ℓ_2-norm in Turán Type Problems

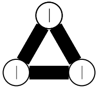
József Balogh Felix Christian Clemen Bernard Lidický

AMS Sectional Meeting #1182
Oct 22, 2022

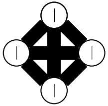
Turán's Theorem

$\mathrm{ex}(n, F) :=$ Maximum number edges in an F-free graph on n vertices.

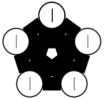
$T_2(n)$



$T_3(n)$



$T_4(n)$



$T_5(n)$

Theorem (Mantel (1907))

$ex(n, K_3) = |E(T_2(n))|$. Moreover, $T_2(n)$ is the unique extremal graph.

Theorem (Turán (1941))

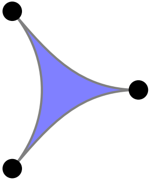
$ex(n, K_{r+1}) = |E(T_r(n))|$ for $r \geq 3$, and $T_r(n)$ is the unique extremal graph.

Theorem (Erdős-Stone (1946), Erdős-Simonovits (1966))

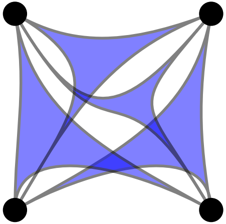
$$ex(n, F) = \left(1 - \frac{1}{\chi(F)-1}\right)\frac{n^2}{2} + o(n^2).$$

Hypergraph Setting

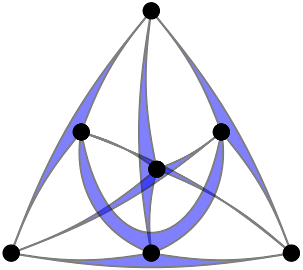
3-uniform hypergraphs have triples of vertices as edges.



One edge, K_3^3



Complete hypergraph K_4^3



Fano Plane

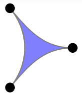
Turán's Tetrahedron problem: Determine $\text{ex}(n, K_4^3)$

\$500 reward by Erdős

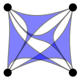
Turán's Tetrahedron problem

Determine $\mathrm{ex}(n, K_4^3)$
Asymptotic setting:

$$\pi(K_4^3) = \lim_{n\to\infty} \mathrm{ex}(n, K_4^3)/\binom{n}{3}$$



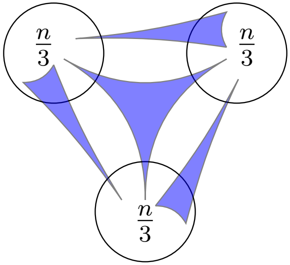
K_3^3



K_4^3

Theorem (Kostochka 1982, Brown 1983, Fon-der-Flaass 1988, Frohmade 2008)

$\pi(K_4^3) \geq 5/9$



At least $6^{n/3}$ extremal hypergraphs

Upper bound

Construction $\pi(K_4^3) \geq 5/9$

Observation

Every 4 vertices miss at least 1 edge $\pi(K_4^3) \leq 3/4$

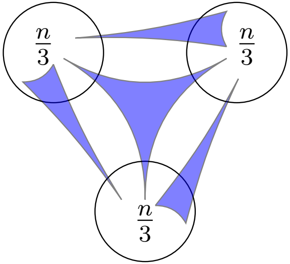
Theorem (Baber 2012)

Flag algebras $\pi(K_4^3) \leq 0.5615$

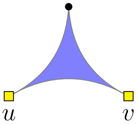
Theorem (Razborov 2010)

Flag algebras $\pi(K_4^3, \textit{few other graphs}) = 5/9$

If 3-partite and all cross edges present
$\pi(K_4^3) \leq 0.5583$



For a hypergraph G
degree of v : $d(v) := |\{e \in E(G) : v \in e\}|$
co-degree of u, v : $d(u, v) := |\{e \in E(G) : u, v \in e\}|$



$$\sum_{u,v \in V} d(u, v) = 3|E(G)|$$

co-degree vector $co(G) = (d(v_1, v_2), d(v_1, v_3), \ldots, d(v_{n-1}, v_n))$

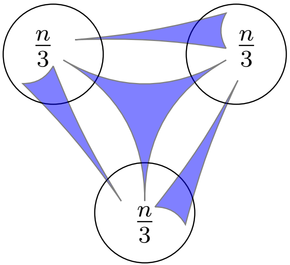
$$\ell_1\text{-norm of } co(G) = ||co(G)||_1 = 3|E(G)|$$

Turán problem is equivalent to maximizing $||co(G)||_1$.
How about ℓ_2-norm of $co(G)$?

