Partitioning a graph into highly connected subgraphs

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Abstract

Given $k \geq 1$, a *k-proper partition* of a graph G is a partition $\mathcal{P}$ of $V(G)$ such that each part P of $\mathcal{P}$ induces a k-connected subgraph of G. We prove that if G is a graph of order n such that $\delta(G) \geq \sqrt{n}$, then G has a 2-proper partition with at most $n/\delta(G)$ parts. The bounds on the number of parts and the minimum degree are both best possible. We then prove that If G is a graph of order n with minimum degree

$$\delta(G) \geq \sqrt{c(k-1)n},$$

where $c = \frac{2123}{180}$, then G has a k-proper partition into at most $\frac{cn}{\delta(G)}$ parts. This improves a result of Ferrara, Magnant and Wenger [Conditions for Families of Disjoint k-connected Subgraphs in a Graph, *Discrete Math.* **313** (2013), 760–764] and both the degree condition and the number of parts are best possible up to the constant c.

1 Introduction

A graph G is *k-connected* if the removal of any collection of fewer than k vertices from G results in a connected graph with at least two vertices. Highly connected

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graphs represent robust networks that are resistant to multiple node failures. When a graph is not highly connected, it is useful to partition the vertices of the graph so that every part induces a highly connected subgraph. For example, Hartuv and Shamir [9] designed a clustering algorithm where the vertices of a graph G are partitioned into highly connected induced subgraphs. It is important in such applications that each part is highly connected, but also that there are not too many parts. Given a simple graph G and an integer $k \geq 1$, we say a partition $\mathcal{P}$ of $V(G)$ is *k-proper* if for every part $P \in \mathcal{P}$, the induced subgraph $G[P]$ is k-connected. We study degree conditions on simple graphs that imply the existence of k-proper partitions into few parts.

Ferrara, Magnant, and Wenger [5] demonstrate that a minimum-degree condition on G guarantees a k-proper partition.

Theorem 1 (Ferrara, Magnant, Wenger [5]). *Let $k \geq 2$ be an integer, and let G be a graph of order n. If $\delta(G) \geq 2k\sqrt{n}$, then G has a k-proper partition $\mathcal{P}$ with $|\mathcal{P}| \leq 2kn/\delta(G)$.*

In addition, they present a graph G with $\delta(G) = (1 + o(1))\sqrt{(k-1)n}$ that contains no k-proper partition. This example, which we make more precise below, leads us to make the following conjecture.

Conjecture 2. *Let $k \geq 2$ be an integer, and let G be a graph of order n. If $\delta(G) \geq \sqrt{(k-1)n}$, then G has a k-proper partition $\mathcal{P}$ with $|\mathcal{P}| \leq \frac{n-k+1}{\delta-k+2}$.*

To see that the degree condition in Conjecture 2, if true, is approximately best possible, let n, ℓ and p be integers such that $\ell = \sqrt{(k-1)(n-1)}$ and $p = \frac{\ell}{(k-1)} = \frac{n-1}{\ell}$. Starting from $H = pK_\ell$, so that $|H| = n-1$, construct the graph G by adding a new vertex v that is adjacent to exactly $k-1$ vertices in each component of H. Then $\delta(G) = \ell - 1$, but there is no k-connected subgraph of G that contains v.

To see that the number of components in Conjecture 2 is best possible, let r and s be integers such that $r = \sqrt{(k-1)n} - k + 2$ and $s = \frac{n-k+1}{r}$. Consider then $G = sK_r \vee K_{k-1}$, which has minimum degree $r + k - 2 = \sqrt{(k-10)n}$, while every k-proper partition has at least $s = \frac{n-k+1}{\delta-k+2}$ parts.

As an interesting comparison, Nikiforov and Shelp [11] give an approximate version of Conjecture 2 with a slightly weaker degree condition. Specifically, they prove that if $\delta(G) \geq \sqrt{2(k-1)n}$, then there exists a partition of $V(G)$ such that $n - o(n)$ vertices are contained in parts that induce k-connected subgraphs.

In Section 2, we verify Conjecture 2 in the case $k = 2$.

Theorem 3. *Let G be a graph of order n. If $\delta(G) \geq \sqrt{n}$, then G has a 2-proper partition $\mathcal{P}$ with $|\mathcal{P}| \leq n/\delta(G)$.*

Ore's Theorem [12] states that if G is a graph of order $n \geq 3$ such that $\sigma_2(G) = \min\{d(u)+d(v) \mid uv \notin E(G)\} \geq n$, then G is hamiltonian, and therefore has a trivial 2-proper partition. As demonstrated by Theorem 3 however, the corresponding minimum degree threshold is considerably different. Note as well that if G has a 2-factor $\mathcal{F}$, then G has a 2-proper partition, as each component of $\mathcal{F}$ induces a hamiltonian, and therefore 2-connected graph. Consequently, the problem of determining if G has a 2-proper partition can also be viewed as an extension of the 2-factor problem

[1, 13], which is itself one of the most natural generalizations of the hamiltonian problem [6, 7, 8].

In Section 3, we improve the bound on the minimum degree to guarantee a k-proper partition for general k, as follows.

Theorem 4. *If G is a graph of order n with*

$$\delta(G) \geq \sqrt{\frac{2123}{180}(k-1)n}$$

then G has a k-proper partition into at most $\frac{2123n}{180\delta}$ parts.

