On the hardness of recognizing triangular line graphs

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Abstract

Given a graph G, its triangular line graph is the graph $T(G)$ with vertex set consisting of the edges of G and adjacencies between edges that are incident in G as well as being within a common triangle. Graphs with a representation as the triangular line graph of some graph G are *triangular line graphs*, which have been studied under many names including anti-Gallai graphs, 2-in-3 graphs, and link graphs. While closely related to line graphs, triangular line graphs have been difficult to understand and characterize. Van Bang Le asked if recognizing triangular line graphs has an efficient algorithm or is computationally complex. We answer this question by proving that the complexity of recognizing triangular line graphs is NP-complete via a reduction from 3-SAT.

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1 Introduction

The line graph operator, L, is well studied. Recognition and characterization have been determined for L (see Lehot [Leh74] and Roussopoulos [Rou73]), but these problems for related operators are less understood. The triangular line graph operator, T, defined by Jarrett [Jar95] is one such operator.

Definition 1.1 (Triangular Line Graph)**.** Given a graph G with edge set $E(G)$, the *triangular line graph* of G, denoted $T(G)$, is the graph with vertex set $E(G)$ and an edge between the edges e_1 and e_2 if and only if

- e_1 and e_2 are incident at a common vertex in G, and
- there is an edge $e_3 \in E(G)$ so that $e_1e_2e_3$ is a triangle in G.

A graph H is a *triangular line graph* if there exists a graph G so that $H \cong T(G)$.

The second restriction is the only difference from the line graph operator, but causes great difference in behavior. For example, the class of triangular line graphs is not closed under taking induced subgraphs, unlike line graphs.

The triangular line graph is studied under many different names. For integers k and ℓ, the k-in-ℓ operator $\Phi_{k,\ell}$ maps k-cliques to vertices which are adjacent if and only if they appear in a common ℓ-clique [Pri95]. For $k = 2$ and $\ell = 3$, the 2-in-3 operator $\Phi_{2,3}$ is equivalent to the triangular line graph operator. For $\ell = k + 1$, the k-in-ℓ operator is identical to the k-anti-Gallai operator [Pri95]. The name comes from the k-Gallai graph operator, where k-cliques are mapped to vertices and adjacencies exist when the k-cliques intersect at $k - 1$ vertices but are not within a $k + 1$-clique. These operators were originally defined by Gallai [Gal67] in connection with comparability graphs. In addition, $T(G)$ is a special case of the H-line graph (when $H = K_3$), defined by Chartrand, Gavlas, and Schultz [CGS95] as a subgraph of $L(G)$ which preserves edge adjacency for edges in an induced subgraph isomorphic to H.

Prior research on triangular line graphs has focused primarily on the properties of iterations of the operator (see Jarret [Jar95], Dorrough [Dor96], and Anand, Escuadro, Gera, Martell [AEGM10b], [AEGM10a]), their relation to perfect graphs [Le96], or the forbidden subgraphs characterization for H-free triangular line graphs [SRV07].

In [Le96], Le provides a characterization for k-Gallai graphs and k-anti-Gallai graphs, inspired by the characterization of line graphs in [Kra43]. With $k = 2$ this yields a characterization for triangular line graphs:

Theorem 1.2 (Van Bang Le, [Le96])**.** *A graph G is a triangular line graph if and only if there is a family of induced subgraphs $\mathcal{F} = \{G_i : i \in I\}$ of G such that:*

1. *Every vertex in G appears in exactly two members of $\mathcal{F}$.*
2. *Every edge in G appears in exactly one member of $\mathcal{F}$.*
3. *$|G_i \cap G_j| \leq 1$, for every $i \neq j$.*
4. *For any distinct $i, j, k \in I$, if $\{v_i\} = G_i \cap G_k$ and $\{v_j\} = G_j \cap G_k$ and $v_i \neq v_j$, then $(v_i, v_j) \in E(G)$ if and only if $G_i \cap G_j \neq \emptyset$.*

This characterization arises when considering a *triangular line graph preimage* H of a graph G so that $T(H) \cong G$. For each vertex v in $V(H)$, the edges incident to v induce one of the subgraphs in the family $\mathcal{F}$. While this is a complete characterization, it is unsatisfying as it is cumbersome and lacks an efficient algorithmic interpretation. In [Le96], Le states that the complexity of recognizing this class is unknown.

We prove that recognition of triangular line graphs is NP-complete. We do so via a reduction from the canonical NP-complete problem 3-SAT (see [GJ90]). The logic of a conjunctive normal formula ϕ with clauses of size three is encoded to a graph G_ϕ whose triangular line graph preimage (if it exists) describes a satisfying assignment of ϕ. This result is in stark contrast to the polynomial-time algorithm of Roussopoulos [Rou73] that recognizes line graphs and constructs the unique line graph preimage. Moreover, it implies simpler characterizations of triangular line graphs that lead to polynomial-time algorithms are extremely unlikely.

Our reduction relies on the fact that certain triangular line graphs have precisely two preimage isomorphism classes, thus providing us with a representation of binary values. We begin with a discussion of these classes, and proceed to show how these classes enforce similar requirements for the preimages of graphs containing these components as subgraphs. From these two observations, we construct gadgets that encode logic of 3-CNF formulas, starting with binary values for each variable. These gadgets are combined in the final construction that requires each clause to be satisfied if and only if a preimage exists.

2 Triangle-Induced Subgraph Closure

An important property of line graphs is that any induced subgraph is also a line graph. Triangular line graphs are not closed under induced subgraphs, but do satisfy a similar property.

Definition 2.1. A subgraph H_1 is a *triangle-induced subgraph* of H_2, denoted $H_1 \lhd H_2$, if

1. H_1 is an induced subgraph of H_2 and
2. If there is an edge uv in H_1 and a triangle uvw in H_2, then the vertex w is in H_1.

The interesting property of this relation is that any triangle-induced subgraph of a triangular line graph is also a triangular line graph.

Lemma 2.2. *Let H_1, H_2 be triangular line graphs, with triangular line graph preimages and edge-to-vertex bijections given by*

$$\mathcal{G}_j = \{(G, f) : T(G) \cong H_j \text{ via the bijection } f : E(G) \to V(H_j)\}, \text{ for } j \in \{1, 2\}.$$

If $H_1 \lhd H_2$, then for any preimage $(G_2, f_2) \in \mathcal{G}_2$, there exists a preimage $(G_1, f_1) \in \mathcal{G}_1$ so that G_1 is isomorphic to a subgraph of G_2 using the edge map $f_2^{-1} \circ f_1 : E(G_1) \to E(G_2)$.

Proof. Let $(G_2, f_2) \in \mathcal{G}_2$ be a preimage for a triangular line graph H_2. Set G_1 to be the graph induced by edge set $F = f_2^{-1}(V(H_1))$. It is clear that the adjacencies in $T(G_1) \subseteq H_2$ are also adjacencies in $T(G_2) = H_2$, as triangles in G_1 are triangles in G_2. Let e_1 and e_2 be edges in G_1 that are adjacent vertices in H_1. There is an edge e_3 in G_2 that completes the triangle with e_1 and e_2. Since e_1 and e_2 are vertices in H_1, and H_1 is a triangle-induced subgraph of H_2, then e_3 must also be in H_1, as these vertices form a triangle in H_2. Hence, e_3 is in G_1 and e_1 and e_2 are adjacent in $T(G_1)$. To complete the proof, define $f_1 = f_2|_{E(G_1)}$. ∎

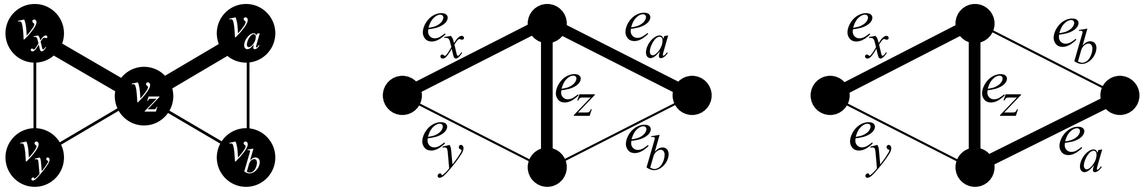
It follows from this lemma that the class of triangular line graphs is closed under taking triangle-induced subgraphs.

