11/4-colorability of subcubic triangle-free graphs

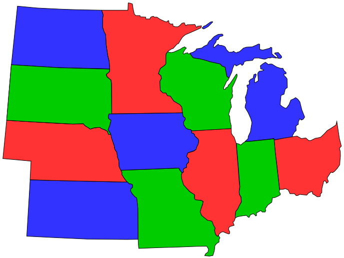
Zdeněk Dvořák   Bernard Lidický   Luke Postle

Warwick
February 28, 2022

Motivation

Theorem (4CT - Appel, Haken (1976))
Every planar graph is 4-colorable.

Also Robertson, Sanders, Seymour, Thomas (1997)



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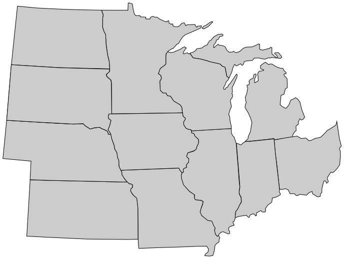

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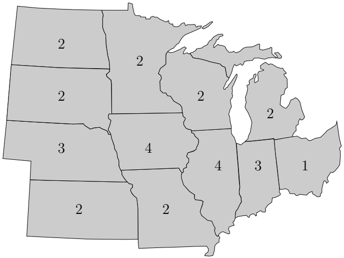

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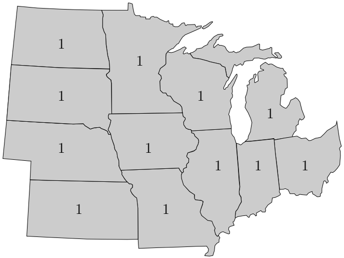

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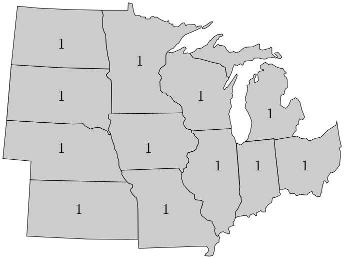

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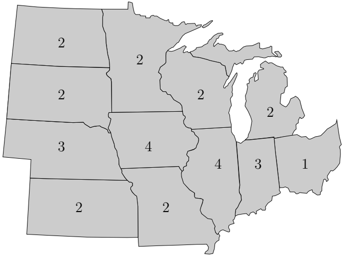

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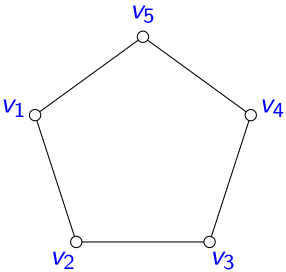
Weighted independent set leads to fractional coloring.

Fractional coloring using linear programming

$\mathcal{I}(G)$ are all independent sets

$$P\begin{cases} \text{minimize} & \sum_{I\in\mathcal{I}(G)} x(I) \\ \text{subject to} & \sum_{I\ni v} x(I) = 1 \\ & x \in \{0,1\}^{\mathcal{I}(G)} \end{cases}$$



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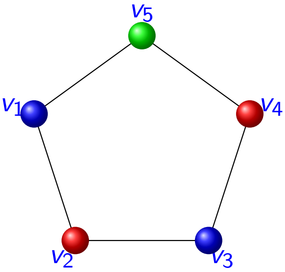

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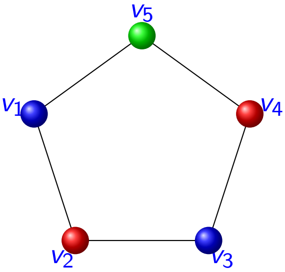

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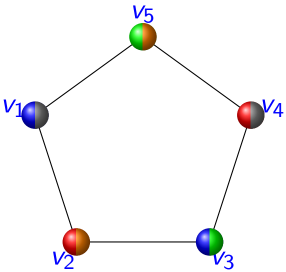

