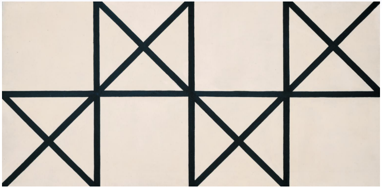


Hill considered crossing
number of complete graphs.



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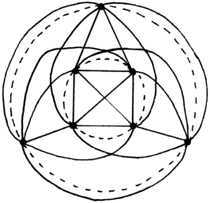

Hill considered crossing number of complete graphs.

Conjecture (Hill 1962)

$$cr(K_n) = \frac{1}{4}\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$$

Conjecture (Hill 1962)

$cr(K_n) = \frac{1}{4} \left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{n-2}{2} \right\rfloor \left\lfloor \frac{n-3}{2} \right\rfloor$

Conjecture is true

- if $n \leq 12$.
- 100% with various additional restrictions on the drawing
- 80% Kleitman 1970
- 83% De Klerk, Maharry, Pasechnik, Richter, Salazar 2006
- 85.9% De Klerk, Pasechnik, Schrijver 2007
- 90.5% Norin, Zwols 2013

Conjecture (Hill 1962)

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Theorem (Balogh, L., Salazar)

Conjecture is 98.5% true.

How is it done?

For multipartite graphs

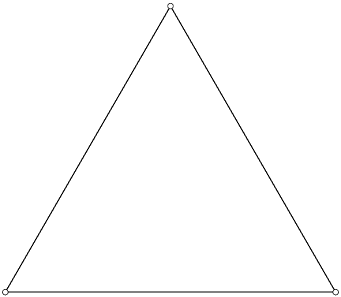
- Color vertices to indicate parts.
- For every $\{(a, b), (c, d)\}$, where a, b, c, d are vertices remember if edges ab and cd cross or not.
- Necessary to generate all (combinatorial) embeddings of graphs on n vertices.

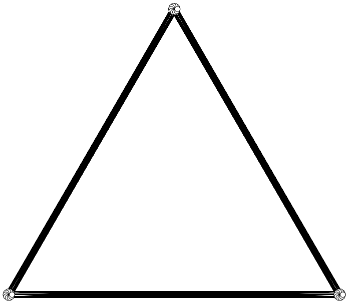
For complete graphs

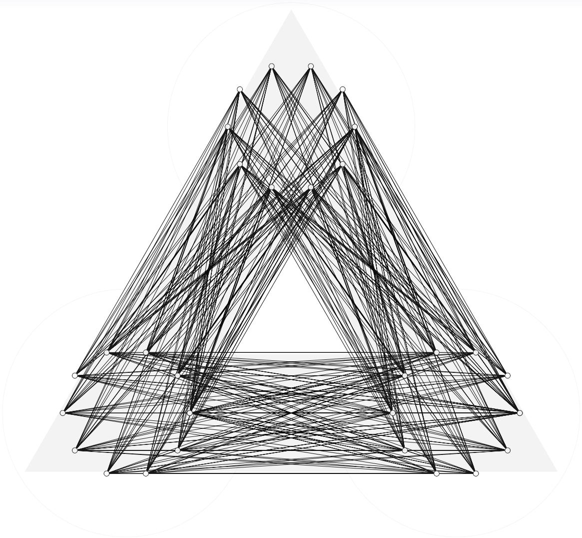
- For every vertex remember clockwise order of its neighbors.
- Necessary to generate all (combinatorial) embeddings of graphs on n vertices.

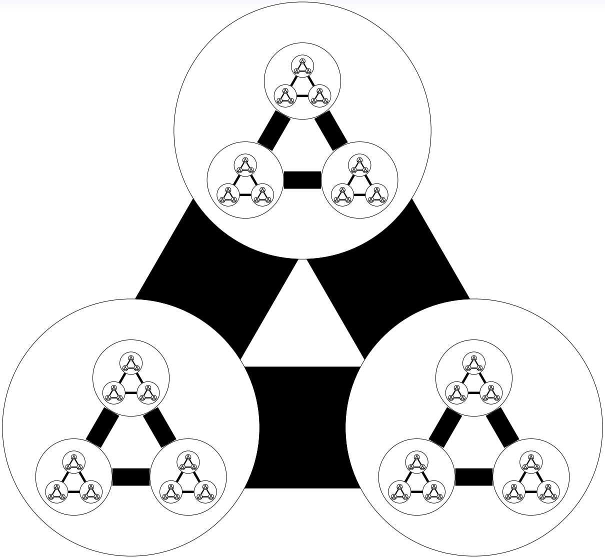
Fractalizers

Application with exact result and iterated extremal construction.