Corollary 2.3. *Let G be any graph and $H = T(G)$ its triangular line graph. Any triangle-induced subgraph $H' \lhd H$ is also a triangular line graph with preimage G' that is a subgraph of G.*

Given a graph G and its line graph $H = T(G)$, we use the notation $T^{-1}(H')$ on any triangle-induced subgraph $H' \lhd H$ to denote the edges in G that map to the vertices of H'. We proceed to define gadgets which become triangle-induced subgraphs of the final graph. By putting restrictions on the class of preimages of each gadget, we prove that the larger graph has a preimage if and only if each gadget satisfies their individual preimage restrictions. Put together, these gadgets encode the logic of a satisfied 3-CNF formula.

3 Cycles, Wheels, and Suns

This section describes a class of triangular line graphs with multiple preimages. The smallest unit of these graphs is the *bowtie*: the graph consisting of two triangles intersecting at a single vertex, called the *center*. The bowtie is the triangular line graph of the *double triangle*: $K_4 - e$. While this is the only preimage (up to isomorphism), there are two different classes of labeled preimages, depending on the incidences of edges between the triangles.

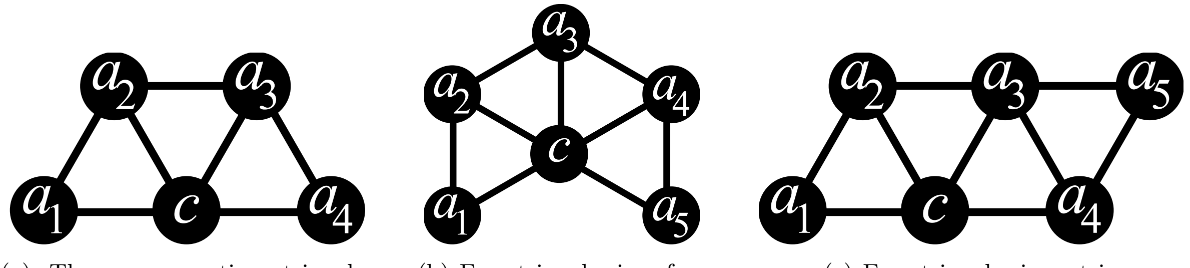


(a) The bowtie. (b) First preimage of the bowtie. (c) Second preimage of the bowtie.

Figure 1: The bowtie and its preimages with three highlighted edges.

Consider two nonadjacent vertices in the bowtie. They are both adjacent to the center, so their edges in the preimage are adjacent to the center edge. The two preimages differ depending on whether or not these edges are incident to each other. See Figure 1 which shows these preimages.

A *triangle trail* is a sequence of triangles where consecutive triangles intersect at an edge. There are two special triangle trails of length k, for $k \geq 3$. The first is the k-fan, where all triangles have a common vertex, but are otherwise disjoint other than the edge between consecutive triangles. The second is the k-triangle strip, where triangles of distance two intersect at a vertex, but triangles of larger distance are disjoint. For $k = 3$, these definitions coincide, and Figure 2 demonstrates the difference at $k = 4$.



(a) Three consecutive triangles make a fan. (b) Four triangles in a fan. (c) Four triangles in a strip

Figure 2: Fans and triangle strips.

Notice that the k-triangle strip and k-fan have isomorphic triangular line graphs composed of a sequence of $k - 1$ bowties where consecutive bowties intersect at a triangle and are otherwise disjoint. By closing these sequences into cyclic orders, we form the base of our gadget constructions.

Definition 3.1. Let $k \geq 4$ be an integer.

- The *k-wheel*, W_k, is the graph formed by a k-cycle and a dominating vertex, called the *center* of the wheel. Edges incident to the center are called *spokes*.
- The *squared k-cycle*, C_k^2, is the graph given by a k-cycle with additional edges added for vertex pairs at distance two in the k-cycle.
- The *k-sun*, S_k, is the graph given by a k-cycle with a new vertex for each edge, adjacent to the endpoints.

Lemma 3.2. *For $k \geq 7$, the triangular line graphs of the k-wheel W_k and squared cycle C_k^2 are both isomorphic to the k-sun S_k. The 7-wheel W_7 and squared cycle C_7^2 are the only two preimages of the 7-sun S_7 up to isomorphism.*

Proof. The first part of the lemma is easy to verify. The second part uses case analysis and is found in Appendix A on page 13. ■

It is necessary for later arguments to understand some structure of the squared cycle C_k^2.

Observation 3.3. *Let $k \geq 7$ be an integer.*

- *If k is even, then C_k^2 is embeddable on the cylinder.*
- *If k is odd, then C_k^2 is embeddable on the Möbius strip.*

The Möbius strip embedding of C_7^2 is given below in Figure 3.

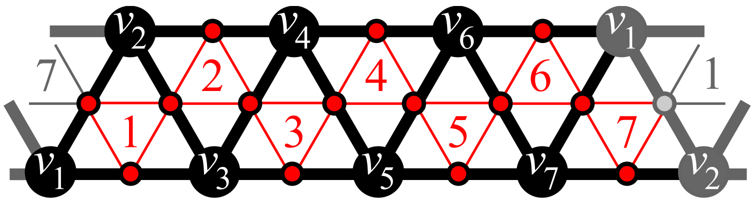


Figure 3: Embedding C_7^2 on the Möbius strip. The triangular line graph S_7 is shown in red.

The two preimages of the 7-sun are very useful for encoding binary value. Larger k-suns have more than two preimages, but it is possible to constrain larger k-suns to have exactly two preimages by embedding 7-suns into the k-sun.

Definition 3.4. Let $k \geq 9$ be an integer. Define the *binary-enforced k-sun*, S_k^b, to be the graph built as follows:

1. Start with a k-sun, S_k.
2. Label the vertices in the k-cycle as $v_1, \ldots, v_k$, with subscripts taken modulo k.
3. For each $i \in \{1, \ldots, k\}$, add a sequence of three triangles between vertices v_i and v_{i+4}, forming a 7-sun (see Figure 4(c) for the case when $k = 12$).

Lemma 3.5. *Let $k \geq 9$ be an integer. A binary-enforced k-sun has exactly two preimages. In the first preimage, the k-sun has a k-wheel preimage and all 7-suns have 7-wheel preimages with a common center vertex. In the second preimage, the k-sun has a squared cycle preimage and all 7-suns have squared cycle preimages.*

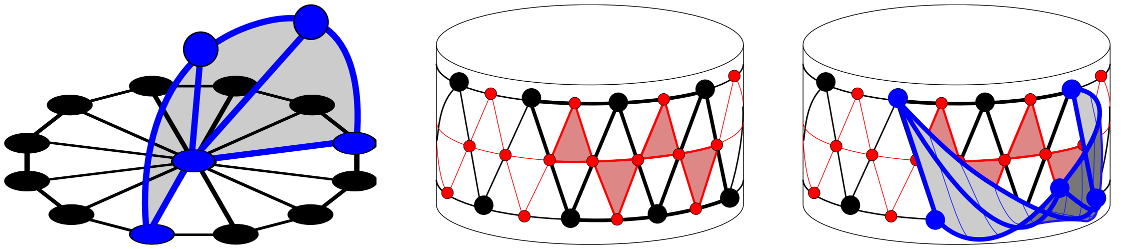
Proof. Suppose S_k^b is a triangular line graph with some preimage. Note that each 7-sun is a triangle-induced subgraph of S_b^k so by Lemmas 2.2 and 3.2, it preimage must be a 7-wheel or a squared cycle. Each sequence of four consecutive triangles in the k-sun forms either a 4-fan or 4-strip in the preimage, depending on whether the preimage of the enclosing 7-sun is a wheel or squared cycle, respectively.

Consider a single 7-sun, S. If S has a 7-wheel preimage, then the four triangles on the k-sun form a 4-fan, with the vertices on the k-cycle corresponding to edges incident to the center of the fan. An adjacent 7-sun S' shares three consecutive triangles with S. The last triangle in the k-cycle and S' (but not in S) has a vertex whose corresponding edge in the preimage is incident to the center of the 4-fan in $T^{-1}(S)$. Hence, the preimage of this triangle and the three triangles in both S and S' create a 4-fan and forces S' to have a wheel preimage. By iterating through all consecutive wheels, all 7-suns in S_b^k have wheel preimages. Hence, the 7-suns have either all wheel preimages or all squared cycle preimages.

If all 7-suns have wheel preimages, then each set of four consecutive triangles have 4-fan preimages. Two consecutive 4-fans in the preimage share the center with the common 3-fan. By iterating through consecutive 4-fans around the k-cycle, all 7-wheel preimages share a common center. Hence, the k-sun preimage must be a wheel.

