Maximizing five cycles in K_r-free graphs

Bernard Lidický* Kyle Murphy

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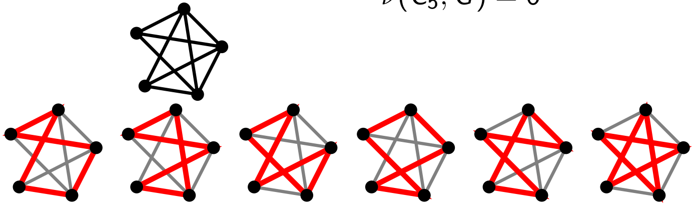
Definitions

- $\nu(H, G)$ is the (possibly non-induced) number of subgraphs of G isomorphic to H.
- G is F-*free* if $\nu(F, G) = 0$.

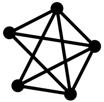
$\nu(C_5, G) = 6$



Definitions

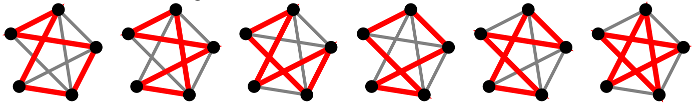
- $\nu(H, G)$ is the (possibly non-induced) number of subgraphs of G isomorphic to H.
- G is F-*free* if $\nu(F, G) = 0$.
- $$\text{ex}(n, H, F) = \max_{G \in \mathcal{G}_n} \nu(H, G)$$

  where $\mathcal{G}_n$ are F-free graphs on n vertices.
  Note: $\text{ex}(n, K_2, F)$ is just $\text{ex}(n, F)$.



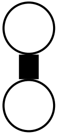
$$\nu(C_5, G) = 6$$

$$\text{ex}(5, C_5, K_5) = 6$$

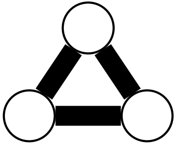


Turán's Theorem

The *Turán Graph* $T_r(n)$ is the *complete r-partite graph* on n vertices so that the sizes of the partite sets are as equal as possible. That is, each set has size $\left\lfloor \frac{n}{r} \right\rfloor$ or $\left\lceil \frac{n}{r} \right\rceil$.



$T_2(n)$



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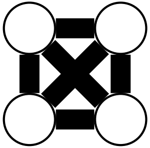
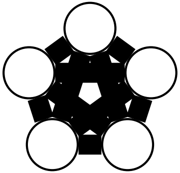


$T_4(n)$



$T_5(n)$

Theorem (Mantel's Theorem (1907))

$ex(n, K_3) = |E(T_2(n))|$. Moreover, $T_2(n)$ is the unique extremal graph.

Theorem (Turán's Theorem (1941))

For $r \geq 3$, $ex(n, K_{r+1}) = |E(T_r(n))|$. Moreover, $T_r(n)$ is the unique extremal graph.

The Erdős-Stone-Simonovits Theorem

Theorem (Erdős-Stone (1946), Erdős-Simonovits (1966))

If F is a graph with chromatic number $\chi(F)$, then

$$ex(n, F) = \left(1 - \frac{1}{\chi(F) - 1}\right) \frac{n^2}{2} + o(n^2).$$

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An edge e in a graph F is *color-critical* if $\chi(F-e) < \chi(F)$.

Theorem (Simonovits (1966))

Let F be a $(r+1)$-chromatic graph. For n large enough, the (unique) extremal graph for $ex(n,F)$ is $T_r(n)$ if and only if F has a color critical edge.

Generalized Turán Numbers

Generalized Turán Problems, which study $\mathrm{ex}(n, H, F)$ when H is not K_2.

Theorem (Zykov (1949))

Let r and t be integers such that $t \leq r$.

$$ex(n, K_t, K_{r+1}) = \nu(K_t, T_r(n)).$$

Theorem (Alon, Shikhelman (2015))

$$ex(n, K_3, C_5) \leq (1 + o(1))\frac{\sqrt{3}}{2}n^{3/2}$$

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For many graphs H and F if $\chi(H) < \chi(F)$

$$\mathrm{ex}(n, H, F) = \nu(H, T_{\chi(F)-1}(n)). \tag{1}$$

Gerbner and Palmer: H is *F-Turán-good* if (1) holds.