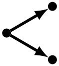
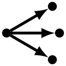
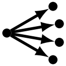
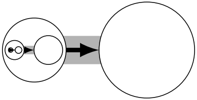






Results with iterated constructions

Theorem (Falgas-Ravry, Vaughan 2012)

Density of ,  *and*  *is maximized by* .

Results with iterated constructions

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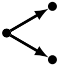
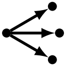
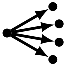
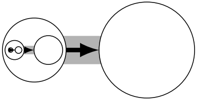
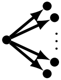
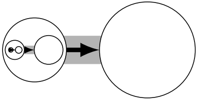
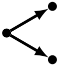
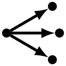
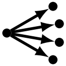
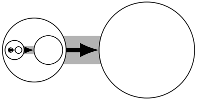
Density of *,*  *and*  *is maximized by* .

Theorem (Huang 2014)

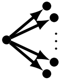
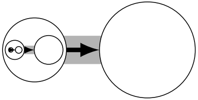
Density of  *is maximized by* .

Results with iterated constructions

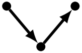
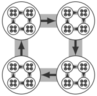
Theorem (Falgas-Ravry, Vaughan 2012)

Density of  ,  *and*  *is maximized by*  .

Theorem (Huang 2014)

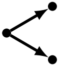
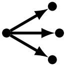
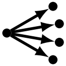
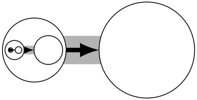
Density of  *is maximized by*  .

Theorem (Hladký, Král', Norin)

Density of  *is maximized by*  .

Results with iterated constructions

Theorem (Falgas-Ravry, Vaughan 2012)

Density of  *,*  *and*  *is maximized by*  *.*

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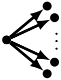
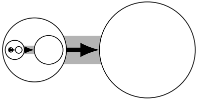
Density of  *is maximized by*  *.*

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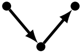
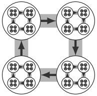
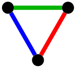
Density of  *is maximized by*  *.*

Theorem (Pikhurko 2014)

Iterated blow-up of any graph is extremal for $\pi(\mathcal{F})$ *for some family* $\mathcal{F}$.

PROBLEM

$F(n) :=$ max  over all 3-edge-colorings of K_n

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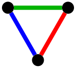
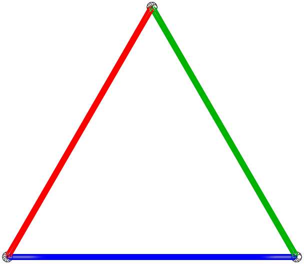
$F(n) :=$ max  over all 3-edge-colorings of K_n



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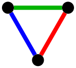
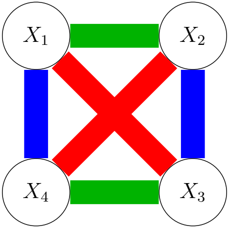
$F(n) :=$ max  over all 3-edge-colorings of K_n



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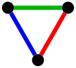
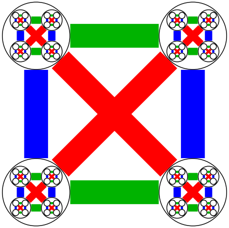
$F(n) :=$ max  over all 3-edge-colorings of K_n



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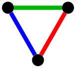
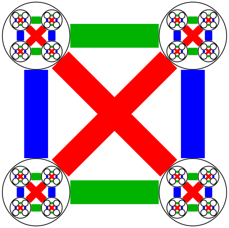
$F(n) :=$ max  over all 3-edge-colorings of K_n



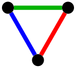
CONJECTURE (ERDŐS, SÓS 1972-)

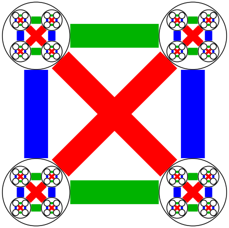
This construction is the best possible. In other words,

$$F(n) = x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 + \sum_i F(x_i),$$

where $x_1 + x_2 + x_3 + x_4 = n$, *and* $|x_i - x_j| \leq 1$.