If all 7-suns have squared cycle preimages, then the consecutive 4-strip preimages intersect at a 3-fan. The edges of the 3-fan which intersect the other triangles of the 4-strips are not incident to the center of the 3-fan. Hence, every sequence of five triangles in the k-sun have triangle-strip preimages. Iterating on all lengths $\ell \in \{5, 6, \ldots, k-1\}$ shows the k-sun must have a squared-cycle preimage, since all sequences of ℓ triangles have triangle-strip preimages.

It remains to show that these two possibilities are realizable with actual preimages. The all-wheels case can be viewed as a k-wheel, W_k, placed in the plane, with 3-fans glued between the ends of 4-fans in the wheel, sharing the center vertex. See Figure 4(a) for an example of this process.


(a) W_{12} with a gluing.

(b) C_{12}^2 and $T(C_{12}^2)$ embedded on a cylinder.

(c) C_{12}^2 and $T(C_{12}^2)$ with twisted gluing.

Figure 4: Visualizing S_{12}^b and its possible preimages

The all-cycles case can be viewed as the squared cycle, C_k^2, embedded on either the cylinder or Möbius strip (depending on the parity of k, see Observation 3.3). Then, a 3-fan can be glued to the last edges of each 4-triangle strip, adding a twist to make a Möbius strip. Figures 4(b) and 4(c) demonstrate this gluing procedure for $k = 12$. ■

Armed with these structures, we proceed to define gadgets that encode the logic of a boolean formula.

4 Encoding a Boolean Formula

Let $\phi = \wedge_{j=1}^{m} C_j$ be a formula in conjunctive normal form with m clauses $C_1, \ldots, C_m$ on the variables $x_1, \ldots, x_n$ where each clause has size three. Label each clause as $C_j = u_{i_{j,1}} \vee u_{i_{j,2}} \vee u_{i_{j,3}}$, where each $u_{i_{j,k}}$ is a variable $x_{i_{j,k}}$ or its complement $\overline{x}_{i_{j,k}}$. For our reduction, we construct graph gadgets that form a graph G_ϕ which is a triangular line graph if and only if ϕ is a satisfiable formula.

We begin by defining a gadget which stores a binary value.

4.1 Variable Gadgets

Begin by assigning variables to distinct copies of a 7-sun. The two preimages of a 7-sun correspond to the two possible values of a variable. If the preimage is a 7-wheel, W_7, then that variable is assigned a false value, and if it is a squared cycle, C_7^2, then that variable is assigned a true value.

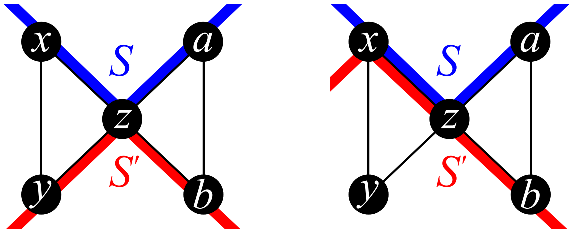
Definition 4.1 (Variable Gadget)**.** For each variable x_i, create a 7-sun labeled H_{x_i}. Associate the value of x_i with the preimage of H_{x_i} as $x_i = 1$ if and only if H_{x_i} has a C_7^2 preimage.

There are two different ways to connect two 7-suns. One connection guarantees the two preimages are isomorphic while the other connection ensures the preimages are not isomorphic. The connections act as logic gates in the graph.

Definition 4.2 (Equal Gadget)**.** Let S and S' be two 7-suns and let B and B' be bowties in S and S', respectively. Identify the vertices of the bowties B and B' in the following way:

1. Identify the centers of the bowties B and B'.
2. In one triangle of B and in one triangle of B', identify the vertices of different degrees in S and S'. (Here, the vertex on the 7-cycle in S is identified with the vertex not on the 7-cycle in S'.)
3. Repeat (2) for the remaining triangles in B and B'.

Figure 5(a) shows how the bowties are identified between two variable gadgets joined by an Equal gadget by highlighting which vertices are associated with the inner cycle of each 7-sun.



(a) The Equal gadget. (b) The Not gadget.

Figure 5: The Equal and Not gadgets.

Definition 4.3 (Not Gadget)**.** Let S and S' be two 7-suns and let B and B' be bowties in S and S', respectively. Identify the vertices of the bowties B and B' in the following way:

1. Identify the centers of the bowties.
2. In one triangle of B and in one triangle of B', identify the vertices of the same degree in S and S'. (Here, the vertices lying on the 7-cycles of S and S' are identified.)

3. In the remaining triangles of B and B', identify the vertices of different degrees in S and S'. (Here, the vertex on the k-cycle in S are identified with the vertex not on the k-cycle in S'.)

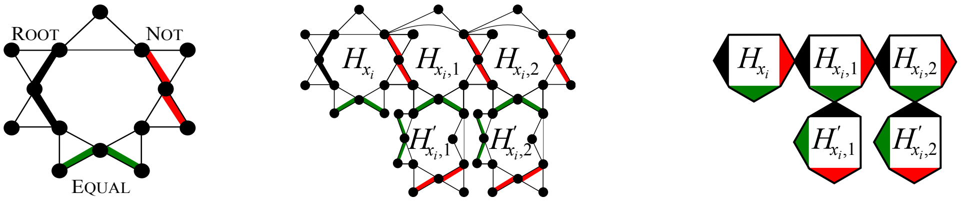
Figure 5(b) shows how the bowties are identified between two variable gadgets joined by a NOT gadget by highlighting which vertices are associated with the inner cycle of each variable gadget.

Lemma 4.4. *Consider two 7-suns S and S' that intersect at a bowtie.*

1. *If the intersection is an* EQUAL *gadget, then $T^{-1}(S)$ is isomorphic to $T^{-1}(S')$.*
2. *If the intersection is a* NOT *gadget, then $T^{-1}(S)$ is not isomorphic to $T^{-1}(S')$.*

Proof. Note that S and S' are triangle-induced subgraphs of the combined graph. By Lemmas 2.2 and 3.2, the preimage of each is either a W_7 or a C_7^2. To show that only the proper two of the four combinations are possible, we use case analysis as shown in Appendix B on page 14. ∎

In order to provide multiple connection points for clauses to access the value of x_i or $\overline{x}_i$, we connect a sequence of 7-suns to each 7-sun H_{x_i}. To that end, we assign three bowties within each 7-sun two of which are for special use with the EQUAL and NOT gadgets. One bowtie is labeled ROOT which is used to connect to other variable gadgets via EQUAL or NOT gadgets, one is labeled EQUAL which is reserved for other variable gadgets to connect via an EQUAL gadget, and one is labeled NOT which is reserved for other variable gadgets to connect via a NOT gadget. Figure 6 shows the three attachment points and a symbolic representation of the connections.



(a) The attachment points

(b) How they attach

(c) A symbolic representation

Figure 6: The ROOT , NOT , and EQUAL attachment points of a 7-sun and how they attach.

Definition 4.5 (Wire)**.** A *wire* is a sequence $H_0, H_1, \ldots, H_k$ of 7-suns such that for $i \in \{1, 2, \ldots, k\}$, H_i is attached at the ROOT to H_{i-1} via a NOT gadget.

Observe that the preimage of a wire is a set of overlapping W_7s and C_7^2s so that $T^{-1}(H_i) \cong T^{-1}(H_0)$ if and only if i is even. Using a wire as a central line storing the value of a variable x_i and its complement $\overline{x}_i$, we build a larger structure (*variable cluster*) that allows connections to other variables via clause gadgets.

4.2 Variable Clusters

For each variable x_i, form a wire of length $2m+1$: $H_{x_i}, H_{x_i,1}, \ldots, H_{x_i,2m}$. For each $j \in \{1, 2, \ldots, 2m\}$, attach a 7-sun $H'_{x_i,j}$ via an EQUAL gadget to the 7-sun $H_{x_i,j}$ in the wire (see Figures 6(b) and 6(c) for an example when $m = 1$). Note that $T^{-1}(H'_{x_i,j}) \cong T^{-1}(H_{x_i,j})$ for all j with $1 \le j \le 2m$. Hence, $T^{-1}(H'_{x_i,j}) \cong T^{-1}(H_{x_i})$ if and only if j is even. Consider the value stored by $T^{-1}(H'_{x_i,j})$ to be x_i if j is even and $\overline{x_i}$ when j is odd.

Our construction of variable clusters is almost complete. However, we need to make these variable clusters interact with each other via clause gadgets that allow all the satisfying cases in the truth table of a clause. To allow all true cases, the clause requires larger suns than the 7-sun, but requires the suns to have only the two canonical preimages. Hence, we must incorporate the binary-enforced k-sun into our construction for large enough k. It is sufficient to have $k = 12$, as we will see in Section 5.