Generalized Turán Numbers

Theorem (Győri, Pach, Simonovits (1991))

The path P_ℓ for $\ell \geq 3$ and the cycle $C_{2\ell}$ for $\ell \geq 2$ are K_3-Turán-good.

Theorem (Győri, Pach, Simonovits (1991))

C_4 and $K_{2,3}$ are K_r-Turán-good.

Theorem (Gerbner, Palmer (2020))

Let M_ℓ be a matching with ℓ egdes. M_ℓ is K_3-Turán-good.

Theorem (Gerbner (2021))

For all $m, \ell \geq 0$, P_m and C_{2m} are $C_{2\ell+1}$-Turán-good.

Theorem (Qian, Xie, Ge (2021))

For all $r \geq 4$, the paths P_4 and P_5 are K_r-Turán-good.

Theorem (Murphy, Nir (2021+))

For all $r \geq 4$, the path P_4 is K_r-Turán-good.

F-Turán-good if
$\mathrm{ex}(n, H, F) = \nu(H, T_{\chi(F)-1}(n))$

Theorem (Gerbner (2021))

There exists an F-Turán-good graph if and only if F has a color-critical vertex.

Theorem (Ma, Qui (2021))

If F has a color-critical edge and $\chi(F) > r$ than r then K_r is F-Turán-good.

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Theorem (Gerbner, Palmer (2020))

For every complete multipartite graph H there is an integer k_0 such that if $k \geq k_0$, then H is K_k-Turán-good.

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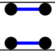
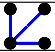
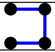
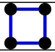
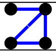
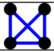
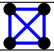
Conjecture (Gerbner, Palmer)

For every graph H, there is an integer k_0 such that if $k \geq k_0$ then H is K_k-Turán good.

Small graphs compiled by Gerbner (2021)

$H \setminus F$	K_2									
K_2	0	E	E	E	E	E	A	E	E	E
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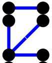
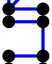
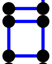
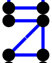
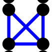
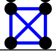
	0	0	E	E	0	E	E	E	E	E
	0	0	E	0	E	0	A	E	E	E
	0	0	E	0	E	0	0	E	E	E
	0	0	0	0	0	0	E	0	B	E
	0	0	0	0	0	0	0	0	0	E
	0	0	0	0	0	0	0	0	0	0

$\mathrm{ex}(n, H, F)$ E. . . Exact result A. . . Asymptotic result B. . . Bounds known

Our result

Theorem (L., Murphy 2021)

For all $r \geq 4$, the cycle C_5 is K_r-Turán good.

Our result

Theorem (L., Murphy 2021)

For all $r \geq 4$, the cycle C_5 is K_r-Turán good.

Kyle and his No Mountain Boys band

The Erdős Pentagon Problem

Question

How far is a triangle-free graph from being bipartite?

Erdős proposed the following three measures of "non-bipartiteness":

1. The minimal possible number of edges in a subgraph spanned by half the vertices.
2. The minimal possible number of edges that have to be removed to make the graph bipartite (max cut).
3. The number of five-cycles contained in the graph.

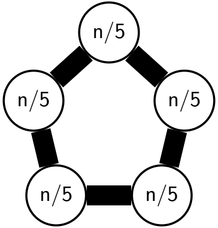
Erdős Pentagon Problem

Question

What is $\mathrm{ex}(n, C_5, K_3)$?

Conjecture (Erdős (1984))

For all $n \geq 5$, the balanced blow-up of a C_5 maximizes the number of (induced) C_5 subgraphs among all triangle-free graphs.



Solved by Grzesik (2012), and independently in Hatami, Hladký, Král', Norin, and Razborov (2013) for large n. All n by L. and Pfender (2018).

Extending Erdős Pentagon Problem

What is $\text{ex}(n, C_5, K_3)$?

Question Palmer

Maximize C_5s in K_r-free graphs for $r \geq 4$.

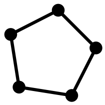
Extending Erdős Pentagon Problem

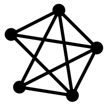
What is $\mathrm{ex}(n, C_5, K_3)$?

Question Palmer

Maximize C_5s in K_r-free graphs for $r \geq 4$.