PROBLEM

$F(n) :=$ max  over all 3-edge-colorings of K_n



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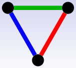
Our result: The conjecture is true for n large and for any $n = 4^k$.

$F(n) :=$ max  over all 3-edge-colorings of K_n

Theorem (Balogh, Hu, L., Pfender, Volec, Young 2017)

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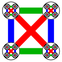
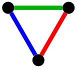
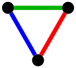
where $x_1 + x_2 + x_3 + x_4 = n$, *and* $|x_i - x_j| \leq 1$ *and* n *is large or* $n = 4^k$.

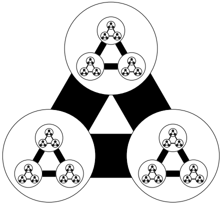
Construction  :  ≥ 0.4

FA:  ≤ 0.40006

Usual stability approach with excluded subgraphs does not work (nothing is excluded). Not having tight result from FA is typical if the extremal construction is iterated.

Use dense structures!

A graph G is a *fractalizer* if the graph maximizing the number of copies of G is an iterated blow-up of G.



Theorem (Fox, Huang, Lee 2015+)

Almost every graph is a fractalizer.

But can you find some?

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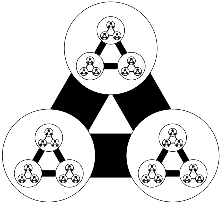

THEOREM (FOX, HUANG, LEE 2015+)

Almost every graph is a fractalizer.

But can you find some? Other than K_n or $\overline{K_n}$, of course.

Desperate junkies search for an alleged " hay in the haystack."

Conjectured hay

Conjecture (Pippinger, Golumbic 1975)

Cycles, except C_4, are fractalizers.

Conjectured hay

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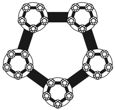
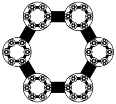
Theorem (Balogh, Hu, L., Pfender, 2016)

C_5 is almost a fractalizer.



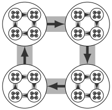
Theorem (Brandt, L., Pfender)

C_6 is almost a fractalizer.



Theorem (Hu, L., Pfender, Volec)

The oriented C_4 is almost a fractalizer.



Almost for C_ℓ means graphs on ℓ^k vertices or in the limit.

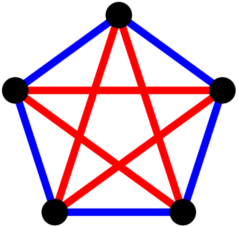
Small Ramsey numbers

Application to something seemingly unrelated (finite).

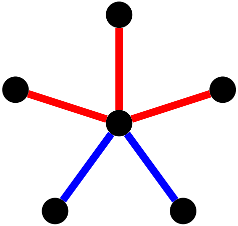
Definition

$R(G_1, G_2, \ldots, G_k)$ is the smallest integer n such that any k-edge coloring of K_n contains a copy of G_i in color i for some $1 \leq i \leq k$.

$R(K_3, K_3) > 5$



$R(K_3, K_3) \leq 6$



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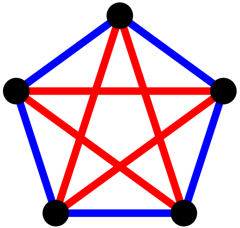

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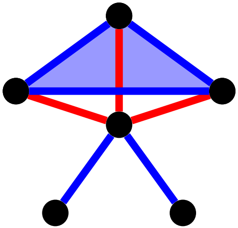

THEOREM (RAMSEY 1930)

$R(K_m, K_n)$ is finite.

$R(G_1, \ldots, G_k)$ is finite

Questions:

- study how $R(G_1, \ldots, G_k)$ grows if $G_1, \ldots, G_k$ grow (large)
- study $R(G_1, \ldots, G_k)$ for fixed $G_1, \ldots, G_k$ (small)