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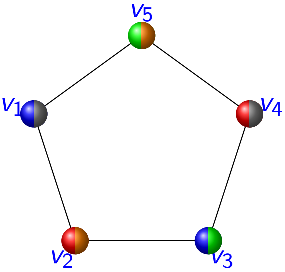

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- $\chi(G) \geq \chi_f(G) \geq |V(G)|/\alpha(G)$

Equivalent definitions for fractional coloring

G is fractionally k-colorable if exists φ

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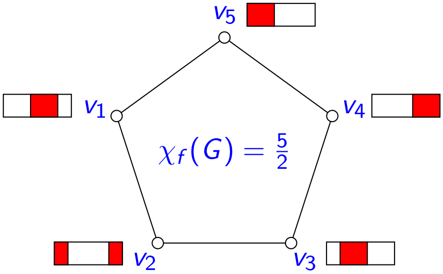
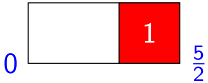
and $\varphi(u) \cap \varphi(v) = \emptyset$ for $uv \in E(G)$





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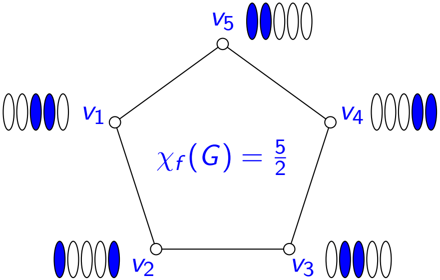

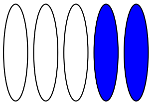

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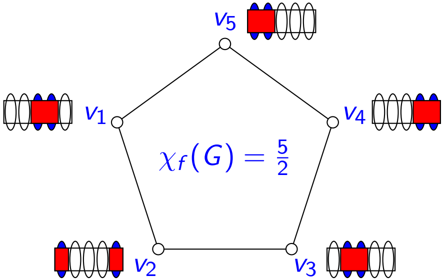

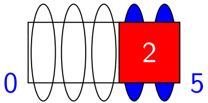

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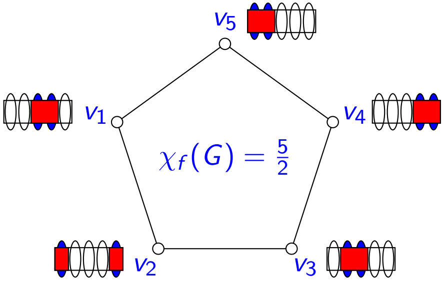
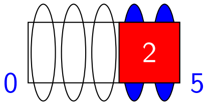


Theorem (Hilton, Rado, Scott (1973))

$\chi_f(G) < 5$ for any planar G.
(But no $c < 5$ with $\chi_f(G) < c$ for all planar graphs G.)



Theorem (Cranston and Rabern (2017))

Planar graphs are $\frac{9}{2}$-colorable. (Without using 4CT.)

Planar triangle-free graphs

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Every triangle-free planar graph G is 3-colorable.

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$\exists G : \alpha(G) \leq n/3 + 1$ (and $\Delta(G) = 4$) Jones (1990)

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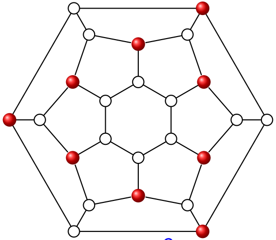

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Question (Albertson, Bollobás, Tucker (1976))

Find $s \in \left(\frac{1}{3}, \frac{3}{8}\right]$ s.t. every subcubic triangle-free planar graph G has $\alpha(G) \geq sn$?