Note: Two different problems. Are five-cycles induced or not necessarily?





Theorem (L., Murphy (2021))

C_5 is K_r-Turán-Good for all $r \geq 4$. (non-induced)

Theorem

Theorem (L., Murphy (2021))

For all $r \geq 3$, the cycle C_5 is K_{r+1}-Turán good.

Proof Outline

- Flag Algebras: asymptotically, $T_r(n)$ is best possible.
- Stability: large extremal graphs must be "close" to $T_r(n)$.
- Exact Result: $T_r(n)$ is the maximizer.

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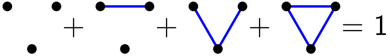
The non-induced *density* of H in G is $d(H, G) = \nu(H, G)/\binom{|V(G)|}{|V(H)|}$.
By a straightforward counting argument, it follows that

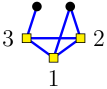
$$\lim_{n\to\infty} d(C_5, T_r(n)) = \frac{1}{r^4}(12r^4 - 60r^3 + 120r^2 - 120r + 48)$$

Razborov's Flag Algebras

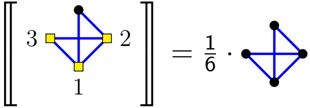
- Think of a (very) large n-vertex graph G.
- $P(H, G)$ is the number of *induced* copies of H in G divided by $\binom{n}{|V(H)|}$.
- Instead of $P(H, G)$, we write just H.



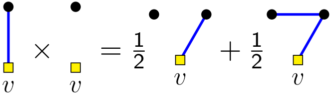
- Yellow vertices are fixed (in G too), black to pick.



- Average over all choices of yellow vertices (unlabeling).



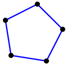
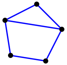
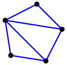
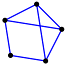
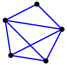
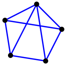
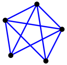
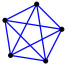
- Multiplication can be simplified.



- There are $o(1)$ errors.

Proof max C_5 in K_{r+1}-free: FA setup

non-induced density of C_5 as induced densities:

$d(C_5, G) =$  $+$  $+ \cdot$  $+ 2 \cdot$  $+ 2 \cdot$  $+ 4 \cdot$  $+ 6 \cdot$  $+ 12 \cdot$ 

More generally:

$$d(C_5, G) = \sum_{H \in \mathcal{F}_5} \nu(C_5, H) P(H, G) = \sum_{H \in \mathcal{F}_5} \nu(C_5, H) H$$

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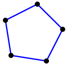
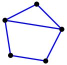
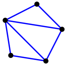
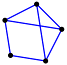
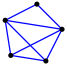
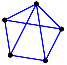
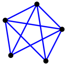
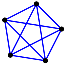
non-induced density of C_5 as induced densities:

$d(C_5, G) =$  $+$  $+ \cdot$  $+ 2 \cdot$  $+ 2 \cdot$  $+ 4 \cdot$  $+ 6 \cdot$  $+ 12 \cdot$ 

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$$d(C_5, G) = \sum_{H \in \mathcal{F}_5} \nu(C_5, H) P(H, G) = \sum_{H \in \mathcal{F}_5} \nu(C_5, H) H$$

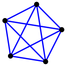
$$\text{Assume } 0 \leq \sum_{H \in \mathcal{F}_5} c_H H$$

$$d(C_5, G) = \sum_{H \in \mathcal{F}_5} \nu(C_5, H) H \leq \sum_{H \in \mathcal{F}_5} (c_H + \nu(C_5, H)) H \leq \max_{H \in \mathcal{F}_5} (c_H + \nu(C_5, H))$$

Notice $\sum_{H \in \mathcal{F}_5} H = 1$

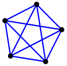
Proof max C_5 in K_{r+1}-free: finding $0 \leq \sum_{H \in \mathcal{F}_5} c_H H$

- Zykov's Theorem

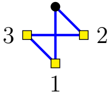
 $$\leq \frac{r^4 - 10r^3 + 35r^2 - 50r + 24}{r^4}$$

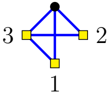
Proof max C_5 in K_{r+1}-free: finding $0 \leq \sum_{H \in \mathcal{F}_5} c_H H$