Conjecture 2 yields that both the degree condition and the number of parts in the partition in Theorem 4 are best possible up to the constant $\frac{2123}{180}$. Our proof of Theorem 4 has several connections to work of Mader [10] and Yuster [15], discussed in Section 3. One interesting aspect of our proof is that under the given conditions, the greedy method of building a partition by iteratively removing the largest k-connected subgraph will produce a k-proper partition.

Definitions and Notation

All graphs considered in this paper are finite and simple, and we refer the reader to [4] for terminology and notation not defined here. For $v \in V(G)$, we define the *eccentricity* $\mathrm{ecc}_G(v)$ of v in G by $\mathrm{ecc}_G(v) = \max\{d_G(v,u) \mid u \in V(G)\}$. The *diameter* of G is defined to be the maximum of $\mathrm{ecc}_G(v)$ as v ranges over $V(G)$, and is denoted by $\mathrm{diam}(G)$. Let H be a subgraph of a graph G, and for a vertex $x \in V(H)$, let $N_H(x) = \{y \in V(H) \mid xy \in E(H)\}$.

A subgraph B of a graph G if a *block* if B is either a bridge or a maximal 2-connected subgraph of G. It is well-known that any connected graph G can be decomposed into blocks. A pair of blocks $B_1 \cap B_2$ are necessarily edge-disjoint, and if two blocks intersect, then their intersection is exactly some vertex v that is necessarily a cut-vertex in G. The *block-cut-vertex graph* of G is defined to be the bipartite graph T with one partite set comprised of all cut-vertices of G and the other partite comprised of all blocks of G. For a cut-vertex v and a block B, v and B are adjacent in T if and only if v is a vertex of B in G.

2 2-Proper Partitions

It is a well-known fact that the block-cut-vertex graph of a connected graph is a tree. This observation makes the block-cutpoint graph, and more generally the block structure of a graph, a useful tool, specifically when studying graphs with connectivity one. By definition, each block of a graph G consists of at least two vertices. A block B of G is *proper* if $|B| \geq 3$. When studying a block decomposition of G, the structure of proper blocks is often of interest. In particular, at times one might hope that the proper blocks will be pairwise vertex-disjoint. In general, however, such an ideal structure is not possible. However, the general problem of determining conditions that ensure a graph has a 2-proper partition, addressed in one of many possible ways by Theorem 3, can be viewed as a vertex analogue to that of determining when a graph has vertex-disjoint proper blocks.

A block of a graph G is *large* if the order of the block is at least $2\delta(G) - 1$. A 2-proper partition $\mathcal{P}$ of a graph G is *strong* if $|P| \geq \delta(G)$ for every $P \in \mathcal{P}$. Note that if G has a strong 2-proper partition $\mathcal{P}$, then $|V(G)| = \sum_{P\in\mathcal{P}} |P| \geq |\mathcal{P}| \cdot \delta(G)$, and so $|\mathcal{P}| \leq |V(G)|/\delta(G)$.

We prove the following stronger form of Theorem 3:

Theorem 5. *Let G be a graph of order n. Suppose that $\delta(G) \geq \sqrt{n}$. Then G has a 2-proper partition $\mathcal{P}$ with $|\mathcal{P}| \leq n/\delta(G)$. Furthermore, if G has no large block, then G has a strong 2-proper partition.*

Proof. Sharpness follows from the sharpness of Conjecture 2, demonstrated above. Since every strong 2-proper partition of G satisfies $|\mathcal{P}| \leq n/\delta(G)$, it suffices to show that

(i) if G has no large block, then G has a strong 2-proper partition, and

(ii) if G has a large block, then G has a (possibly strong) 2-proper partition $\mathcal{P}$ with $|\mathcal{P}| \leq n/\delta(G)$.

We proceed by induction on n, with the base cases $n \leq 4$ being trivial. Thus we may assume that $n \geq 5$.

First suppose that G is disconnected, and let $G_1, \cdots, G_m$ be the components of G. For each $1 \leq i \leq m$, since

$$\delta(G_i) \geq \delta(G) \geq \sqrt{n} > \sqrt{|V(G_i)|},$$

G_i has a 2-proper partition $\mathcal{P}_i$ with at most $\frac{|V(G_i)|}{\delta}$ parts, by induction. Therefore, $\mathcal{P} = \bigcup_{1\leq i\leq m} \mathcal{P}_i$ is a 2-proper partition of G with

$$|\mathcal{P}'| = \sum_{1\leq i\leq m} |\mathcal{P}'_i| \leq \sum_{1\leq i\leq m} |V(G_i)|/\delta(G) = n/\delta(G).$$

Note that since $\delta(G) \leq \delta(G_i)$, any large block B of G_i is also a large block of G. Thus, if G contains no large block, then neither does any G_i. Hence by induction each $\mathcal{P}_i$ is a strong 2-partition and therefore so too is $\mathcal{P}$.

Hence G is connected. If G is 2-connected, then the trivial partition $\mathcal{P} = \{V(G)\}$ is a strong 2-proper partition of G, so we proceed by supposing that G has at least one cut-vertex.