Definition 4.6 (Large Variable Gadget)**.** Given a variable x_i and integer $j \in \{1, \ldots, m\}$, the *large variable gadget* $V_{x_i,j}$ is given as a binary-enforced 12-sun, with one of its 7-suns labeled $H'_{x_i,j}$.

Lemma 3.5 demonstrated that a large variable gadget has exactly two preimages. To complete the variable cluster for x_i, take the $2m$ large variable gadgets $V_{x_i,1}, \ldots, V_{x_i,2m}$ and identify each $H'_{x_i,j}$ with the 7-sun having the same subscript in the the wire gadget for x_i.

Definition 4.7 (Variable Clusters)**.** Each variable x_i is represented by a *variable cluster* composed of a wire $H_{x_i}, H_{x_i,1}, \ldots, H_{x_i,2m}$ of length $2m + 1$, variable clusters $V_{x_i,1}, \ldots, V_{x_i,2m}$ with the 7-suns $H'_{x_i,j}$ attached via EQUAL gadgets to $H_{x_i,j}$ in the wire.

Lemma 4.8. *A variable cluster has exactly two preimages:*

(a) H_{x_i} has a wheel preimage while $H_{x_i,j}$ has a wheel preimage when j is even and a squared cycle preimage when j is odd. The variable gadget $V_{x_i,j}$ has a preimage that contains a 12-wheel if and only if j even.

(b) H_{x_i} has a squared cycle preimage while $H_{x_i,j}$ is a squared cycle preimage when j is even and a wheel preimage when j is odd. The variable gadget $V_{x_i,j}$ has a preimage that contains a 12-wheel if and only if j is odd.

Proof. Since each H_{x_i}, $H_{x_i,j}$, or $H'_{x_i,j}$ is a triangle-induced subgraph isomorphic to a 7-sun, its preimage is either a wheel or a squared cycle by Lemmas 2.2 and 3.2. By the properties of the NOT gadget (Lemma 4.4), the preimages of $H_{x_i,j}$ in the wire alternate between a wheel and a squared cycle depending on the preimage of H_{x_i}. Since $H'_{x_i,j}$ is connected to $H_{x_i,j}$ via an EQUAL gadget, the preimage of $H'_{x_i,j}$ is isomorphic to the preimage of $H_{x_i,j}$. Moreover, the preimage of $H'_{x_i,j}$ determines the preimage of $V_{x_i,j}$ by Lemma 3.5. ■

Figure 7 shows a diagram of the variable cluster for x_i and the associated variable value correspondence.

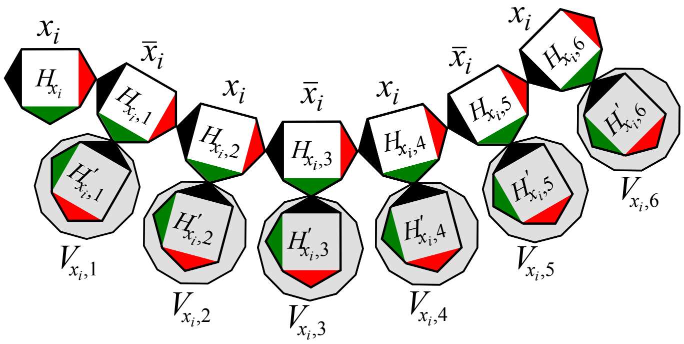


Figure 7: The variable cluster for x_i with $m = 3$.

5 Enforcing Clause Satisfaction

It remains to show how to encode each clause C_j of ϕ into a graph H so that H is a triangular line graph if and only if ϕ is satisfiable. The method combines large variable gadgets using clause gadgets so that the large variable gadgets corresponding to the three literals in a clause cannot all have wheels as preimages. This prevents a preimage of H from existing unless the variable gadget preimages correspond to an assignment of the variables that satisfies ϕ.

Consider three 12-suns G_1, G_2, and G_3. For $\ell \in \{1, 2, 3\}$, let $\langle a_{\ell,1}, a_{\ell,2}, a_{\ell,3}\rangle$ and $\langle b_{\ell,1}, b_{\ell,2}, b_{\ell,3}\rangle$ be two triangles of maximum distance apart in G_ℓ where $a_{\ell,3}$ and $b_{\ell,3}$ are vertices of degree 2 in G_ℓ. It follows that that are five triangles between $\langle a_{\ell,1}, a_{\ell,2}, a_{\ell,3}\rangle$ and $\langle b_{\ell,1}, b_{\ell,2}, b_{\ell,3}\rangle$ in G_ℓ for each $\ell \in \{1, 2, 3\}$ and $a_{\ell,1}, a_{\ell,2}, b_{\ell,1}$ and $b_{\ell,2}$ all lie on the 12-cycle of G_ℓ. Assume that the vertices on the 12-cycle of G_ℓ appear in the order $a_{\ell,1}, a_{\ell,2}, b_{\ell,1}, b_{\ell,2.}$ Identify triangles by identifying the vertices (a) $a_{\ell,1}$ with $b_{\ell+1,1}$, (b) $a_{\ell,2}$ with $b_{\ell+1,3}$, and (c) $a_{\ell,3}$ with $b_{\ell+1,2}$ for $\ell \in \{1, 2, 3\}$ (with subscripts taken modulo 3). This construction is shown in Figure 8.

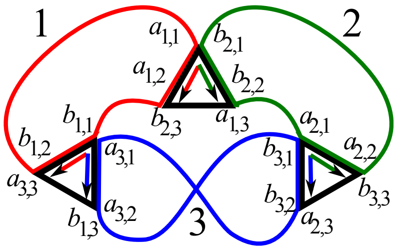


Figure 8: Twisting three 12-suns together

Definition 5.1 (Clause Gadget)**.** Let $C_j = u_{i_{j,1}} \vee u_{i_{j,2}} \vee u_{i_{j,3}}$ be the jth clause in ϕ. If $u_{i_{j,\ell}} = x_{i_{j,\ell}}$, set $k_\ell = 2j$, otherwise $u_{i_{j,\ell}} = \overline{x}_{i_{j,\ell}}$ and set $k_\ell = 2j - 1$, $(\ell \in \{1, 2, 3\}$. The *clause gadget* for C_j joins the 12-suns within the three large variable gadgets $V_{x_{i_{j,1}},k_1}, V_{x_{i_{j,2}},k_2}, V_{x_{i_{j,3}},k_3}$ using the construction above.

The important property of this clause gadget is that the possible preimages correspond to the satisfying assignments of a disjunction of three literals.

Lemma 5.2. *Let G_1, G_2, and G_3 be the 12-suns used in a clause gadget. If the clause gadget has a preimage, then at least one of G_1, G_2 and G_3 has C_{12}^2 as a preimage.*

Proof. The existence of preimages to the clause gadget with zero, one, or two wheels is shown by the explicit construction found in Appendix C and Tables 1-4. It remains to show that the clause gadget does not have a preimage where each G_ℓ has a wheel preimage. We will show that this is not a triangular line graph, as more adjacencies are required.

Suppose for the sake of contradiction that there exists a preimage of the clause gadget where each 12-sun has a wheel preimage. Since the preimage of the induced 12-sun G_ℓ is a wheel, the vertices $a_{\ell,1}$, $a_{\ell,2}$, $b_{\ell,1}$, and $b_{\ell,2}$ correspond to spokes in $T^{-1}(G_\ell)$ for $\ell \in \{1, 2, 3\}$. These spokes share a common endpoint which is the center of the wheel $T^{-1}(G_\ell)$. Since $a_{1,1} = b_{2,1}$, $a_{2,1} = b_{3,1}$, and $a_{3,1} = b_{1,1}$, it follows that the centers of the wheels $T^{-1}(G_1)$, $T^{-1}(G_2)$, and $T^{-1}(G_3)$ induce a triangle in the preimage. However, $a_{1,1}$ is not adjacent to either $a_{2,1}$ nor $a_{3,1}$ in the clause gadget, forming a contradiction. ∎

With this clause gadget, we prove the main theorem.

Theorem 5.1. Recognizing if a graph is a triangular line graph is NP-complete.

Proof. Given a graph H on n vertices $v_1, \dots, v_n$, non-deterministically select pairs $e_1, \dots, e_n$ from the set $\{1, \dots, 2n\}$ forming a graph $G = (\{1, \dots, 2n\}, \{e_1, \dots, e_n\})$. Then, check that the adjacencies of $e_i e_j$ in $T(G)$ match those of $v_i v_j$ in H, for each $i, j \in \{1, \dots, n\}$. This takes polynomial time, so the problem is in NP.