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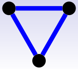
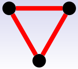
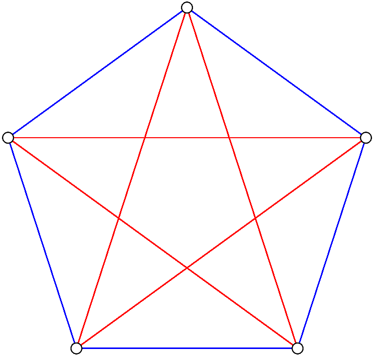
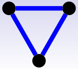
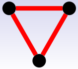
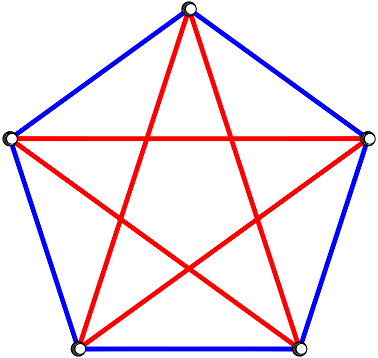
Radziszowski - *Small Ramsey Numbers*
Electronic Journal of Combinatorics - Survey

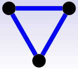
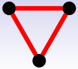
[Erdős] Suppose aliens invade the earth and threaten to obliterate it in a year's time unless human beings can find the Ramsey number for red five and blue five. We could marshal the world's best minds and fastest computers, and within a year we could probably calculate the value. If the aliens demanded the Ramsey number for red six and blue six, however, we would have no choice but to launch a preemptive attack.

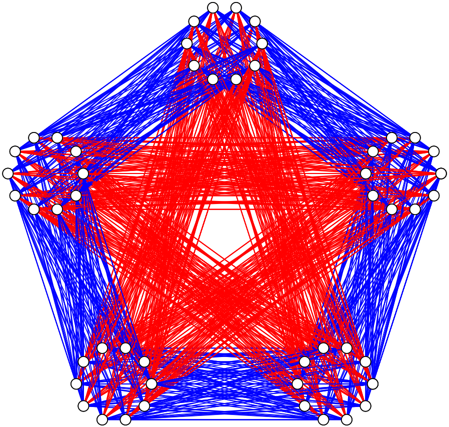
Take any graph G with no  and  .

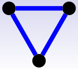
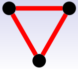


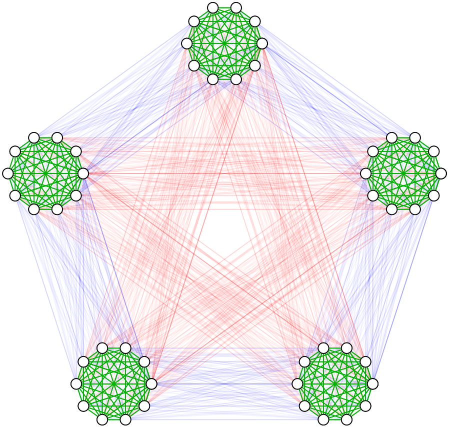
Take any graph G with no  and  .



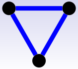
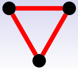
Take any graph G with no  and  .

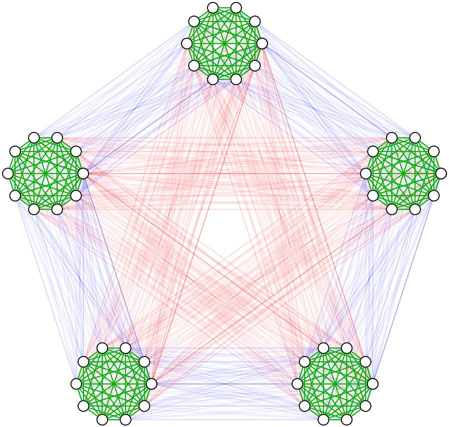


Take any graph G with no  and  .



If G has k vertices, then the blow-up has density of non-edges $\geq \frac{1}{k}$.

Take any graph G with no  and  .



If G has k vertices, then the blow-up has density of non-edges $\geq \frac{1}{k}$.
If any blow-up has density of non-edges $\geq \frac{1}{k}$ then G has $\leq k$ vertices.