Claim 1. *If G has a large block, then G has a 2-proper partition $\mathcal{P}$ with $|\mathcal{P}| \leq n/\delta(G)$.*

Proof. Let B be a large block of G. It follows that

$$|V(G - V(B))| \leq n - 2\delta(G) + 1 \leq n - 2\sqrt{n} + 1,$$

and

$$\delta(G - V(B)) \geq \delta(G) - 1 \geq \sqrt{n} - 1.$$

Since $\sqrt{n}-1=\sqrt{n-2\sqrt{n}+1}$,

$$\delta(G-V(B))\geq\sqrt{n}-1\geq\sqrt{|V(G-V(B))|}.$$

Applying the induction hypothesis, $G-V(B)$ has a 2-proper partition $\mathcal{P}$ with

$$|\mathcal{P}|\leq(n-|V(B)|)/\delta(G-V(B))\leq(n-(2\delta(G)-1))/(\delta(G)-1).$$

Since $n(\delta(G)-1)-(n-\delta(G))\delta(G)=\delta(G)^2-n\geq n-n=0$, $n/\delta(G)\geq(n-\delta(G))/(\delta(G)-1)$, and hence

$$|\mathcal{P}\cup\{V(B)\}|\leq\frac{n-(2\delta(G)-1)}{\delta(G)-1}+1=\frac{n-\delta(G)}{\delta(G)-1}\leq\frac{n}{\delta(G)}.$$

Consequently $\mathcal{P}\cup\{V(B)\}$ is a 2-proper partition of G with $|\mathcal{P}\cup\{V(B)\}|\leq n/\delta(G)$. □

By Claim 1, we may assume that G has no large block. Let $\mathcal{B}$ be the set of blocks of G. For each $B\in\mathcal{B}$, let $X_B=\{x\in V(B)\mid x\text{ is not a cut-vertex of }G\}$. Note that $N_G(x)\subseteq V(B)$ for every $x\in X_B$. Let $X=\bigcup_{B\in\mathcal{B}}X_B$. For each vertex x of G, let $\mathcal{B}_x=\{B\in\mathcal{B}\mid x\in V(B)\}$. In particular, for each cut-vertex x of G we have that $|\mathcal{B}_x|\geq 2$.

Claim 2. *Let x be a cut-vertex of G, and let C be a component of $G-x$. Then $|V(C)|\geq\delta(G)$. In particular, every end-block of G has order at least $\delta(G)+1$.*

Proof. Let $y\in V(C)$. Note that $d_C(y)\geq d_G(y)-1\geq\delta(G)-1$. Since $N_C(y)\cup\{y\}\subseteq V(C)$, $(\delta(G)-1)+1\leq d_C(y)+1\leq|V(C)|$. □

Claim 3. *Let B be an end-block of G, and let $x\in V(B)$ be the unique cut-vertex of G in B. Then $B-x$ is 2-connected.*

Proof. Recall that $n\geq 5$. Since $|V(B)|\geq\delta(G)+1\geq\sqrt{n}+1$ by Claim 2, we have $|V(B-x)|\geq 3$. Suppose that $B-x$ is not 2-connected and let B_1 and B_2 be distinct blocks in $B-x$. For each $i\in\{1,2\}$, let $y_i\in V(B_i)-V(B_{3-i})$. Since $|(N_{B-x}(y_1)\cup\{y_1\})\cap(N_{B-x}(y_2)\cup\{y_2\})|\leq|V(B_1)\cap V(B_2)|\leq 1$,

$$\begin{aligned}|V(B-x)|&\geq|(N_{B-x}(y_1)\cup\{y_1\})\cup(N_{B-x}(y_2)\cup\{y_2\})|\\&\geq|N_{B-x}(y_1)\cup\{y_1\}|+|N_{B-x}(y_2)\cup\{y_2\}|-1\\&\geq 2\delta(G)-1,\end{aligned}$$

and hence $|V(B)|\geq 2\delta(G)$, which contradicts the assumption that G has no large block. Therefore $B-x$ is 2-connected. □

Claim 4. *For each $x\in V(G)$, $|N_G(x)\cap X|\geq 2$. In particular, for a block B of G, if $X_B\neq\emptyset$, then $|X_B|\geq 3$.*

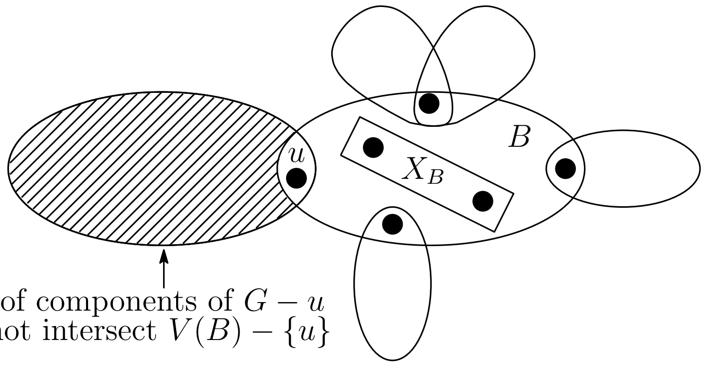

Figure 1: Near end block B of G

Proof. Suppose that $|N_G(x) \cap X| \le 1$. Each vertex $y \in N_G(x) - X$ is a cut-vertex of G, so there exists a component C_y of $G - y$ such that $x \notin V(C_y)$. For distinct $y, y' \in N_G(x) - X$, we therefore have that $V(C_y) \cap (V(C_{y'}) \cup N_G(x)) = \emptyset$. By Claim 2, $|V(C_y)| \ge \delta(G)$ so