To show hardness, we reduce from 3-SAT by converting a 3-CNF formula ϕ on n variables and m clauses into the graph G_ϕ:

1. For each variable x_i, $i = 1, 2, \dots, n$, form a variable cluster using a wire of length $2m + 1$.
2. Combine these n variable clusters via the clause gadgets for clauses $C_1, C_2, \dots, C_m$ to obtain graph G_ϕ.

This process can be done in polynomial time. Note that each gadget is a triangle-induced subgraph of G_ϕ and so Lemma 2.2 restricts the preimage of G_ϕ to include only allowed preimages of each gadget.

We now show that ϕ is satisfiable if and only if G_ϕ is a triangular line graph. Given an assignment $\mathbf{x}$, consider a possible preimage for G_ϕ by setting the variable cluster for x_i to have $T^{-1}(H_{x_i}) = W_7$ if $x_i = 0$ and $T^{-1}(H_{x_i}) \cong C_7^2$ if $x_i = 1$. The rest of the preimage of the variable cluster for x_i propagates according to Lemma 4.8. If $\mathbf{x}$ satisfies ϕ, then every clause C_j is simultaneously satisfied. Hence, the clause gadget for C_j has at least one satisfying variable whose large variable gadget has preimage containing the squared cycle C_{12}^2. Thus, each clause gadget has a preimage as well.

On the other hand, if there is a triangular line graph preimage for G_ϕ, then each clause gadget has at least one large variable gadget whose preimage contains the squared cycle C_{12}^2. Hence, a satisfying assignment $\mathbf{x}$ can be formed by setting each $x_i = 1$ if and only if $T^{-1}(H_{x_i}) \cong C_7^2$. ∎

This shows that the general problem is NP-complete, but our construction shows hardness for a special class of triangular line graphs.

Corollary 5.2. Let $\mathcal{G}$ be the class of graphs G where each edge in $E(G)$ is in exactly one triangle in G. Deciding if a graph $G \in \mathcal{G}$ is a triangular line graph is NP-complete.

Proof. The construction of Theorem 5.1 produced a graph G_ϕ which is a triangular line graph if and only if ϕ is a satisfiable 3-CNF formula. This G_ϕ has every edge in a unique triangle, so G_ϕ is in the class $\mathcal{G}$, and the above reductions works for this problem. ∎

This corollary is interesting since if a preimage exists in this class, it is immediately clear which edges form triangles and which subgraphs are triangle-induced subgraphs. In this class, preimages have no cliques of size four.

6 Conclusion

The hardness of recognizing triangular line graphs is perhaps surprising given the polynomial-time algorithms which recognize line graphs. This hardness is consistent with the complexity of Le's characterization, which is unlikely to be substantially simplified for triangular line graphs. It is interesting to consider Gallai graphs, which are constructed from line graphs and triangular line graphs by $\Gamma(G) = L(G) - E(T(G))$. Even though these graphs are closely related to triangular line graphs, the constructions and proofs in this work do not immediately generalize to show the hardness of recognizing these graphs. The complexity for other generalizations, such as k-Gallai and k-anti-Gallai graphs, or k-in-ℓ graphs, remains unknown.

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A Proof of Lemma 3.2

Lemma 3.2. *For $k \geq 7$, the triangular line graphs of the k-wheel W_k and squared cycle C_k^2 are both isomorphic to the k-sun S_k. The 7-wheel W_7 and squared cycle C_7^2 are the only two preimages of the 7-sun S_7 up to isomorphisms.*

Proof (continued from page 5). Let G be a preimage of S_7. As mentioned above, G is composed of seven triangles where each triangle intersects exactly two triangles at two distinct shared edges. Note that G cannot contain an $n-$wheel where $n \leq 6$. Moreover, since $T(G) \cong S_7$ has exactly 14 vertices, it follows that G has 14 edges. We consider two cases:

Case 1: G does not contain a 4-fan.

It follows that each set of four consecutive triangles in G forms an alternating strip of triangles, as in Figure 2(c). Put together, we find that all seven triangles can be placed in a strip of seven alternating triangles, given by vertices $v_1, v_2, \ldots, v_9$, where v_i is adjacent to $v_{i-2}, v_{i-1}, v_{i+1}, v_{i+2}$. The edges v_1v_2 and v_8v_9 must be identified to close the cycle in the 7-sun. If v_2 and v_8 are identified, the vertices $v_2v_4v_6$ form a triangle which is not represented in the 7-sun. Hence, v_1 and v_8 are identified and the vertices form a squared cycle C_7^2.

Case 2: G contains a 4-fan.

It follows that there are four consecutive triangles in G that share a common vertex c. Let $a_1, \ldots, a_5$ be the endpoints around this 4-fan centered at c as in Figure 2(b). Since the triangles $\langle a_1, c, a_2 \rangle$ and $\langle a_4, c, a_5 \rangle$ intersect other triangles at the edges ca_2 and ca_4, respectively, these edges cannot be used to intersect $\langle a_1, c, a_2 \rangle$ and $\langle a_4, c, a_5 \rangle$ with other adjacent triangles. We consider two cases according to which edges of $\langle a_1, c, a_2 \rangle$ and $\langle a_4, c, a_5 \rangle$ are shared with neighboring triangles (different from the triangles that are in the 4-fan shown in Figure 2(b)).

Case 2.1: Both edges ca_1 and ca_5 are shared with neighboring triangles.

We have three possibilities. If $a_1a_5 \in E(G)$, then G contains a *5-wheel* which is impossible. If a_1, a_5 and c are adjacent to some new vertex x, then G contains a *6-wheel* which is impossible. Lastly, if a_1 and c are adjacent to some new vertex x, and a_5 and c are adjacent to some new vertex $y \neq x$, then we have accounted for 13 edges of G so far. Since G has 14 edges, it follows that $xy \in E(G)$ and so $G \cong W_7$.

Case 2.2: Only one of the edges ca_1 and ca_5 is shared with neighboring triangles.

Without loss of generality, let the edge ca_1 be shared with a neighboring triangle of $\langle a_1, c, a_2 \rangle$. It follows that the edge a_4a_5 must be shared with a neighboring triangle of $\langle a_4, c, a_5 \rangle$. Since no edge a_ia_j $(1 \leq i \neq j \leq 5)$ can be introduced as we would obtain a W_n with $n \leq 6$, it follows that both a_1 and c must adjacent to a new vertex x and both a_4 and a_5 must be adjacent to a new vertex $y \neq x$. Since we have already accounted for 13 of the 14 edges of G, we do not have enough edges left to form the other neighboring triangles of $\langle a_1, c, x \rangle$ and $\langle a_4, a_5, y \rangle$. ■

B Proof of Lemma 4.4

Lemma 4.4. Consider two 7-suns S and S' that intersect at a bowtie.

1. If the intersection is an Equal gadget, then $T^{-1}(S)$ is isomorphic to $T^{-1}(S')$.
2. If the intersection is a Not gadget, then $T^{-1}(S)$ is not isomorphic to $T^{-1}(S')$.

Proof. Let x, y, z, a, b be the vertices in the bowtie of the gadget between S and S' as given in Figure 5 and let e_x, e_y, e_z, e_a and e_b, respectively, be the corresponding edges in G. Recall that $T^{-1}(S)$ and $T^{-1}(S')$ are isomorphic to either the 7-wheel W_7 or the squared cycle C_7^2.

Case 1: S and S' intersect at an Equal gadget.

Consider the case where $T^{-1}(S)$ is a wheel. Then, the edges e_x, e_z, and e_a are spokes of $T^{-1}(S)$ while e_y and e_b are consecutive edges on the 7-cycle of $T^{-1}(S)$, both of which are incident to e_z at the same vertex. Hence, the edges e_y and e_b are incident. Since y and b each have degree 4 in S' and e_y and e_b are incident in G (and therefore also in $T^{-1}(S')$), it follows that $T^{-1}(S')$ is not a squared cycle. Thus, $T^{-1}(S')$ is the wheel W_7.

On the other hand, if $T^{-1}(S)$ is a squared cycle, then the edges e_x and e_a are not incident in G. Moreover, the edges e_y and e_b are not incident in G. Hence, $T^{-1}(S')$ is not a wheel and must be the squared cycle C_7^2.

Case 2: S and S' intersect at a Not gadget.