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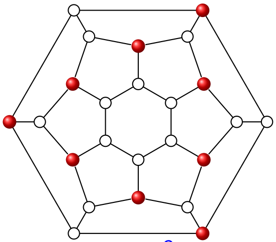

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- $s = \frac{5}{14} \approx 0.35714$ Staton (1979) No planarity condition!
- $s = \frac{3}{8} = 0.375$ Heckman and Thomas (2006)

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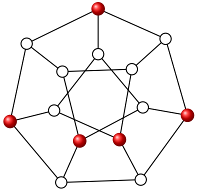

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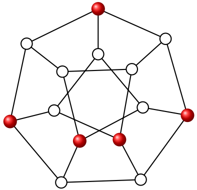
- $\alpha(G) \geq \frac{3n}{8} = 0.375n$ Cames van Batenburg, Goedgebeur, Joret (2020)
  if G is avoids 6 exceptional graphs. All non-planar, containing 5-cycles.
  (Infinitely many 3-connected tight examples.)



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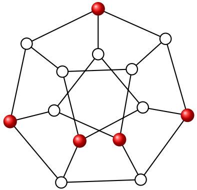

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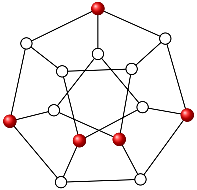

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$$i(g) := \inf\left\{\frac{\alpha(G)}{|V(G)|} : G \text{ subcubic graph with girth at least } g\right\}$$

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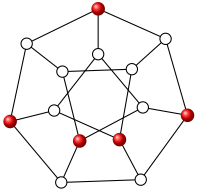

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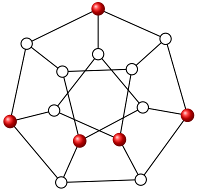

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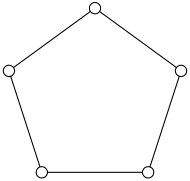
$$0.44533 \leq \lim_{g \to +\infty} i(g) \leq 0.454$$

By Csóka (2016) and Balogh, Kostochka, X. Liu (2019).

From α to χ_f



If G is fractionally $\frac{1}{s}$-colorable, it has $\alpha(G) \geq sn$.

$\alpha(C_5) = \frac{2}{5}n \quad \chi_f(C_5) = \frac{5}{2}$

If $\alpha(G) = sn$, is G fractionally $\frac{1}{s}$-colorable?

Conjecture (Heckman and Thomas (2001))

Every subcubic triangle-free graph is fractionally $14/5$*-colorable.*

Conjecture (Heckman and Thomas (2006))

Every subcubic triangle-free planar graph is fractionally $8/3$*-colorable.*

Conjecture (Heckman and Thomas (2001))

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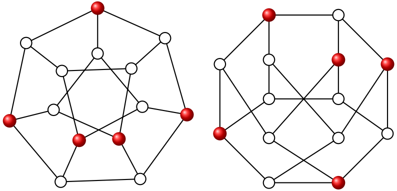
$\chi_f(F_{14}^{(1)}) = \chi_f(F_{14}^{(1)}) = 14/5$



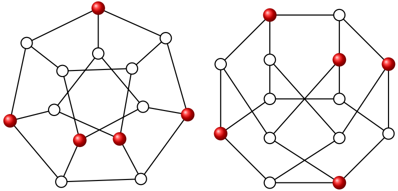
$F_{14}^{(1)}$ $F_{14}^{(2)}$

Conjecture (Heckman and Thomas (2001))

Every subcubic triangle-free graph is fractionally 14/5-colorable.

$\chi_f(F_{14}^{(1)}) = \chi_f(F_{14}^{(1)}) = 14/5$

- $3 - \frac{3}{64} \approx 2.953$ Hatami, Zhu (2009)
- $3 - \frac{3}{43} \approx 2.930$ Lu, Peng (2012)
- $\frac{32}{11} \approx 2.909$ Furgeson, Kaiser, Král' (2014)
- $\frac{43}{15} \approx 2.867$ Chun-Hung Liu (2014)
- $\frac{14}{5} = 2.8$ Dvořák, Sereni, Volec (2014)



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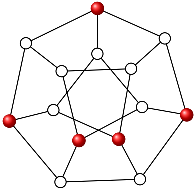
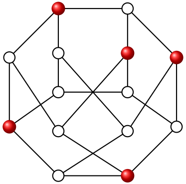
- $\frac{11}{4} = 2.75$ Dvořák, L., Postle, if not $\rightarrow$