$$\begin{aligned} n &\ge |N_G(x) \cup \{x\}| + \sum_{y \in N_G(x)-X} |V(C_y)| \\ &\ge (\delta(G)+1) + |N_G(x) - X|\delta(G) \\ &\ge (\delta(G)+1) + (\delta(G)-1)\delta(G) \\ &= \delta(G)^2 + 1 \\ &\ge n+1, \end{aligned}$$

which is a contradiction. □

A block B of G is a *near end-block* if B is an inner-block of G such that there exists a cut-vertex $u \in V(B)$ of G such that, for every $x \in V(B) - (\{u\} \cup X_B)$, $\mathcal{B}_x - \{B\}$ is a set of end-blocks of G (see Figure 1). We call the cut vertex u the *internal* cut vertex of B. Let x be a cut vertex in G and let $\{B_1, B_2, \ldots, B_t\} \subseteq \mathcal{B}_x$. Define

$$\mathcal{P}(\mathcal{B}_x; B_1, \ldots, B_t) = \bigcup_{\substack{B' \in \mathcal{B}_x \\ B' \notin \{B_1, \ldots, B_t\}}} (V(B') - \{x\}).$$

Claim 5. *Let B be a near end-block of G and let u be the internal cut vertex of B. If C is the unique component of $G - u$ such that $V(B) - \{u\} \subseteq V(C)$, then C has a 2-proper partition $\mathcal{P}$ such that $|P| \ge \delta(G)$ for every $P \in \mathcal{P}$.*

Proof. For each cut vertex x in B distinct from u, fix a block $B_x \in \mathcal{B}_x - \{B\}$. Since B_x is an end-block of G, $|V(B_x)| \ge \delta(G) + 1$ by Claim 2.

Suppose that $X_B = \emptyset$ and let $\mathcal{C}$ denote the set of components of $C - \bigcup_{x \in V(B-u)} B_x$. Then each B_x and each component of $\mathcal{C}$ is 2-connected by Claim 3, which implies that

$$\mathcal{P} = \left(\bigcup_{x \in V(B-u)} V(B_x) \right) \cup \left(\bigcup_{F \in \mathcal{C}} V(F) \right)$$

is a 2-proper partition of G. Furthermore, for every $P \in \mathcal{P}$, $|P| \geq \delta(G)$ by Claim 2. Thus we may assume that $X_B \neq \emptyset$.

We first show that there exists a block of $B - u$ which contains every vertex in X_B. Suppose otherwise, so that there are distinct vertices x and x' in X_B that belong to different blocks of $B - u$. Then $|N_{B-u}(x) \cap N_{B-u}(x')| \leq 1$ and $xx' \notin E(G)$. Hence

$$\begin{aligned}
|V(B) - \{u\}| &\geq |N_{B-u}(x) \cup N_{B-u}(x') \cup \{x, x'\}| \\
&= |N_{B-u}(x)| + |N_{B-u}(x')| - |N_{B-u}(x) \cap N_{B-u}(x')| + 2 \\
&\geq 2(\delta(G) - 1) - 1 + 2 \\
&= 2\delta(G) - 1,
\end{aligned}$$

which contradicts the assumption that G has no large block. Thus there exists a block B^* of $B - u$ such that $X_B \subseteq V(B^*)$. Since $|X_B| \geq 3$ by Claim 4, B^* is 2-connected.

We next show that $|V(B^*)| \geq \delta(G)$. Suppose that $|V(B^*)| \leq \delta(G) - 1$. By the definition of a block, for any $x, x' \in X_B$ with $x \neq x'$, $N_{B-u}(x) \cap N_{B-u}(x') \subseteq V(B^*)$, and so $|(N_{B-u}(x) - (V(B^*) - \{x\})) \cup (N_{B-u}(x) - (V(B^*) - \{x\}))| = |N_{B-u}(x) - (V(B^*) - \{x\})| + |N_{B-u}(x') - (V(B^*) - \{x'\})|$ (see Figure 2). Hence

$$\begin{aligned}
n &\geq |V(C)| \\
&\geq \left| (V(B) - \{u\}) \cup \left(\bigcup_{x \in V(B) - (X_B \cup \{u\})} (V(B_x) - \{x\}) \right) \right| \\
&= |V(B) - \{u\}| + \sum_{x \in V(B) - (X_B \cup \{u\})} |V(B_x) - \{x\}| \\
&\geq |V(B) - \{u\}| + \delta(G) \left(|V(B) - \{u\}| - |X_B| \right) \\
&= (\delta(G) + 1)|V(B) - \{u\}| - \delta(G)|X_B| \\
&\geq (\delta(G) + 1) \left| V(B^*) \cup \left(\bigcup_{x \in X_B} (N_{B-u}(x) - (V(B^*) - \{x\})) \right) \right| - \delta(G)|X_B| \\
&= (\delta(G) + 1) \left(|V(B^*)| + \sum_{x \in X_B} |N_{B-u}(x) - (V(B^*) - \{x\})| \right) - \delta(G)|X_B| \\
&\geq (\delta(G) + 1) \left(|V(B^*)| + \sum_{x \in X_B} (\delta(G) - 1 - (|V(B^*)| - 1)) \right) - \delta(G)|X_B| \\
&= (\delta(G) + 1)(|V(B^*)| + |X_B|(\delta(G) - |V(B^*)|)) - \delta(G)|X_B| \\
&= |X_B|\delta(G)^2 - |V(B^*)|(\delta(G) + 1)(|X_B| - 1) \\
&\geq |X_B|\delta(G)^2 - (\delta(G) - 1)(\delta(G) + 1)(|X_B| - 1) \\
&= \delta(G)^2 + |X_B| - 1 \\
&\geq n + 3 - 1,
\end{aligned}$$

which is a contradiction. Thus $|V(B^*)| \geq \delta(G)$.