Consider the case where $T^{-1}(S)$ is a wheel. Then the edges e_x, e_z, and e_a are spokes and e_b is on the 7-cycle of $T^{-1}(S)$. Hence, the edges for e_x and e_b are not incident in G (and therefore also in $T^{-1}(S')$). It follows that e_x and e_b do not form spokes in $T^{-1}(S')$ and so $T^{-1}(S')$ is not a wheel. Hence, $T^{-1}(S')$ is the squared cycle C_7^2.

Now assume that $T^{-1}(S)$ is a squared cycle C_7^2. Then edges e_x, e_y, and e_z form a triangle, e_a is incident to e_z but not e_x, and e_b is incident to e_x in G. This means that e_x, e_b, and e_z share a common vertex in G. Moreover, e_x, e_b and e_z are incident in G while e_y and e_b are not. Thus, $T^{-1}(S')$ is not the squared cycle C_7^2 (since if $T^{-1}(S')$ is a squared cycle, then e_b and e_y would be incident while e_b and e_x would not be). Consequently, $T^{-1}(S')$ is isomorphic to the 7-wheel W_7. ■

C Clause Gadgets and Preimages

To construct the clause gadget explicitly, we must first describe three 12-suns, S_1, S_2, S_3. Each S_ℓ has 24 vertices, $S_{\ell,i}$ for $i \in \{0, \dots, 23\}$. The first index of $S_{\ell,i}$ is modulo three while the second index is modulo 24. The even i vertices $S_{\ell,i}$ are in the 12-cycle and are adjacent to vertices $S_{\ell,j}$ where $j \in \{i-2, i-1, i+1, i+2\}$. The odd i vertices have degree two and are adjacent to vertices $S_{\ell,j}$ where $j \in \{i-1, i+1\}$.

The triangles in S_ℓ used for the clause gadget are given by

$$\begin{aligned} a_{\ell,1} &= S_{\ell,0} & \qquad b_{\ell,1} &= S_{\ell,12} \\ a_{\ell,2} &= S_{\ell,2} & \qquad b_{\ell,2} &= S_{\ell,14} \\ a_{\ell,3} &= S_{\ell,1} & \qquad b_{\ell,3} &= S_{\ell,13} \end{aligned}$$

Then, by merging the a-triangle in S_ℓ to the b-triangle in $S_{\ell+1}$, the following vertices are identified by equality:

$$\begin{aligned} S_{\ell,0} = a_{\ell,1} = b_{\ell+1,1} = S_{\ell+1,12} \\ S_{\ell,2} = a_{\ell,2} = b_{\ell+1,3} = S_{\ell+1,13} \\ S_{\ell,1} = a_{\ell,3} = b_{\ell+1,2} = S_{\ell+1,14} \end{aligned}$$

The full adjacency list for the clause gadget given by these S_1, S_2, S_3 is given in Table 1 on page 16. Preimages for the clause gadget are given in the following tables. Each of these preimages labels the adjacencies with the corresponding vertex in the clause gadget that edge represents.

- Table 2 on page 17 shows the preimage with zero wheels and three cycles.
- Table 3 on page 18 shows the preimage with one wheel and two cycles.
- Table 4 on page 19 shows the preimage with two wheels and one cycle.

Vertex	Adjacencies			
$S_{1,0\equiv 2,12}$	$S_{1,1\equiv 2,14}$	$S_{1,23}$	$S_{2,10}$	$S_{2,11}$
	$S_{1,22}$	$S_{1,2\equiv 2,13}$		
$S_{1,1\equiv 2,14}$	$S_{1,0\equiv 2,12}$	$S_{1,2\equiv 2,13}$	$S_{2,16}$	$S_{2,15}$
$S_{1,2\equiv 2,13}$	$S_{1,0\equiv 2,12}$	$S_{1,1\equiv 2,14}$	$S_{1,4}$	$S_{1,3}$
$S_{1,3}$	$S_{1,2\equiv 2,13}$	$S_{1,4}$		
$S_{1,4}$	$S_{1,2\equiv 2,13}$	$S_{1,6}$	$S_{1,5}$	$S_{1,3}$
$S_{1,5}$	$S_{1,6}$	$S_{1,4}$		
$S_{1,6}$	$S_{1,8}$	$S_{1,7}$	$S_{1,5}$	$S_{1,4}$
$S_{1,7}$	$S_{1,8}$	$S_{1,6}$		
$S_{1,8}$	$S_{1,7}$	$S_{1,6}$	$S_{1,10}$	$S_{1,9}$
$S_{1,9}$	$S_{1,8}$	$S_{1,10}$		
$S_{1,10}$	$S_{3,0\equiv 1,12}$	$S_{1,8}$	$S_{1,11}$	$S_{1,9}$
$S_{1,11}$	$S_{3,0\equiv 1,12}$	$S_{1,10}$		
$S_{1,15}$	$S_{3,1\equiv 1,14}$	$S_{1,16}$		
$S_{1,16}$	$S_{3,1\equiv 1,14}$	$S_{1,18}$	$S_{1,17}$	$S_{1,15}$
$S_{1,17}$	$S_{1,18}$	$S_{1,16}$		
$S_{1,18}$	$S_{1,20}$	$S_{1,17}$	$S_{1,16}$	$S_{1,19}$
$S_{1,19}$	$S_{1,20}$	$S_{1,18}$		
$S_{1,20}$	$S_{1,19}$	$S_{1,21}$	$S_{1,18}$	$S_{1,22}$
$S_{1,21}$	$S_{1,20}$	$S_{1,22}$		
$S_{1,22}$	$S_{1,0\equiv 2,12}$	$S_{1,20}$	$S_{1,21}$	$S_{1,23}$
$S_{1,23}$	$S_{1,0\equiv 2,12}$	$S_{1,22}$		
$S_{2,0\equiv 3,12}$	$S_{2,1\equiv 3,14}$	$S_{2,22}$	$S_{2,23}$	$S_{3,10}$
	$S_{2,2\equiv 3,13}$	$S_{3,11}$		
$S_{2,1\equiv 3,14}$	$S_{2,0\equiv 3,12}$	$S_{2,2\equiv 3,13}$	$S_{3,15}$	$S_{3,16}$
$S_{2,2\equiv 3,13}$	$S_{2,0\equiv 3,12}$	$S_{2,1\equiv 3,14}$	$S_{2,3}$	$S_{2,4}$
$S_{2,3}$	$S_{2,2\equiv 3,13}$	$S_{2,4}$		
$S_{2,4}$	$S_{2,2\equiv 3,13}$	$S_{2,3}$	$S_{2,5}$	$S_{2,6}$
$S_{2,5}$	$S_{2,4}$	$S_{2,6}$		
$S_{2,6}$	$S_{2,4}$	$S_{2,5}$	$S_{2,7}$	$S_{2,8}$
$S_{2,7}$	$S_{2,6}$	$S_{2,8}$		
$S_{2,8}$	$S_{2,10}$	$S_{2,6}$	$S_{2,7}$	$S_{2,9}$
$S_{2,9}$	$S_{2,10}$	$S_{2,8}$		
$S_{2,10}$	$S_{1,0\equiv 2,12}$	$S_{2,11}$	$S_{2,8}$	$S_{2,9}$
$S_{2,11}$	$S_{1,0\equiv 2,12}$	$S_{2,10}$		
$S_{2,15}$	$S_{1,1\equiv 2,14}$	$S_{2,16}$		
$S_{2,16}$	$S_{1,1\equiv 2,14}$	$S_{2,18}$	$S_{2,17}$	$S_{2,15}$
$S_{2,17}$	$S_{2,18}$	$S_{2,16}$		
$S_{2,18}$	$S_{2,20}$	$S_{2,16}$	$S_{2,17}$	$S_{2,19}$
$S_{2,19}$	$S_{2,18}$	$S_{2,20}$		
$S_{2,20}$	$S_{2,22}$	$S_{2,21}$	$S_{2,19}$	$S_{2,18}$
$S_{2,21}$	$S_{2,22}$	$S_{2,20}$		
$S_{2,22}$	$S_{2,0\equiv 3,12}$	$S_{2,21}$	$S_{2,20}$	$S_{2,23}$
$S_{2,23}$	$S_{2,0\equiv 3,12}$	$S_{2,22}$		
$S_{3,0\equiv 1,12}$	$S_{3,1\equiv 1,14}$	$S_{3,2\equiv 1,13}$	$S_{1,11}$	$S_{1,10}$
	$S_{3,22}$	$S_{3,23}$		
$S_{3,1\equiv 1,14}$	$S_{3,0\equiv 1,12}$	$S_{3,2\equiv 1,13}$	$S_{1,16}$	$S_{1,15}$
$S_{3,2\equiv 1,13}$	$S_{3,1\equiv 1,14}$	$S_{3,0\equiv 1,12}$	$S_{3,4}$	$S_{3,3}$
$S_{3,3}$	$S_{3,2\equiv 1,13}$	$S_{3,4}$		
$S_{3,4}$	$S_{3,2\equiv 1,13}$	$S_{3,5}$	$S_{3,6}$	$S_{3,3}$
$S_{3,5}$	$S_{3,4}$	$S_{3,6}$		
$S_{3,6}$	$S_{3,5}$	$S_{3,4}$	$S_{3,7}$	$S_{3,8}$
$S_{3,7}$	$S_{3,6}$	$S_{3,8}$		
$S_{3,8}$	$S_{3,10}$	$S_{3,7}$	$S_{3,6}$	$S_{3,9}$
$S_{3,9}$	$S_{3,10}$	$S_{3,8}$		
$S_{3,10}$	$S_{2,0\equiv 3,12}$	$S_{3,11}$	$S_{3,8}$	$S_{3,9}$
$S_{3,11}$	$S_{2,0\equiv 3,12}$	$S_{3,10}$		
$S_{3,15}$	$S_{2,1\equiv 3,14}$	$S_{3,16}$		
$S_{3,16}$	$S_{2,1\equiv 3,14}$	$S_{3,18}$	$S_{3,15}$	$S_{3,17}$
$S_{3,17}$	$S_{3,18}$	$S_{3,16}$		
$S_{3,18}$	$S_{3,20}$	$S_{3,17}$	$S_{3,16}$	$S_{3,19}$
$S_{3,19}$	$S_{3,18}$	$S_{3,20}$		
$S_{3,20}$	$S_{3,18}$	$S_{3,21}$	$S_{3,22}$	$S_{3,19}$
$S_{3,21}$	$S_{3,20}$	$S_{3,22}$		
$S_{3,22}$	$S_{3,0\equiv 1,12}$	$S_{3,20}$	$S_{3,21}$	$S_{3,23}$
$S_{3,23}$	$S_{3,0\equiv 1,12}$	$S_{3,22}$		