$F_{14}^{(1)}$



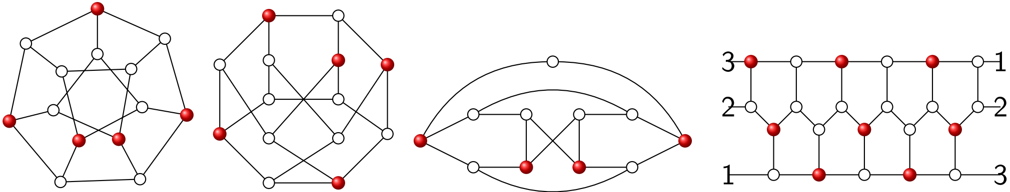
$F_{14}^{(2)}$

Theorem (Dvořák, L., Postle (2021+))

Every subcubic triangle-free graph avoiding $F_{14}^{(1)}$ and $F_{14}^{(1)}$ is fractionally 11/4-colorable.

Theorem (Dvořák, L., Postle (2021+))

Every subcubic triangle-free graph avoiding $F^{(1)}_{14}$ and $F^{(1)}_{14}$ is fractionally $11/4$-colorable.



$F^{(1)}_{14}$ $F^{(2)}_{14}$ F_{11} F_{22}

$\chi_f(F^{(1)}_{14}) = \chi_f(F^{(2)}_{14}) = 14/5$ $\chi_f(F_{22}) = \chi_f(F_{11}) = 11/4$

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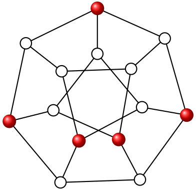

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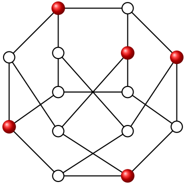

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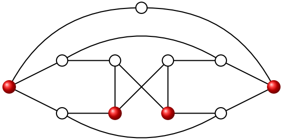
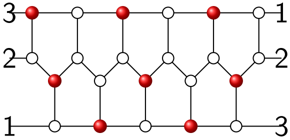


F_{11}



F_{22}

$\chi_f(F_{14}^{(1)}) = \chi_f(F_{14}^{(2)}) = 14/5$ $\chi_f(F_{22}) = \chi_f(F_{11}) = 11/4$

Corollary (Dvořák, L., Postle (2021+))

Every subcubic triangle-free planar graph is fractionally $11/4$-colorable.

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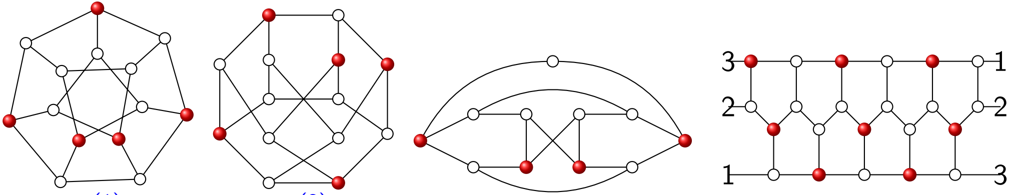
Every subcubic triangle-free graph avoiding $F_{14}^{(1)}$ *and* $F_{14}^{(1)}$ *is fractionally* $11/4$*-colorable.*



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Corollary (Dvořák, L., Postle (2021+))

Every subcubic triangle-free planar graph is fractionally $11/4$*-colorable.*

Conjecture (Dvořák, L., Postle (2021+))

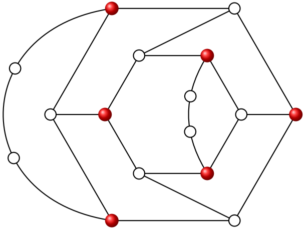
Every subcubic triangle-free graph avoiding $F_{14}^{(1)}$, $F_{14}^{(1)}$, F_{11}, *and* F_{22} *is fractionally* $19/7$*-colorable.*