New upper bounds (L., Pfender)

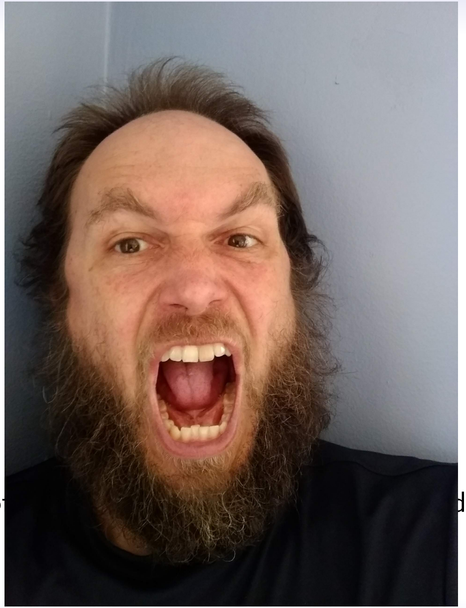
Problem	Lower	New upper	Old upper
$R(K_4^-, K_8^-)$	29	32	38
$R(K_4^-, K_9^-)$	31	46	53
$R(K_4, K_7^-)$	37	49	52
$R(K_5^-, K_6^-)$	31	38	39
$R(K_5^-, K_7^-)$	40	65	66
$R(K_5, K_6^-)$	43	62	66
$R(K_5, K_7^-)$	58	102	110
$R(K_6^-, K_7^-)$	59	124	135
$R(K_7, K_4^-)$	28	29	30
$R(K_8, K_4^-)$	29	39	42
$R(K_8, C_5)$	29	29	33
$R(K_9, C_5)$	33	36	
$R(K_9, C_6)$	41	41	
$R(K_9, C_7)$	49	58	
$R(K_{2,2,2}, K_{2,2,2})$	30	32	60?

Problem	Lower	New upper	Old upper
$R(K_{3,4}, K_{2,5})$		20	21
$R(K_{3,4}, K_{3,3})$		20	25
$R(K_{3,4}, K_{3,4})$		25	30
$R(K_{3,5}, K_{1,6})$	17	17	
$R(K_{3,5}, K_{2,4})$	16	20	
$R(K_{3,5}, K_{2,5})$	21	23	
$R(K_{3,5}, K_{3,3})$		24	28
$R(K_{3,5}, K_{3,4})$		29	33
$R(K_{3,5}, K_{3,5})$	30	33	38
$R(K_{4,4}, K_{4,4})$	30	49	62
$R(W_7, W_4)$		21	
$R(W_7, W_5)$		16	
$R(W_7, W_6)$		19	
$R(B_4, B_5)$	17	19	20
$R(B_3, B_6)$	17	19	22
$R(B_5, B_6)$	22	24	26

Problem	Lower	New upper	Old upper
$R(W_5, K_6)$	33	36	
$R(W_5, K_7)$	43	50	
$R(Q_3, Q_3)$	13	13	14
$R(K_3, C_5, C_5)$	17	17	21?
$R(K_3, C_4, C_4, C_4)$	24	29	
$R(K_4, C_4, C_4)$	52	71	72
$R(K_4^-, K_4^-, K_4^-)$	28	28	30
$R(K_3, K_4^-, K_4^-)$	21	23	27
$R(K_4, K_4^-, K_4^-)$	33	47	59
$R(K_4, K_4, K_4^-)$	55	104	113
$R(K_3, K_4, K_4^-)$	30	40	41
$R(K_4^-, K_5^-; 3)$	12	12	
$R(K_4^-, K_5; 3)$	14	16	
$R(K_4^-, K_4^-, K_4^-; 3)$	13	14	16

Erdős Pentagon Problem

Combination of ideas from results on fractalizers and Ramsey numbers.



Combination of ... d Ramsey numbers.

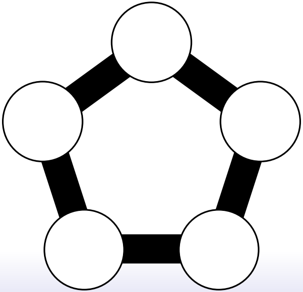
Pentagons in triangle-free graphs

Problem (Erdős, 83)

Is it true that a triangle-free graph on $5n$ *vertices can contain at most* n^5 *pentagons?*

Theorem (Grzesik '12 & Hatami, Hladký, Kráľ, Norin, Razborov '13)

For all $n > n_0$ *or* $5|n$ *, the balanced blow-up of* C_5 *maximizes the number of* C_5*s over all triangle free graphs, and it is unique.*



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Is it true that a triangle-free graph on $5n$ vertices can contain at most n^5 pentagons?