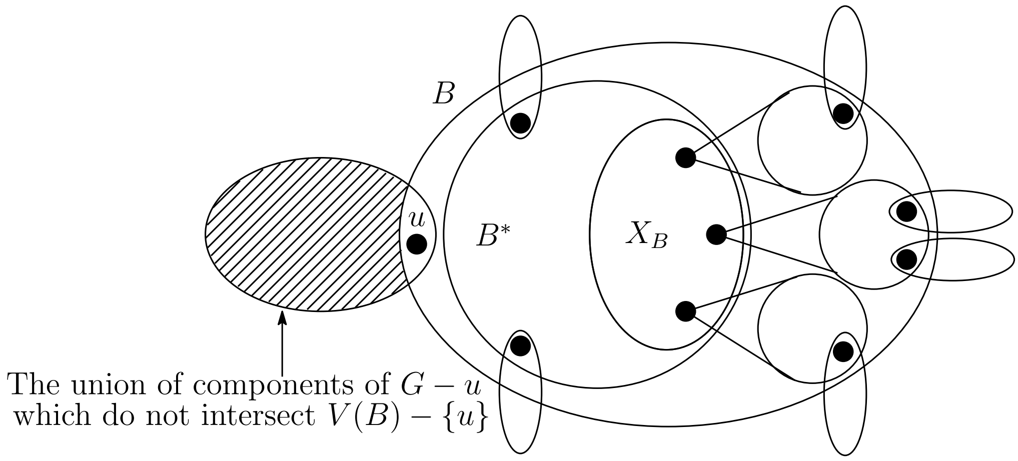

Figure 2: Block B of G and block B^* of $B - u$

By Claims 2 and 3,

$$V(B^*) \cup \left(\bigcup_{x \in V(B^*) - X_B} \mathcal{P}(\mathcal{B}_x; B) \right) \cup \left(\bigcup_{x \in V(B) - (V(B^*) \cup \{u\})} (\{V(B_x)\} \cup \mathcal{P}(\mathcal{B}_x; B, B_x)) \right)$$

is a 2-proper partition of C such that $|P| \geq \delta(G)$ for every $P \in \mathcal{P}'$. $\square$

We divide the remainder of the proof into three cases.

Case 1: G has no inner-block.

Let x be the unique cut-vertex of G and note that $\mathcal{B}_x = \mathcal{B}$. Then for any $B_1 \in \mathcal{B}$, $\mathcal{P}(\mathcal{B}; B_1)$ is a strong 2-proper partition of G by Claims 2 and 3.

Case 2: G has exactly one inner-block.

Let B_0 be the unique inner-block of G, so that necessarily B_0 is a near end-block and $V(B_0) - X_{B_0} \neq \emptyset$. Further, let $u \in V(B_0) - X_{B_0}$ and let C be the unique component of $G - u$ such that $V(B_0) - \{u\} \subseteq V(C)$. Then by Claim 5, C has a 2-proper partition $\mathcal{P}$ such that $|P| \geq \delta(G)$ for every $P \in \mathcal{P}$. Let $B_1 \in \mathcal{B}_u - \{B_0\}$. Then $\mathcal{P} \cup \{V(B_1)\} \cup \mathcal{P}(\mathcal{B}_u; B_0, B_1)$ is a strong 2-proper partition of G by Claims 2 and 3.

Case 3: G has at least two inner-blocks.

Claim 6. *There exists a cut-vertex $u \in V(G)$ of G such that*

(i) *$\mathcal{B}_u$ contains at least two inner-blocks, and*

(ii) *there exists a block $B_0 \in \mathcal{B}_u$ such that each block in $\mathcal{B}_u - \{B_0\}$ is an end-block or a near end-block of G.*

Proof. Let T be the block-cut-vertex graph of G. Note that T is a tree with $\mathrm{diam}(T) \geq 6$. Choose a block A of G so that $\mathrm{ecc}_T(A) = \mathrm{diam}(T)$. Then A is an end-block of G. Let $a \in V(A)$ be the unique cut-vertex of G. By the choice of

A, we see that $\mathcal{B}_a$ contains exactly one inner-block A' which is a near end-block of G. Let u be the internal cut vertex of A'. Since G has at least two inner-blocks, $\mathcal{B}_u$ contains at least two inner-blocks. By the choice of A, $\mathcal{B}_u$ contains at most one block which is neither an end-block nor a near end-block of G. Therefore we get the desired result. $\square$

Let u and B_0 be as in Claim 6. Let $\mathcal{C}$ be the set of components of $G - u$ which do not intersect $V(B_0) - \{u\}$. For each $C \in \mathcal{C}$, C has a 2-proper partition $\mathcal{P}_C$ such that $|P| \geq \delta(G)$ for every $P \in \mathcal{P}_C$ by Claims 2, 3 and 5. Going forward, fix C to be the element of $\mathcal{C}$ with largest order. Since $\mathcal{B}_u - \{B_0\}$ contains a near end-block, $|V(C)| \geq \delta(G) + 1$ by Claim 2.