$11/4 = 2.75$ $19/7 \approx 2.7143$ $8/3 \approx 2.6666$

Knowledge overview

For subcubic triangle-free graph G that avoids $\mathcal{F}$, following is know or conjectured

$\mathcal{F}$	α	coloring
$\emptyset$	$5n/14$	$14/5$
$\{F_{14}^{(1)}, F_{14}^{(2)}\}$	$4n/11$	$11/4$
$\{F_{11}, F_{22}, F_{14}^{(1)}, F_{14}^{(2)}\}$	$7n/19$	$?19/7?$
$\{F_{19}^{(1)}, F_{19}^{(2)}, F_{11}, F_{22}, F_{14}^{(1)}, F_{14}^{(2)}\}$	$3n/8$	$?8/3?$
all non-planar	$3n/8$	$?8/3?$

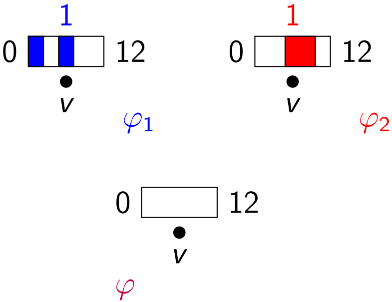


$\alpha = \frac{3n}{8}$

Conjecture (Cames van Batenburg, Goedgebeur, Joret (2020))
Every subcubic triangle-free graph avoiding $F_{14}^{(1)}, F_{14}^{(2)}, F_{11}, F_{22}, F_{19}^{(1)}, F_{19}^{(2)}$ is fractionally $8/3$-colorable.

Combining fractional colorings

Let φ_1, φ_2 be fractional r-colorings of G.



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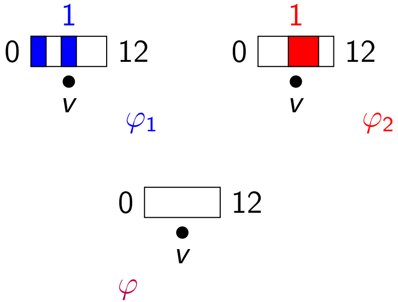
Let φ_1, φ_2 be fractional r-colorings of G.

$0 \leq \lambda_1, \lambda_2$ such that $\lambda_1 + \lambda_2 = 1$
then

$$\text{“}\varphi(v) = \lambda_1\varphi_1(v) \cup (\lambda_2\varphi_1(v) + r\lambda_1)\text{”}$$

is a fractional r-coloring of G.



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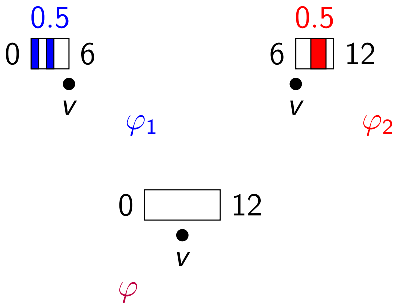

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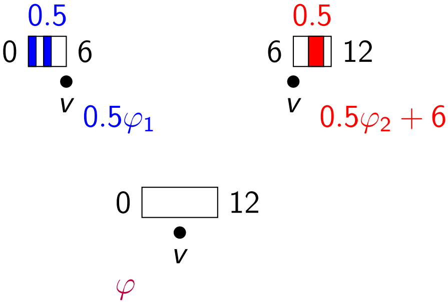