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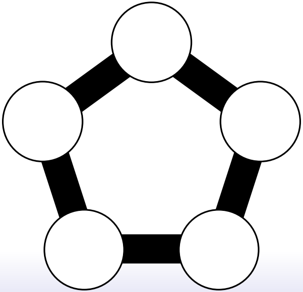
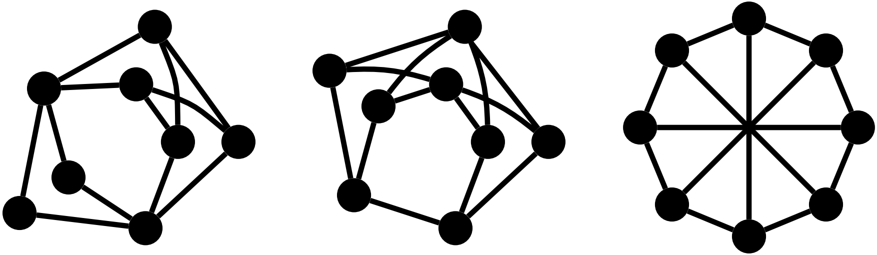
For all n $\cancel{> n_0 \text{ or } 5|n}$, the balanced blow-up of C_5 maximizes the number of C_5s over all triangle free graphs, and it is unique unless $n < 5$ or $n = 8$.

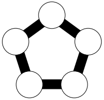


Extremal examples on 8 vertices



Maximize C_5 in a triangle-free graph

Let G be a triangle-free graph maximizing C_5 on $n \geq 10$ vertices.

Goal is to show that G is .

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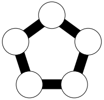
Goal is to show that G is .

Idea: Blow-up G as in Ramsey application
Then use dense structures as in fractalizers.

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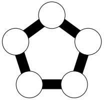
Goal is to show that G is .

Idea: Blow-up G as in Ramsey application
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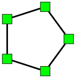
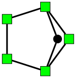
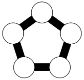
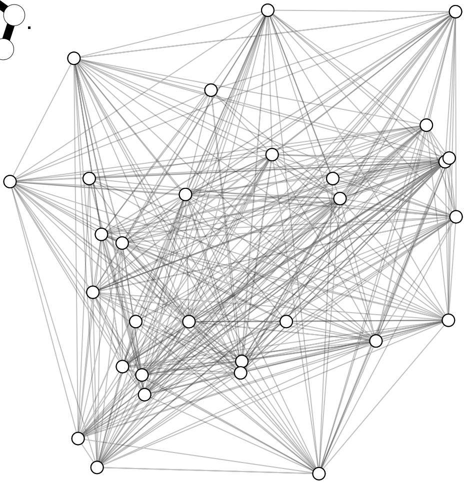
Lemma

There exists  *in G that gives*  *for ANY other* $\bullet$.

Maximize C_5 in a triangle-free graph

Goal: G is .



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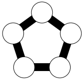
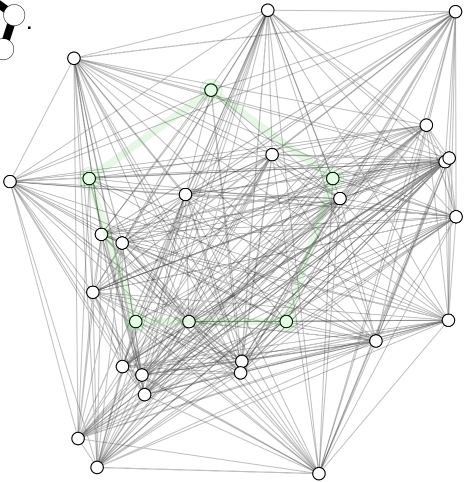
Goal: G is .



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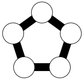
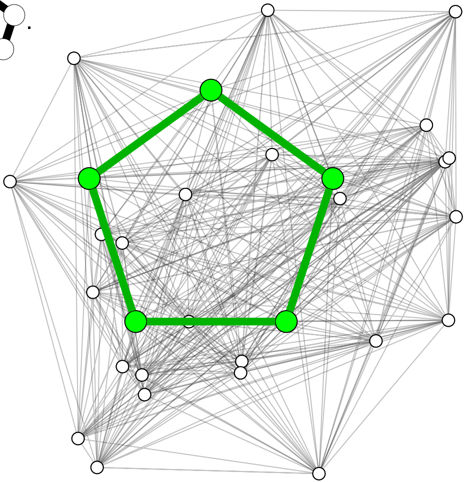
Goal: G is .