Why not degree vector?

By symmetrization, most extremal hypergraphs are almost vertex regular.

$$co(G) = (\underbrace{n/3, \cdots, n/3}_{\sim n^2/6}, \underbrace{2n/3 \cdots 2n/3}_{\sim n^2/3})$$



Problem

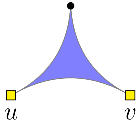
For n-vertex hypergraph G

co-degree of u, v : $d(u,v) := |\{e \in E(G) : u, v \in e\}|$

co-degree vector

$co(G) = (d(v_1, v_2), d(v_1, v_3), \dots, d(v_{n-1}, v_n))$

$$co_2(G) = \sum_{u,v} d(u,v)^2$$



Problem

Determine $exco_2(n, F) := \max\{co_2(G) : G \text{ is } n\text{-vertex and } F\text{-free}\}$

Related concepts for F-free graphs

- uniform Turán density $\pi_u(F)$
- max min co-degree $\pi_2(F)$
- max min positive co-degree $\gamma^+(F)$

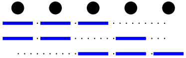
Main properties of $\text{exco}_2(n, F)$

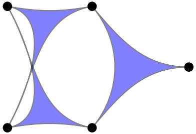
$$\text{exco}_2(n, F) := \max\{co_2(G) = \textstyle\sum_{uv} d(u, v)^2 : G \text{ is } n\text{-vertex and } F\text{-free}\}$$

- scaled limit exists: $\lim_{n\to\infty} \frac{\text{exco}_2(n, F)}{\binom{n}{2} n^2} =: \sigma(F)$
- bounded by Turán density: $\pi(F)^2 \leq \sigma(F) \leq \pi(F)$
- jump from 0: $\sigma(F) = 0$ or $\sigma(F) \geq 2/27$
- supersaturation $\forall \varepsilon > 0 \exists \delta > 0$
  if n-vertex G has $co_2(G) > \text{exco}_2(n, F) + \varepsilon n^4$
  then G contains at least $\delta\binom{n}{|F|}$ copies of F

F_5

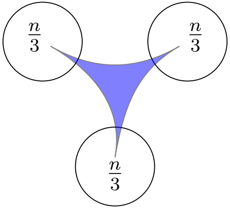
$F_5 = \{123, 124, 345\}$





Theorem (Balogh, Clemen, L. 2022)

$\sigma(F_5) = 2/9$ *with stability and unique extremal example for large n*



$\pi(F_5) = 2/9$ Frankl-Füredi (1983)
$\pi_u(F_5) = 0$ Reiher, Rödl, Schacht (2018)
$\pi_2(F_5) = 0$
$\gamma^+(F_5) = 1/3$ Halfpap, Lemons, Palmer, 4:00pm room 215

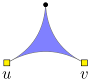
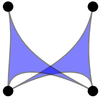
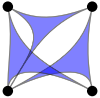
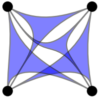
Flag Algebras

Tool for exploring identities on densities of small graphs in large graphs (graph limits).

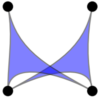
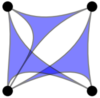
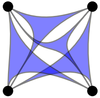
$d(H, G) :=$ (#of subgraphs of G isomorphic to H)$/\binom{|G|}{|H|}$

$d(H, G)$ depicted as H

$co_2(G)$ is eqivalent to

$\frac{1}{\binom{n}{2}} \sum_{uv}$  $^2 = \frac{1}{6}$  $+ \frac{1}{2}$  $+$  $+o(1)$

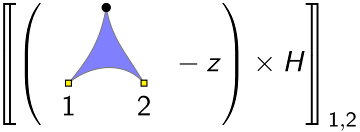
Let $\mathcal{G}_n$ be F-free 3-uniform hypergraphs on n vertices.