Table 1: The vertex-labeled clause gadget

Vertex	Adjacencies (Labels)									
0	1	$(S_{1,1\equiv2,14})$	5	$(S_{1,11})$	6	$(S_{1,22})$	7	$(S_{1,0\equiv2,12})$	14	$(S_{2,5})$
	19	$(S_{2,4})$	20	$(S_{2,6})$						
1	0	$(S_{1,1\equiv2,14})$	2	$(S_{1,3})$	7	$(S_{1,0\equiv2,12})$	8	$(S_{1,2\equiv2,13})$	15	$(S_{2,8})$
	16	$(S_{2,9})$	20	$(S_{2,7})$						
2	8	$(S_{1,2\equiv2,13})$	1	$(S_{1,3})$	3	$(S_{1,5})$	9	$(S_{1,4})$		
3	2	$(S_{1,5})$	4	$(S_{1,7})$	9	$(S_{1,4})$	10	$(S_{1,6})$	21	$(S_{3,1\equiv1,14})$
	24	$(S_{3,0\equiv1,12})$								
4	3	$(S_{1,7})$	5	$(S_{1,9})$	10	$(S_{1,6})$	11	$(S_{1,8})$	22	$(S_{3,5})$
	25	$(S_{3,4})$								
5	0	$(S_{1,11})$	11	$(S_{1,8})$	4	$(S_{1,9})$	6	$(S_{1,10})$		
6	0	$(S_{1,22})$	11	$(S_{1,9})$	5	$(S_{1,10})$	7	$(S_{1,23})$		
7	0	$(S_{1,0\equiv2,12})$	1	$(S_{1,0\equiv2,12})$	6	$(S_{1,23})$	8	$(S_{1,1\equiv2,14})$	15	$(S_{2,9})$
	20	$(S_{2,6})$								
8	1	$(S_{1,2\equiv2,13})$	2	$(S_{1,2\equiv2,13})$	9	$(S_{1,3})$	7	$(S_{1,1\equiv2,14})$		
9	8	$(S_{1,3})$	2	$(S_{1,4})$	3	$(S_{1,4})$	10	$(S_{1,5})$		
10	3	$(S_{1,6})$	4	$(S_{1,6})$	9	$(S_{1,5})$	11	$(S_{1,7})$	24	$(S_{3,1\equiv1,14})$
	25	$(S_{3,3})$								
11	10	$(S_{1,7})$	4	$(S_{1,8})$	5	$(S_{1,8})$	6	$(S_{1,9})$		
12	13	$(S_{2,1\equiv3,14})$	15	$(S_{2,11})$	16	$(S_{2,22})$	17	$(S_{2,0\equiv3,12})$	18	$(S_{3,9})$
	22	$(S_{3,7})$	26	$(S_{3,6})$						
13	12	$(S_{2,1\equiv3,14})$	14	$(S_{2,3})$	17	$(S_{2,0\equiv3,12})$	18	$(S_{2,2\equiv3,13})$	23	$(S_{3,9})$
	26	$(S_{3,7})$								
14	0	$(S_{2,5})$	18	$(S_{2,2\equiv3,13})$	19	$(S_{2,4})$	13	$(S_{2,3})$		
15	16	$(S_{2,10})$	1	$(S_{2,8})$	12	$(S_{2,11})$	7	$(S_{2,9})$		
16	1	$(S_{2,9})$	12	$(S_{2,22})$	17	$(S_{2,23})$	15	$(S_{2,10})$		
17	16	$(S_{2,23})$	18	$(S_{2,1\equiv3,14})$	12	$(S_{2,0\equiv3,12})$	13	$(S_{2,0\equiv3,12})$		
18	12	$(S_{3,9})$	13	$(S_{2,2\equiv3,13})$	14	$(S_{2,2\equiv3,13})$	17	$(S_{2,1\equiv3,14})$	19	$(S_{2,3})$
	21	$(S_{3,11})$	23	$(S_{3,10})$						
19	0	$(S_{2,4})$	18	$(S_{2,3})$	20	$(S_{2,5})$	14	$(S_{2,4})$		
20	0	$(S_{2,6})$	1	$(S_{2,7})$	19	$(S_{2,5})$	7	$(S_{2,6})$		
21	24	$(S_{3,0\equiv1,12})$	18	$(S_{3,11})$	3	$(S_{3,1\equiv1,14})$	23	$(S_{3,22})$		
22	12	$(S_{3,7})$	26	$(S_{3,6})$	4	$(S_{3,5})$	25	$(S_{3,4})$		
23	24	$(S_{3,23})$	18	$(S_{3,10})$	13	$(S_{3,9})$	21	$(S_{3,22})$		
24	10	$(S_{3,1\equiv1,14})$	3	$(S_{3,0\equiv1,12})$	21	$(S_{3,0\equiv1,12})$	23	$(S_{3,23})$		
25	10	$(S_{3,3})$	4	$(S_{3,4})$	26	$(S_{3,5})$	22	$(S_{3,4})$		
26	25	$(S_{3,5})$	12	$(S_{3,6})$	13	$(S_{3,7})$	22	$(S_{3,6})$		