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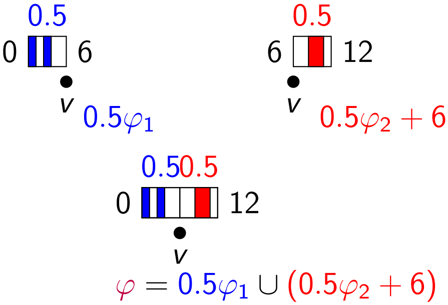
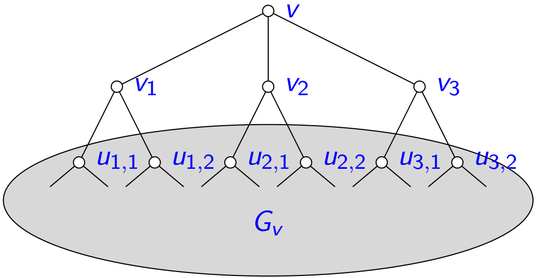


fractional r-colorings of G can be convexly combined

Final part of the proof

Allowed 11 colors, each vertex needs 4. (φ to $[0, 11)$, $|\varphi(v)| = 4$)
Let G be a *nice* minimum counterexample. Remove $N[v]$, color rest.



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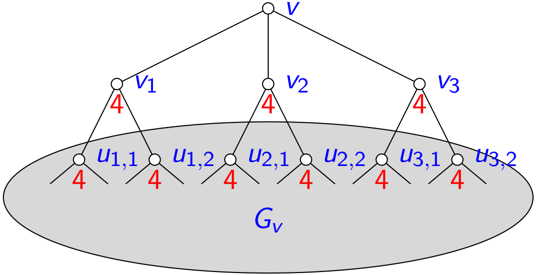

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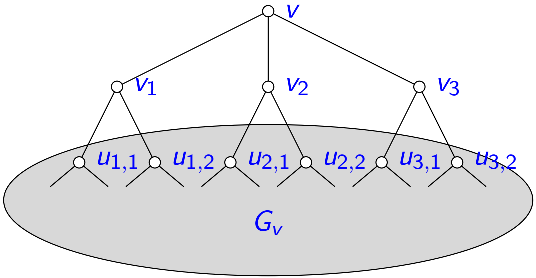

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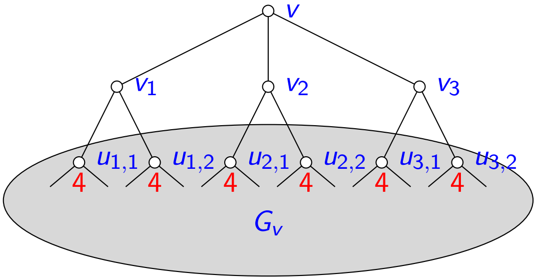

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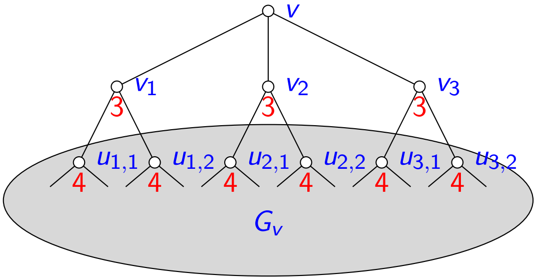

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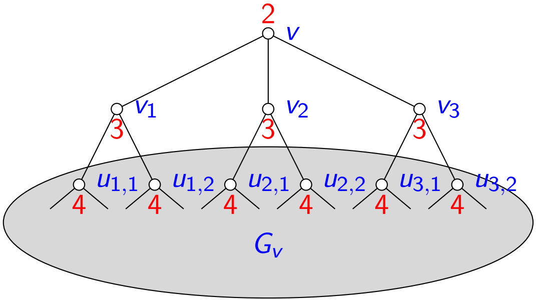

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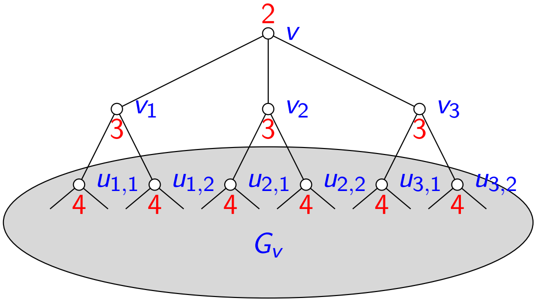
For all $v \in V$, get a coloring φ_v as $\rightarrow$
Let $n = |V(G)|$
Combination of colorings

$$\varphi = \sum_{v \in V(G)} \frac{1}{n} \varphi_v$$



$$|\varphi(v)| = \frac{2 + 3 \times 3 + (n-4) \times 4}{n} = 4 - \frac{5}{n} < 4$$