- Zykov's Theorem

 $\leq \dfrac{r^4 - 10r^3 + 35r^2 - 50r + 24}{r^4}$

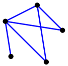
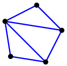
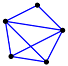
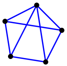
- Average of squares $(5\times)$

$0 \leq \Big[\!\Big[\Big((r-2)$ 

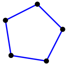
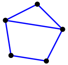
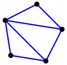
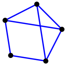
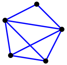
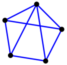
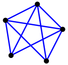
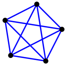
$-$ 

$\Big)^2 \Big]\!\Big]$,

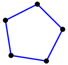
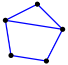
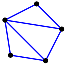
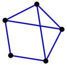
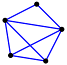
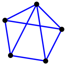
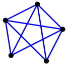
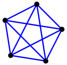
which after expanding and averaging gives:

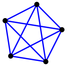
$0 \leq (3r^2 - 12r + 12) \cdot$  $+ (r^2 - 6r + 8) \cdot$  $+ (-4r + 10) \cdot$  $+ 3 \cdot$ 

Proof max C_5 in K_{r+1}-free: density bound

$d(C_5, G) =$  $+$  $+ \cdot$  $+ 2 \cdot$  $+ 2 \cdot$  $+ 4 \cdot$  $+ 6 \cdot$  $+ 12 \cdot$ 

Proof max C_5 in K_{r+1}-free: density bound

$d(C_5, G) \leq$  $+$  $+ \cdot$  $+ 2 \cdot$  $+ 2 \cdot$  $+ 4 \cdot$  $+ 6 \cdot$  $+ 12 \cdot$ 

$+z(r) \cdot \Bigg[\sum_{H \in \mathcal{F}_5 \setminus \{K_5\}} \left(\frac{r^4 - 10r^3 + 35r^2 - 50r + 24}{r^4} \right) H + (-10r^3 + 35r^2 - 50r + 24)$  $\Bigg]$

$+a(r) \cdot [(10r^2 - 20r + 10)$  $+ (r^2 - 2r + 1)$  $+ (-r + 1)$  $+ (-4r + 4)$  $+$  $+$  $]$

$+b(r) \cdot [(3r^2 - 12r + 12)$  $+ (r^2 - 6r + 8)$  $+ (-4r + 10)$  $+ 3$  $]$

$+c(r) \cdot [6$  $-$  $+ 2$  $- 4$  $]$

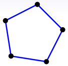
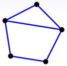
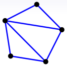
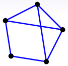
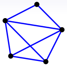
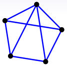
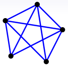
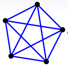
$+d(r) \cdot [(6r^2 - 36r + 54)$  $+ (2r^2 - 20r + 42)$  $+ (4r^2 - 24r + 36)$  $+ (-24r + 84)$  $+ 130$  $]$

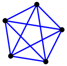
$+e(r) \cdot [(3r^2 - 12r + 12)$  $+ (r^2 - 6r + 8)$  $+ (-4r + 10)$  $+ 3$  $]$

Proof max C_5 in K_{r+1}-free: coefficients

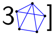
- $z(r) = \frac{120(5r^3-20r^2+30r-16)}{5r^3-35r^2+75r-48}$
- $a(r) = \frac{60(r^5-8r^4+22r^3-24r^2+8r)}{5r^3-35r^2+75r-48}$
- $b(r) = \frac{20(10r^5-60r^4+109r^3-76r^2+18r)}{5r^3-35r^2+75r-48}$
- $c(r) = \frac{20(5r^5-28r^4+45r^3-28r^2+6r)}{5r^3-35r^2+75r-48}$
- $d(r) = \frac{5(5r^7-30r^6+53r^5-52r^4+94r^3-96r^2+24r)}{5r^3-35r^2+75r-48}$
- $e(r) = \frac{5(15r^5-60r^4+78r^3-40r^2+8r)}{5r^3-35r^2+75r-48}$

Proof max C_5 in K_{r+1}-free: density bound

$d(C_5, G) \le$  $+$  $+ \cdot$  $+ 2 \cdot$  $+ 2 \cdot$  $+ 4 \cdot$  $+ 6 \cdot$  $+ 12 \cdot$  $+$