Let Z be a set of $\delta(G)$ vertices, disjoint from $V(G)$, that induces a clique, and let G' be the graph defined by $V(G') = (V(G) - V(C)) \cup Z$ and $E(G') = E(G - V(C)) \cup \{ux \mid x \in Z\} \cup E(Z)$. Then $\delta(G') = \delta(G)$ and, since $|V(C)| \geq \delta(G) + 1$, $|V(G')| < n$. Consequently, $\delta(G') = \delta(G) \geq \sqrt{n} > \sqrt{|V(G')|}$. Furthermore, since G has no large block, G' also has no large block. Thus, by the induction hypothesis, G' has a strong 2-proper partition $\mathcal{P}$. If possible, choose $\mathcal{P}$ so that $Z \in \mathcal{P}$. If $Z \in \mathcal{P}$, then $(\mathcal{P} - \{Z\}) \cup \mathcal{P}_C$ is a strong 2-proper partition of G, as desired. Thus we may assume that $Z \notin \mathcal{P}$. By the construction of G', we then necessarily have that $Z \cup \{u\} \in \mathcal{P}$.

Claim 7. $|N_G(u) \cap X_{B_0}| \leq 1$.

Proof. Suppose that $|N_G(u) \cap X_{B_0}| \geq 2$ and choose distinct vertices x and x' in $N_G(u) \cap X_{B_0}$. Let P_x and $P_{x'}$ be the unique sets in $\mathcal{P}$ with that contain x and x', respectively. Then $P_x, P_{x'} \subseteq V(B_0)$ and either $P_x \cap P_{x'} = \emptyset$ or $P_x = P_{x'}$. We claim that the latter holds. Indeed $\mathcal{P}$ is a strong 2-proper partition of G', so $|P_x| \geq \delta(G')$ and $|P_{x'}| \geq \delta(G')$. However, B_0 is not a large block of G', $|V(B_0)| \leq 2\delta(G') - 2$. Hence $P_x = P_{x'}$. Since $ux, ux' \in E(G')$, $P_x \cup \{u\}$ induces a 2-connected subgraph of G'. Thus, by the construction of G', $(\mathcal{P} - \{P_x, Z \cup \{u\}\}) \cup \{P_x \cup \{u\}, Z\}$ is a strong 2-proper partition of G' and contains Z, which is a contradiction. $\square$

By Claims 4 and 7, there exists a block $B_1 \in \mathcal{B}_u - \{B_0\}$ such that $N_G(u) \cap X_{B_1} \neq \emptyset$. Let C^* be the component of $G - u$ such that $V(B_1) - \{u\} \subseteq V(C^*)$. Then $C^* \in \mathcal{C}$. Suppose that B_1 is an end-block of G. Note that $C^* = B_1 - u$. Since $|V(B_1)| \geq \delta(G) + 1$ by Claim 2, $(\mathcal{P} - \{Z \cup \{u\}\}) \cup (\bigcup_{C \in \mathcal{C} - \{C^*\}} \mathcal{P}_C) \cup \{V(B_1)\}$ is a strong 2-proper partition of G, as desired. Thus we may assume that B_1 is a near end-block of G. Since $X_{B_1} \neq \emptyset$, $\delta(G) + 1 \leq d_G(y) + 1 \leq |V(B_1)|$ where $y \in X_{B_1}$. Hence

$$\mathcal{P}^* = \{V(B_1)\} \cup \left(\bigcup_{x \in V(B_1) - (X_{B_1} \cup \{u\})} \mathcal{P}(\mathcal{B}_x, B_1) \right)$$

is a 2-proper partition of C^* such that $|P| \geq \delta(G)$ for every $P \in \mathcal{P}^*$. Therefore $(\mathcal{P} - \{Z \cup \{u\}\}) \cup (\bigcup_{C \in \mathcal{C} - \{C^*\}} \mathcal{P}_C) \cup \mathcal{P}^*$ is a strong 2-proper partition of G. This completes the proof of Theorem 5. $\square$

3 k-Proper Partitions

Let $e(k,n)$ be the maximum number of edges in a graph of order n with no k-connected subgraph. Define $d(k)$ to be

$$\sup\left\{\frac{2e(k,n)+2}{n} : n > k\right\}$$

and

$$\gamma = \sup\{d(k)/(k-1) : k \geq 2\}.$$

Recall that the average degree of a graph G of order n with $e(G)$ edges is $\frac{2e(G)}{n}$. This leads to the following useful observation.

Observation 6. *If G is a graph with average degree at least $\gamma(k-1)$, then G contains an k-connected subgraph.*

In [10], Mader proved that $3 \leq \gamma \leq 4$ and constructed a graph of order n with no k-connected subgraph and $\left(\frac{3}{2}k-2\right)(n-k+1)$ edges. This led him to make the following conjecture.

Conjecture 7. *If $k \geq 2$, then $e(k,n) \leq \left(\frac{3}{2}k-2\right)(n-k+1)$. Consequently, $d(k) \leq 3(k-1)$ and $\gamma = 3$.*

Note that Conjecture 7 holds when $k=2$, as it is straightforward to show that $e(2,n) = n-1$. The most significant progress towards Conjecture 7 is due to Yuster [15].