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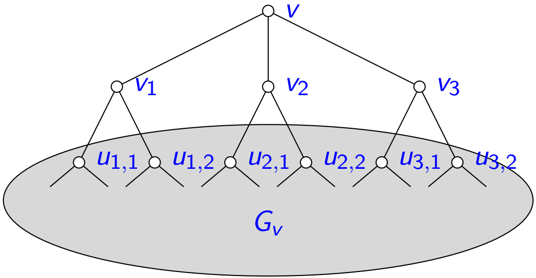

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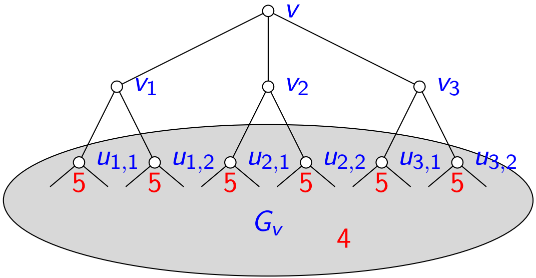

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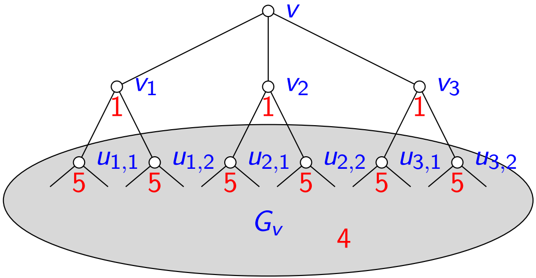

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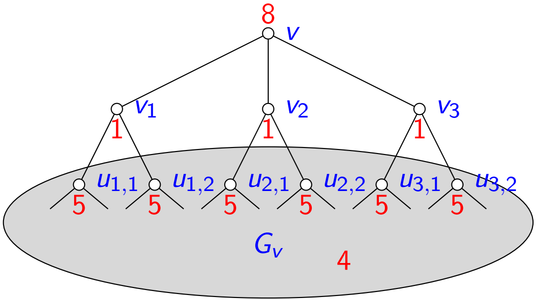

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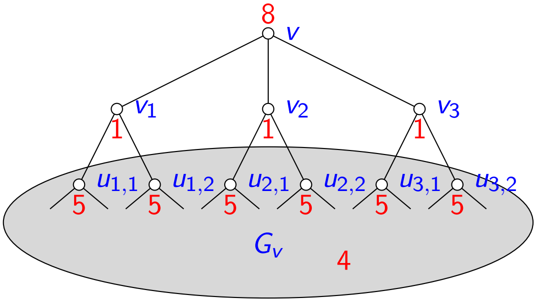

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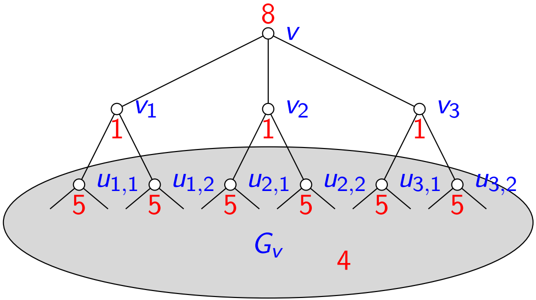

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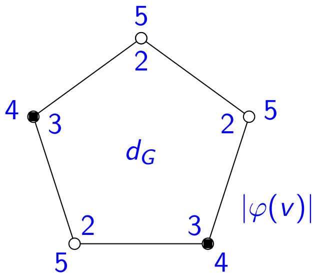
G is nice after 40 pages, 176 counterexamples, and some computer calculations.