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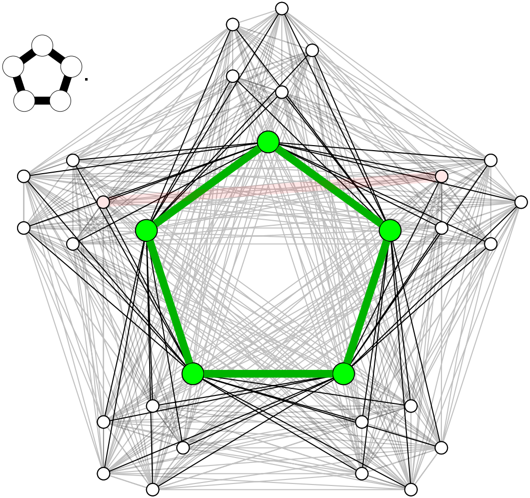
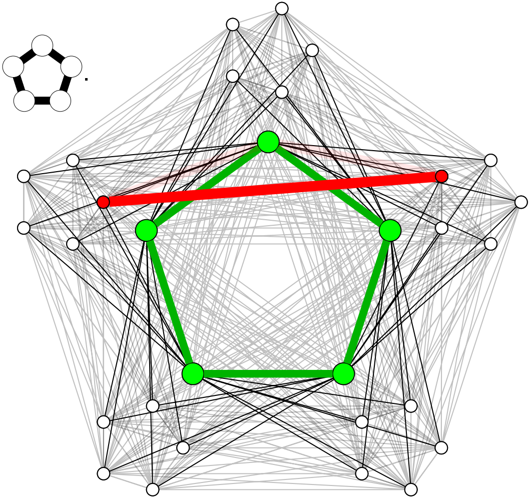
Goal: G is .

Maximize C_5 in a triangle-free graph

Goal: G is .



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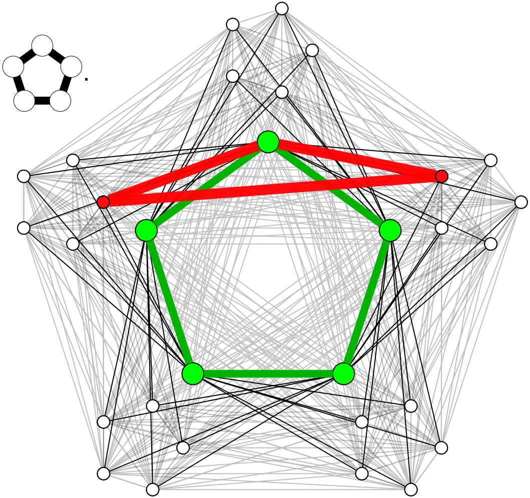
Goal: G is .

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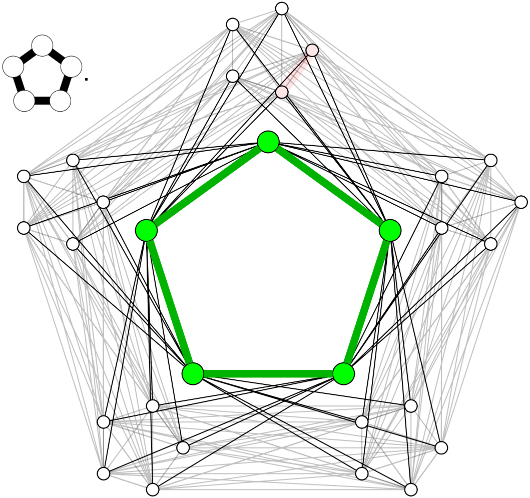
Goal: G is .

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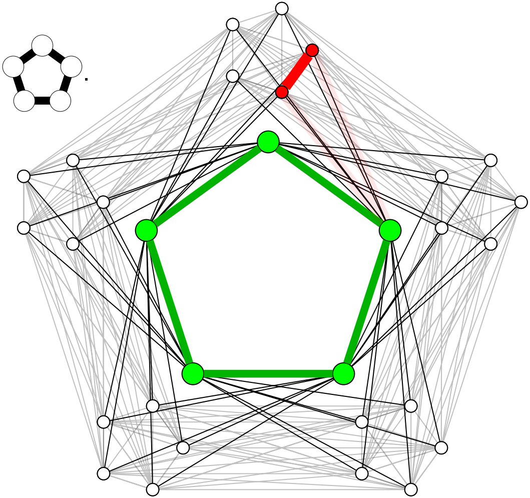
Goal: G is .