$\sigma(F) := \lim_{n\to\infty} \max_{G\in\mathcal{G}_n} \frac{1}{6}$  $+ \frac{1}{2}$  $+$ 

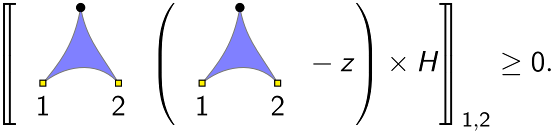
Flag Algebras for other variants

min codegree at least z

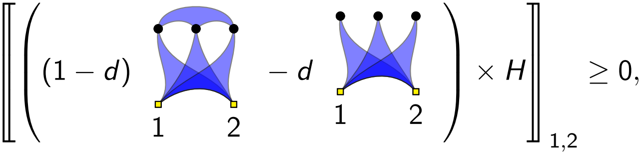
Falgas-Ravry, Pikhurko, Vaughan, Volec (2017)



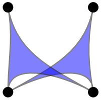
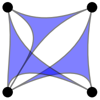
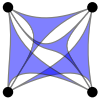
min positive codegree at least z

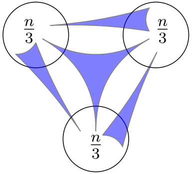


Uniform density at least d formulation by Glebov, Král', Volec (2016)



Let $\mathcal{G}_n$ be F-free 3-uniform hypergraphs on n vertices.

$$\sigma(F) := \lim_{n\to\infty} \max_{G\in\mathcal{G}_n} \frac{1}{6}$$  $$+ \frac{1}{2}$$  $$+$$ 

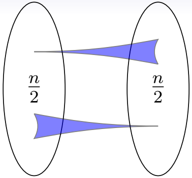


Theorem (Balogh, Clemen, L. 2022)

$\sigma(K_4^3) = \frac{1}{3}$ *with stability*

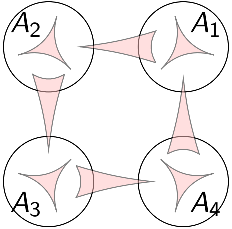
$5/9 \leq \pi(K_4^3) \leq 0.5615$ with many extremal constructions
$1/2 \leq \pi_u(K_4^3) \leq 0.531$ Rödl (1986)
$1/2 \leq \pi_2(K_4^3) \leq 0.529$ Czygrinow, Nagle (2001)
$1/2 \leq \gamma^+(K_4^3) \leq 0.55$ Halfpap, Lemons, Palmer, today 4:00pm room 215

K_5^3



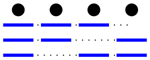
Theorem (Balogh, Clemen, L. 2022)
$\sigma(K_5^3) = \frac{5}{8}$ *including stability*

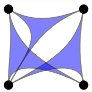
$3/4 \leq \pi(K_5^3) \leq 0.769533$
$\frac{2}{3} \leq \pi_2(K_5^3) \leq 0.74$ LB by Falgas-Ravry (2013) and Lo and Markström (2014)
$2/3 \leq \pi_u(K_5^3) \leq 0.758$

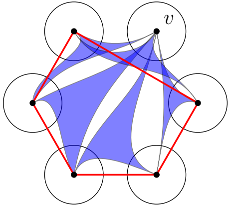
complement of 

K_4^{3-}

$K_4^{3-} = F_4 = \{123, 124, 134\}$





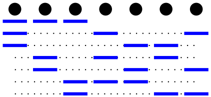


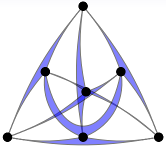
$4/43 \leq \sigma(K_4^{3-}) \leq 4/43 + 0.00004$