Theorem 8. *If $n \geq \frac{9}{4}(k-1)$, then $e(k,n) \leq \frac{193}{120}(k-1)(n-k+1)$.*

Note that Theorem 8 requires $n \geq \frac{9}{4}(k-1)$, which means that we cannot immediately obtain a bound on γ. The following corollary, however, shows that we can use this result in a manner similar to Observation 6.

Corollary 9. *Let G be a graph of order n with average degree $\overline{d}$. Then G contains a $\lfloor\frac{60\overline{d}}{193}\rfloor$-connected subgraph.*

Proof. Let $k = \lfloor\frac{60\overline{d}}{193}\rfloor$ and suppose that G does not contain a k-connected subgraph. If $n \geq \frac{9}{4}(k-1)$, then Theorem 8 implies

$$\frac{1}{2}\overline{d}n = e(G) \leq \frac{193}{120}(k-1)(n-k+1) < \frac{193}{120}\left(\frac{60}{193}\overline{d}\right)n = \frac{1}{2}\overline{d}n.$$

Thus, assume that $n < \frac{9}{4}(k-1)$. This implies that

$$n < \frac{9}{4}(k-1) < \frac{9}{4}\frac{60}{193}\overline{d} < \frac{7}{10}\Delta(G),$$

a contradiction. $\square$

Finally, prior to proving our main result, we require the following simple lemma, which we present without proof.

Lemma 10. *If G is a graph of order $n \geq k+1$ such that $\delta(G) \geq \frac{n+k-2}{2}$, then G is k-connected.*

We prove the following general result, and then show that we may adapt the proof to improve Theorem 4.

Theorem 11. *Let $k \geq 2$ and $c \geq \frac{11}{3}$. If G is a graph of order n with minimum degree δ with $\delta \geq \sqrt{c\gamma(k-1)n}$, then G has a k-proper partition into at most $\lfloor \frac{c\gamma n}{\delta} \rfloor$ parts.*

Proof. Since $n > \delta \geq \sqrt{c\gamma(k-1)n}$, we have $n^2 > c\gamma(k-1)n$ and hence $n > c\gamma(k-1) \geq 11(k-1)$. Therefore, by Lemma 10, it follows that

$$\delta < \frac{n+k-2}{2} < \frac{n+(k-1)}{2} \leq \frac{n+\frac{1}{11}n}{2} \leq \frac{6}{11}n.$$

Let $G_0 = G$, $\delta_0 = \delta$, and $n_0 = |V(G)|$. We will build a sequence of graphs G_i of order n_i and minimum degree δ_i by selecting a k-connected subgraph H_i of largest order from G_i and assigning $G_{i+1} = G_i - V(H_i)$. This process terminates when either G_i is k-connected or G_i does not contain a k-connected subgraph. We claim the process terminates when G_i is k-connected and $H_i = G_i$.

By Observation 6, G_i contains a $\lfloor \frac{\delta_i}{\gamma} \rfloor$-connected subgraph H_i. If $\frac{\delta_i}{\gamma} \geq k-1$, then H_i is k-connected and has order at least $\lfloor \frac{\delta_i}{\gamma} \rfloor + 1 > \frac{\delta_i}{\gamma}$. Since H_i is a maximal k-connected subgraph in G_i, every vertex $v \in V(G_i) \setminus V(H_i)$ has at most $k-1$ edges to H_i by a simple consequence of Menger's Theorem. Therefore, we have

$$\delta_{i+1} \geq \delta_i - (k-1)$$

and

$$n_{i+1} = n_i - |H_i| < n_i - \delta_i/\gamma.$$

This gives us the estimates on δ_i and n_i of

$$\delta_i \geq \delta - i(k-1),$$

and

$$n_i \leq n - \sum_{j=0}^{i-1} \delta_j/\gamma \leq n - \frac{1}{\gamma}\sum_{j=0}^{i-1}[\delta - j(k-1)] = n - \frac{1}{\gamma}\left[i\delta - (k-1)\binom{i}{2}\right].$$

Let $t = \lceil \frac{c\gamma n}{\delta} - 4 \rceil = \frac{c\gamma n}{\delta} - (4-x)$, where $x \in [0,1)$. We claim that the process terminates with a k-proper partition at or before the $(t+1)^{\text{st}}$ iteration (that is, at or before the point of selecting a k-connected subgraph from G_t). First, we have that

$$\delta_{t-1} \geq \delta - (t-1)(k-1) > \delta - \left(\frac{c\gamma n}{\delta} - 4\right)(k-1) = \delta - \frac{c\gamma(k-1)n}{\delta} + 4(k-1).$$

Note that $\delta^2 \geq c\gamma(k-1)n$ and hence $\delta - \frac{c\gamma(k-1)n}{\delta} \geq 0$. Therefore,

$$\delta_{t-1} > 4(k-1) \geq \gamma(k-1) \quad \text{and} \quad \delta_t \geq 3(k-1).$$

As δ_i is a decreasing function of i, we have that $\delta_i > 4(k-1)$ for all $0 \le i \le t-1$. Thus each G_i with $i < t$ contains a k-connected subgraph. Next, consider n_t.