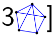
$+z(r) \cdot \left[\sum_{H \in \mathcal{F}_5 \setminus \{K_5\}} H \left(\frac{r^4 - 10r^3 + 35r^2 - 50r + 24}{r^4} \right) + (-10r^3 + 35r^2 - 50r + 24) \right.$  $\left. \right]$

$+a(r) \cdot [(10r^2 - 20r + 10)$  $+ (r^2 - 2r + 1)$  $+ (-r + 1)$  $+ (-4r + 4)$  $+$  $+$  $]$

$+b(r) \cdot [(3r^2 - 12r + 12)$  $+ (r^2 - 6r + 8)$  $+ (-4r + 10)$  $+ 3$  $]$

$+c(r) \cdot [6$  $-$  $+ 2$  $- 4$  $]$

$+d(r) \cdot [(6r^2 - 36r + 54)$  $+ (2r^2 - 20r + 42)$  $+ (4r^2 - 24r + 36)$  $+ (-24r + 84)$  $+ 130$  $]$

$+e(r) \cdot [(3r^2 - 12r + 12)$  $+ (r^2 - 6r + 8)$  $+ (-4r + 10)$  $+ 3$  $]$

$$= \sum_{H \in \mathcal{F}_5} c'_H H \le \max\{c'_H : H \in \mathcal{F}_5\}$$

$$= \frac{1}{r^4}(12r^4 - 60r^3 + 120r^2 - 120r + 48)$$

Proof max C_5 in K_{r+1}-free: Stability

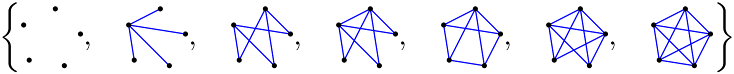
- Suppose G satisfies

$$d(C_5, G) = \frac{1}{r^4}(12r^4 - 60r^3 + 120r^2 - 120r + 48) + o(1),$$

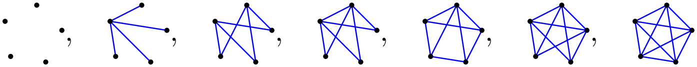
- Since

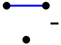
$$d(C_5, G) \leq \sum_{H \in \mathcal{F}_5} c'_H H \leq \max_{H \in \mathcal{F}_5} c'_H$$

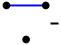
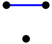
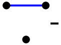
  the only graphs for which $P(H, G) > 0$ are those with $c'_H = \frac{1}{r^4}(12r^4 - 60r^3 + 120r^2 - 120r + 48)$.
- Only $H \in T$ may have $P(H, G) > 0$ other $H \in \mathcal{F}_5$ is $o(1)$

$T =$ 

Proof max C_5 in K_{r+1}-free: Stability

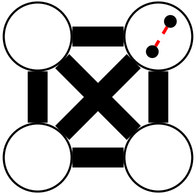
$T = \{$  $\}$

T is  - free.

- If G is  - free, it is a complete k-partite graph.
- $o(1)$ errors give extremal G contains at most δn^3 copies of .
- **Induced Removal Lemma**: G can be made  - free by changing εn^2 adjacencies. This means that G is "close" to being k-partite.
- Counting shows that $T_r(n)$ is best among k-partite graphs, where $k \le r$.

Proof max C_5 in K_{r+1}-free: Exact Result

- **Goal:** Transforming G into $T_r(n)$ increases the number of C_5 subgraphs.
- Partition $V(G)$ to look like $T_r(n)$, with only εn^2 a few "bad edges".
- All vertices are in nearly the same number of C_5's by symmetrization.
- Vertices incident with bad edges have low degrees leading to not enough C_5s
- Thus, we can remove all bad edges and make G complete r-partite, which means $G = T_r(n)$.



□

Open Problem

Conjecture (Gerbner, Palmer)

For every graph H, there is an integer k_0 such that if $k \geq k_0$ then H is K_k-Turán-good.

Question

What is $\mathrm{ex}(n, H, F)$ if $\chi(H) = \chi(F)$?

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Question

What is $\mathrm{ex}(n, H, F)$ if $\chi(H) = \chi(F)$?

Thank you!