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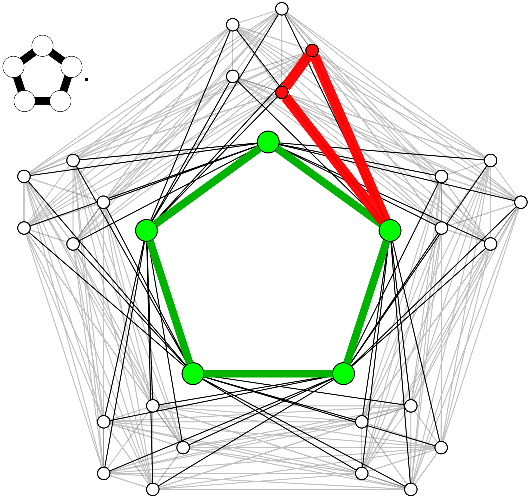
Goal: G is .

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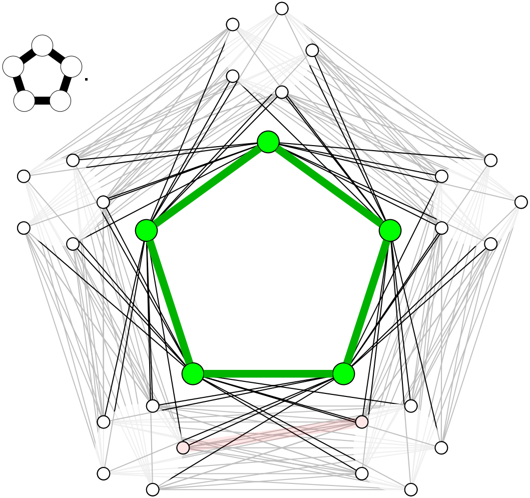
Goal: G is .

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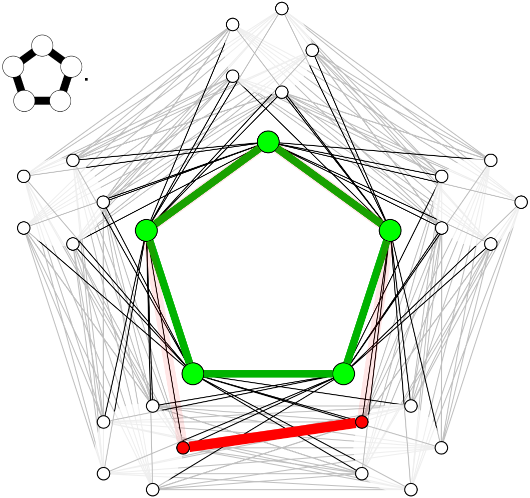
Goal: G is .

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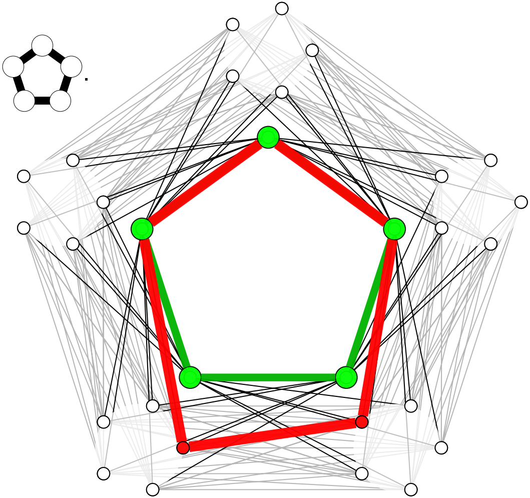

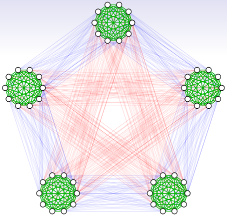
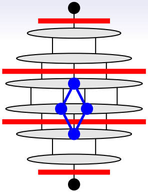
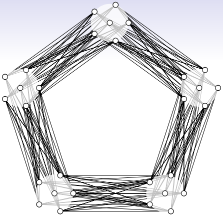

Thank you for your attention!