$2/7 \leq \pi(K_4^{3-}) \leq 2/7 + 0.00117$
$\pi_u(K_4^{3-}) = 1/4$ Glebov, Král', Volec (2016), Reiher, Rödl, Schacht (2018)
$\pi_2(K_4^{3-}) = 1/4$ Falgas-Ravry, Pikhurko, Vaughan, Volec (2017)
$\gamma^+(K_4^{3-}) = 1/3$ Halfpap, Lemons, Palmer, today 4:00pm room 215

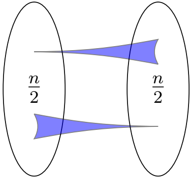
Fano Plane

$\mathbb{F} = \{123, 345, 156, 246, 147, 257, 367\}$





$\pi(\mathbb{F}) = 3/4$ Sós 1976, de Caen Füredi (2000)
$\pi_u(\mathbb{F}) = 0$ Reiher, Rödl, Schacht (2018)
$\pi_2(\mathbb{F}) = 1/2$ Mubayi (2005)



$\gamma^+(\mathbb{F}) = 2/3$ Halfpap, Lemons, Palmer, today 4:00pm room 215

How about $\sigma(\mathbb{F})$? Too large for flag algebras.

H	$\pi(H) \geq$	$\pi(H) \leq$	$\sigma(H) \geq$	$\sigma(H) \leq$	$\pi_2(H) \geq$	$\pi_2(H) \leq$	$\pi_u(H) \geq$	$\pi_u(H) \leq$
K_4^3	5/9 [67]	0.5615 [1]	1/3 [3]	1/3 [3]	1/2 [56]	0.529	1/2	0.529
K_5^3	3/4 [62, 67]	0.7696 [68]	5/8 [3]	5/8 [3]	2/3 [24, 50]	0.74	2/3	0.758
K_6^3	0.84 [62, 67]	0.8584 [68]	0.7348 [3]	0.7536	3/4 [24, 50]	0.838	3/4	0.853
$F_{3,2}$	4/9 [54]	4/9 [35]	1/4	$1/4 + 10^{-9}$	1/3 [25]	1/3 [25]	0	0 [60]
$F_{3,3}$	3/4 [54]	3/4 [54]	5/8 [3]	5/8 [3]	1/2	0.604	1/4	0.605
F_5	2/9 [5]	2/9 [30]	2/27	2/27	0	0	0	0 [60]
$\mathbb{F}$	3/4 [65]	3/4 [16]	5/8	3/4	1/2 [53]	1/2 [53]	0	0 [60]
K_4^{3-}	2/7 [31]	0.28689 [68]	4/43	0.09307	1/4 [56]	1/4 [26]	1/4 [39]	1/4 [39]
$K_4^{3-}, F_{3,2}, C_5$	12/49 [27]	12/49 [27]	2/27	2/27	1/12	0.186	0	0 [60]
$K_4^{3-}, F_{3,2}$	5/18 [27]	5/18 [27]	5/54	5/54	1/12	0.202	0	0 [60]
K_4^{3-}, C_5	1/4 [27]	0.25108 [27]	1/13	0.07695	1/12	0.204	0	0.251
$F_{3,2}, J_4$	3/8 [27]	3/8 [27]	3/16	3/16	1/12	0.274	0	0 [60]
$F_{3,2}, J_5$	3/8 [27]	3/8 [27]	3/16	3/16	1/12	0.28	0	0 [60]
J_4	1/2 [6]	0.50409 [68]	0.28	0.2808	1/4	0.473	1/4	0.49
J_5	1/2 [6]	0.64475	0.28	0.44275	1/4	0.613	1/4	0.631
C_5	$2\sqrt{3} - 3$ [54]	0.46829 [68]	0.25194	0.25311	1/3	0.3993	1/27	0.402
C_5^-	1/4 [54]	0.25074 [68]	1/13	0.07726	0	0.136	0	0 [60]
$K_5^=$	0.58656	0.60962	0.35794	0.38873	1/2	0.569	1/2	0.567
$K_5^<$	5/9	0.57705	1/3	0.34022	1/2	0.560	1/2	0.568
K_5^-	0.58656	0.64209	0.35794	0.41962	1/2	0.621	1/2	0.626

Codegree Squared

Flag algebras work | for small graphs | and exactness?

https://timothyandersondesign.com/