Actual Result to Prove

Let $G = (V, E)$ be a subcubic graph.
Let $d_G : V \to \{2, 3\}$, where $\deg \leq d_G$

An $11/4$-coloring is a fractional coloring φ using $[0, 11)$, such that

$$|\varphi(v)| = \begin{cases} 4 & \text{if } d_G(v) = 3 \\ 5 & \text{if } d_G(v) = 2 \end{cases}$$



THEOREM (DVOŘÁK, L., POSTLE)

If (G, d_G) *is critical for* $11/4$*-coloring, then it is isomorphic to one of* 176 *examples in* $\mathcal{C}$*.*

Out of these 176, only 2 correspond to sub-cubic graphs with $d_G = 3$ and these are $F_{14}^{(1)}, F_{14}^{(2)}$.

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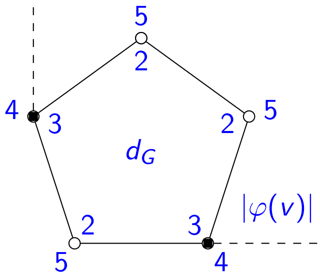

Theorem (Dvořák, L., Postle)

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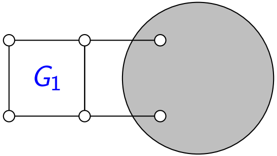
Out of these 176, only 2 correspond to sub-cubic graphs with $d_G = 3$ and these are $F_{14}^{(1)}, F_{14}^{(2)}$.

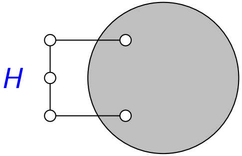
Making nice counterexamples

For contradiction let $(G, d_G) \notin \mathcal{C}$ be the smallest critical graph for $11/4$-coloring.

Exclude small subgraphs such as small cuts, 4-cycles, 2-vertices,... as follows



- Find a pesky structure G_1
- Replace it with some smaller H
- One of these
  - Find $F \in \mathcal{C}$, which gives $G \in \mathcal{C}$
  - Find an $11/4$-coloring and color G



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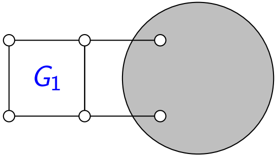
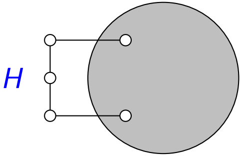




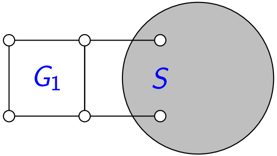
How to extend $11/4$-coloring?
In usual coloring, brute forcing may work.

How to extend the coloring?

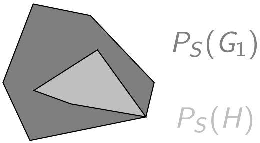
What extends to H extends to G_1?

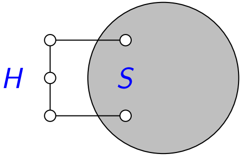
- Consider polytope from LP:

$$P(G)\begin{cases}\sum_{I\in\mathcal{I}(G)} x(I) = 11 & \\ \sum_{I\ni v} x(I) = 4 & \text{if } d_G(v) = 3\\ \sum_{I\ni v} x(I) = 5 & \text{if } d_G(v) = 2\\ x\in[0,1]^{\mathcal{I}(G)} & \end{cases}$$



- P restricted to S is P_S
- Test $P_S(H) \subseteq P_S(G_1)$
- Can be tested on computer by considering vertices of $P_S(H)$.





For subcubic triangle-free graphs avoiding $\mathcal{F}$

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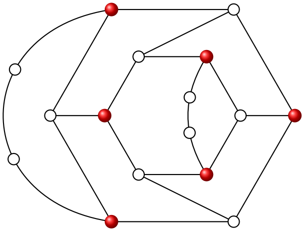

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Thank You!