Table 2: The edge-labeled preimage of a clause with zero wheels

Vertex	Adjacencies (Labels)									
0	1	$(S_{1,0\equiv 2,12})$	2	$(S_{1,2\equiv 2,13})$	3	$(S_{1,4})$	4	$(S_{1,6})$	5	$(S_{1,8})$
	6	$(S_{1,10})$	7	$(S_{1,12})$	8	$(S_{1,14})$	9	$(S_{1,16})$	10	$(S_{1,18})$
	11	$(S_{1,20})$	12	$(S_{1,22})$	16	$(S_{2,9})$	21	$(S_{2,8})$	22	$(S_{3,3})$
	25	$(S_{3,2\equiv 1,13})$								
1	0	$(S_{1,0\equiv 2,12})$	20	$(S_{2,4})$	2	$(S_{1,1\equiv 2,14})$	12	$(S_{1,23})$	15	$(S_{2,5})$
2	0	$(S_{1,2\equiv 2,13})$	1	$(S_{1,1\equiv 2,14})$	3	$(S_{1,3})$	20	$(S_{2,5})$	21	$(S_{2,7})$
3	0	$(S_{1,4})$	2	$(S_{1,3})$	4	$(S_{1,5})$				
4	0	$(S_{1,6})$	3	$(S_{1,5})$	5	$(S_{1,7})$				
5	0	$(S_{1,8})$	4	$(S_{1,7})$	6	$(S_{1,9})$				
6	0	$(S_{1,10})$	5	$(S_{1,9})$	7	$(S_{1,11})$				
7	0	$(S_{1,12})$	8	$(S_{1,13})$	24	$(S_{3,22})$	6	$(S_{1,11})$	23	$(S_{3,11})$
8	0	$(S_{1,14})$	9	$(S_{1,15})$	25	$(S_{3,1\equiv 1,14})$	24	$(S_{3,23})$	7	$(S_{1,13})$
9	0	$(S_{1,16})$	8	$(S_{1,15})$	10	$(S_{1,17})$				
10	0	$(S_{1,18})$	9	$(S_{1,17})$	11	$(S_{1,19})$				
11	0	$(S_{1,20})$	10	$(S_{1,19})$	12	$(S_{1,21})$				
12	0	$(S_{1,22})$	1	$(S_{1,23})$	11	$(S_{1,21})$				
13	14	$(S_{2,1\equiv 3,14})$	16	$(S_{2,11})$	17	$(S_{2,22})$	18	$(S_{2,0\equiv 3,12})$	22	$(S_{3,5})$
	26	$(S_{3,4})$								
14	13	$(S_{2,1\equiv 3,14})$	15	$(S_{2,3})$	18	$(S_{2,0\equiv 3,12})$	19	$(S_{2,2\equiv 3,13})$	26	$(S_{3,5})$
	27	$(S_{3,7})$								
15	1	$(S_{2,5})$	19	$(S_{2,2\equiv 3,13})$	20	$(S_{2,4})$	14	$(S_{2,3})$		
16	0	$(S_{2,9})$	17	$(S_{2,10})$	13	$(S_{2,11})$	21	$(S_{2,8})$		
17	16	$(S_{2,10})$	18	$(S_{2,23})$	13	$(S_{2,22})$	21	$(S_{2,9})$		
18	13	$(S_{2,0\equiv 3,12})$	14	$(S_{2,0\equiv 3,12})$	17	$(S_{2,23})$	19	$(S_{2,1\equiv 3,14})$	23	$(S_{3,9})$
	27	$(S_{3,8})$								
19	18	$(S_{2,1\equiv 3,14})$	20	$(S_{2,3})$	14	$(S_{2,2\equiv 3,13})$	15	$(S_{2,2\equiv 3,13})$		
20	1	$(S_{2,4})$	2	$(S_{2,5})$	19	$(S_{2,3})$	15	$(S_{2,4})$		
21	0	$(S_{2,8})$	16	$(S_{2,8})$	2	$(S_{2,7})$	17	$(S_{2,9})$		
22	0	$(S_{3,3})$	25	$(S_{3,2\equiv 1,13})$	26	$(S_{3,4})$	13	$(S_{3,5})$		
23	24	$(S_{3,10})$	18	$(S_{3,9})$	27	$(S_{3,8})$	7	$(S_{3,11})$		
24	8	$(S_{3,23})$	27	$(S_{3,9})$	23	$(S_{3,10})$	7	$(S_{3,22})$		
25	0	$(S_{3,2\equiv 1,13})$	8	$(S_{3,1\equiv 1,14})$	26	$(S_{3,3})$	22	$(S_{3,2\equiv 1,13})$		
26	25	$(S_{3,3})$	22	$(S_{3,4})$	13	$(S_{3,4})$	14	$(S_{3,5})$		
27	24	$(S_{3,9})$	18	$(S_{3,8})$	14	$(S_{3,7})$	23	$(S_{3,8})$		

Table 3: The edge-labeled preimage of a clause with one wheel

Vertex	Adjacencies (Labels)									
0	1	$(S_{1,0\equiv 2,12})$	2	$(S_{1,2\equiv 2,13})$	3	$(S_{1,4})$	4	$(S_{1,6})$	5	$(S_{1,8})$
	6	$(S_{1,10})$	7	$(S_{1,12})$	8	$(S_{1,14})$	9	$(S_{1,16})$	10	$(S_{1,18})$
	11	$(S_{1,20})$	12	$(S_{1,22})$	19	$(S_{2,15})$	23	$(S_{3,3})$	26	$(S_{3,2\equiv 1,13})$
1	0	$(S_{1,0\equiv 2,12})$	2	$(S_{1,1\equiv 2,14})$	12	$(S_{1,23})$	18	$(S_{2,11})$		
2	0	$(S_{1,2\equiv 2,13})$	1	$(S_{1,1\equiv 2,14})$	3	$(S_{1,3})$	13	$(S_{2,0\equiv 3,12})$	14	$(S_{2,2\equiv 3,13})$
	15	$(S_{2,4})$	16	$(S_{2,6})$	17	$(S_{2,8})$	18	$(S_{2,10})$	19	$(S_{2,16})$
	20	$(S_{2,18})$	21	$(S_{2,20})$	22	$(S_{2,22})$	24	$(S_{3,9})$	28	$(S_{3,8})$
3	0	$(S_{1,4})$	2	$(S_{1,3})$	4	$(S_{1,5})$				
4	0	$(S_{1,6})$	3	$(S_{1,5})$	5	$(S_{1,7})$				
5	0	$(S_{1,8})$	4	$(S_{1,7})$	6	$(S_{1,9})$				
6	0	$(S_{1,10})$	5	$(S_{1,9})$	7	$(S_{1,11})$				
7	0	$(S_{1,12})$	8	$(S_{1,13})$	25	$(S_{3,22})$	6	$(S_{1,11})$	24	$(S_{3,11})$
8	0	$(S_{1,14})$	9	$(S_{1,15})$	26	$(S_{3,1\equiv 1,14})$	25	$(S_{3,23})$	7	$(S_{1,13})$
9	0	$(S_{1,16})$	8	$(S_{1,15})$	10	$(S_{1,17})$				
10	0	$(S_{1,18})$	9	$(S_{1,17})$	11	$(S_{1,19})$				
11	0	$(S_{1,20})$	10	$(S_{1,19})$	12	$(S_{1,21})$				
12	0	$(S_{1,22})$	1	$(S_{1,23})$	11	$(S_{1,21})$				
13	2	$(S_{2,0\equiv 3,12})$	27	$(S_{3,4})$	22	$(S_{2,23})$	14	$(S_{2,1\equiv 3,14})$	23	$(S_{3,5})$
14	2	$(S_{2,2\equiv 3,13})$	27	$(S_{3,5})$	28	$(S_{3,7})$	13	$(S_{2,1\equiv 3,14})$	15	$(S_{2,3})$
15	16	$(S_{2,5})$	2	$(S_{2,4})$	14	$(S_{2,3})$				
16	17	$(S_{2,7})$	2	$(S_{2,6})$	15	$(S_{2,5})$				
17	16	$(S_{2,7})$	2	$(S_{2,8})$	18	$(S_{2,9})$				
18	1	$(S_{2,11})$	2	$(S_{2,10})$	17	$(S_{2,9})$				
19	0	$(S_{2,15})$	2	$(S_{2,16})$	20	$(S_{2,17})$				
20	2	$(S_{2,18})$	19	$(S_{2,17})$	21	$(S_{2,19})$				
21	2	$(S_{2,20})$	20	$(S_{2,19})$	22	$(S_{2,21})$				
22	2	$(S_{2,22})$	21	$(S_{2,21})$	13	$(S_{2,23})$				
23	0	$(S_{3,3})$	26	$(S_{3,2\equiv 1,13})$	27	$(S_{3,4})$	13	$(S_{3,5})$		
24	25	$(S_{3,10})$	2	$(S_{3,9})$	28	$(S_{3,8})$	7	$(S_{3,11})$		
25	8	$(S_{3,23})$	24	$(S_{3,10})$	28	$(S_{3,9})$	7	$(S_{3,22})$		
26	0	$(S_{3,2\equiv 1,13})$	8	$(S_{3,1\equiv 1,14})$	27	$(S_{3,3})$	23	$(S_{3,2\equiv 1,13})$		
27	26	$(S_{3,3})$	13	$(S_{3,4})$	14	$(S_{3,5})$	23	$(S_{3,4})$		
28	24	$(S_{3,8})$	25	$(S_{3,9})$	2	$(S_{3,8})$	14	$(S_{3,7})$		

Table 4: The edge-labeled preimage of a clause with two wheels