$$\begin{aligned}
n_t &\le n - \frac{1}{\gamma}\left[t\delta - (k-1)\binom{t}{2}\right] \\
&= n - \frac{1}{\gamma}\left[c\gamma n - (4-x)\delta - \frac{1}{2}(k-1)\left(\frac{c\gamma n}{\delta} - (4-x)\right)\left(\frac{c\gamma n}{\delta} - (5-x)\right)\right] \\
&= n - \frac{1}{\gamma}\left[c\gamma n - (4-x)\delta - \frac{c^2\gamma^2(k-1)}{2}\frac{n^2}{\delta^2} + \frac{c\gamma(9-2x)(k-1)}{2}\frac{n}{\delta} - \frac{1}{2}(k-1)(4-x)(5-x)\right] \\
&= \frac{1}{\delta^2}\left[\frac{4-x}{\gamma}\delta^3 + \frac{c^2\gamma(k-1)}{2}n^2 - (c-1)n\delta^2\right] + \frac{(4-x)(5-x)}{2\gamma}(k-1) - \frac{c(9-2x)(k-1)}{2}\frac{n}{\delta}.
\end{aligned}$$

We have that $\delta^2 \ge c\gamma(k-1)n$ and $(c-1)^2 \ge c^2/2$, so

$$\frac{(c-1)}{c}((c-1)n\delta^2) \ge (c-1)^2\gamma(k-1)n^2 \ge \frac{c^2}{2}\gamma(k-1)n^2.$$

Also, we have $n > \frac{11}{6}\delta$, and $\frac{c-1}{c} \ge \frac{8}{11}$, hence

$$\frac{1}{c}((c-1)n\delta^2) > \frac{8}{11}\cdot\frac{11}{6}\delta^3 = \frac{4}{3}\delta^3 \ge \frac{4-x}{\gamma}\delta^3.$$

Summing these inequalities, we get that

$$\left[\frac{4-x}{\gamma}\delta^3 + \frac{c^2\gamma(k-1)}{2}n^2 - (c-1)n\delta^2\right] < 0$$

and hence $n_t < \frac{(4-x)(5-x)}{2\gamma}(k-1) \le \frac{20}{2\gamma}(k-1) \le \frac{10}{2}(k-1)$. However, $\delta_t \ge 3(k-1)$, so if the process has not terminated prior to the $(t+1)^{\text{st}}$ iteration, G_t is k-connected by Lemma 10. □

Theorem 11 immediately yields the following.

Corollary 12. *Suppose Conjecture 7 holds. We then have that if G is a graph with minimum degree δ where $\delta \ge \sqrt{11(k-1)n}$, then G has a k-proper partition into at most $\frac{11n}{\delta}$ parts.*

We are now ready to prove Theorem 4.

Proof. Observe that the proof of Theorem 11 holds at every step when substituting $\gamma = \frac{193}{60}$ by using Corollary 9 to imply that G_i contains a $\lfloor\frac{60\delta_i}{193}\rfloor$-connected subgraph. Finally, note that $\left(\frac{11}{3}\right)\frac{193}{60} = \frac{2123}{180}$. □

4 Application: Edit Distance to the Family of k-connected Graphs

Define the *edit distance* between two graphs G and H to be the number of edges one must add or remove to obtain H from G (edit distance was introduced independently in [2, 3, 14]). More generally, the edit distance between a graph G and a set of graphs $\mathcal{G}$ is the minimum edit distance between G and some graph in $\mathcal{G}$.

Utilizing Theorem 4 and observing that $2123/180 = 11.79\overline{4} < 11.8$ we obtain the following corollary, which is a refinement of Corollary 11 in [5] for large enough k.

Corollary 13. *Let $k \geq 2$ and let G be a graph of order n. If $\delta(G) \geq \sqrt{11.8(k-1)n}$, then the edit distance between G and the family of k-connected graphs of order n is at most $\frac{11.8kn}{\delta(G)} - k < k(4\sqrt{n} - 1)$.*

Proof. Let $H_1, \ldots, H_l$ be the k-connected subgraphs of the k-proper partition of G guaranteed by Theorem 4; note that $l \leq \frac{11.8n}{\delta(G)}$. For each $i \in \{1, \ldots, l-1\}$, it is possible to produce a matching of size k between H_i and H_{i+1} by adding at most k edges between H_i and H_{i+1}. Thus, adding at most $k\left(\frac{11.8n}{\delta(G)}\right)$ edges yields a k-connected graph. □

5 Conclusion

We note here that it is possible to slightly improve the degree conditions in Theorems 4 and 5 at the expense of the number of parts in the partition. In particular, a greedy approach identical to that used to prove Theorem 5 can be used to prove the following.

Theorem 14. *Let $k \geq 2$, $c_k \geq \frac{k-1}{k} \cdot 2\gamma$, and $p = \sqrt{\frac{c_k n}{k}}$. If G is a graph of order n with $\delta(G) \geq kp = \sqrt{c_k kn}$, then G has a k-proper partition into at most $\frac{k}{k-1}p$ parts.*

This gives rise to the following, which improves on the degree condition in Theorem 4.

Theorem 15. *If G is a graph of order n with minimum degree*

$$\delta(G) \geq kp = \sqrt{\frac{193}{30}(k-1)n},$$

then G has a k-proper partition into at most $\frac{k}{k-1}p$ parts.

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