Searching for uniquely saturated (and strongly regular) graphs with coupled augmentations[1]

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[2] Supported by an AMS travel grant.

Uniquely K_r-Saturated Graphs

Definition

A graph G is **uniquely K_r-saturated** if G contains no K_r and for every edge $e \in \overline{G}$ admits exactly one copy of K_r in $G + e$.

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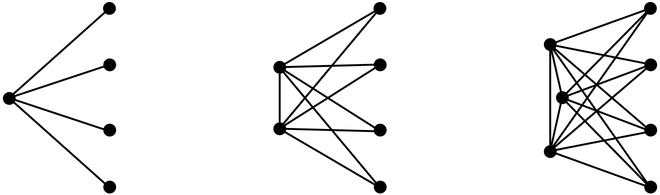

(a) 1-book (b) 2-book (c) 3-book

Figure: The $(r-2)$-books are uniquely K_r saturated.

Dominating Vertices

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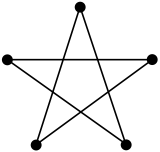
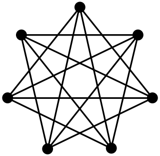


$\overline{C_5}$



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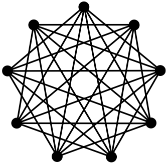
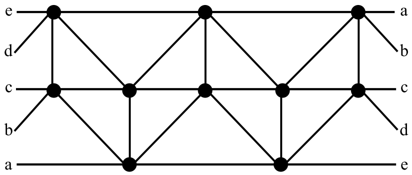
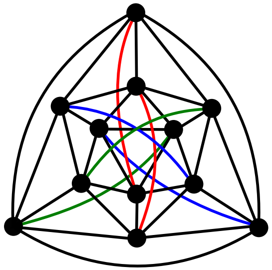


$\overline{C_9}$

For $r \geq 1$, $\overline{C_{2r-1}}$ is r-primitive.





Previously known 4-primitive graphs

Joshua Cooper

Paul Wenger

Two Questions:

Joshua Cooper   Paul Wenger

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1. Fix $r \geq 3$. Are there a **finite number** of r-primitive graphs?

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NO! Exists an irregular 5-primitive graph on 16 vertices!

Variables

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Use variables $x_{i,j} \in \{0, 1, *\}$ where

- $x_{i,j} = 0$ fixes $v_i v_j \notin E(G)$.
- $x_{i,j} = 1$ fixes $v_i v_j \in E(G)$.
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- $x_{i,j} = *$ is **unassigned**.

If $x_{i,j} = *$ for some i, j, the vector $\mathbf{x}$ is a **partial assignment**.

If $x_{i,j} = *$ for all i, j, the vector $\mathbf{x}$ is the **empty assignment**.

Symmetries of the System

The constraints

- There is no r-clique in G.
- Every non-edge e of G has exactly one r-clique in $G + e$.

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Value-preserving permutations reflect the automorphisms of a partial assignment.

Orbital Branching

Generalizes branch-and-bound strategy.

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Introduced by Ostrowski, Linderoth, Rossi, and Smriglio (2007) for symmetric optimization problems such as covering and packing.

K_r-Completions

In addition to the usual constraints, we guarantee:

$x_{i,j} = 0$ **if and only if** there exists a set $S \subset [n]$ so that $x_{i,a} = x_{j,a} = x_{a,b} = 1$ for all $a, b \in S$.

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i.e. for every non-edge we add, we add a ***K_r*-completion**.

Also, we set $x_{i,j} = 0$ if it has a K_r-completion.

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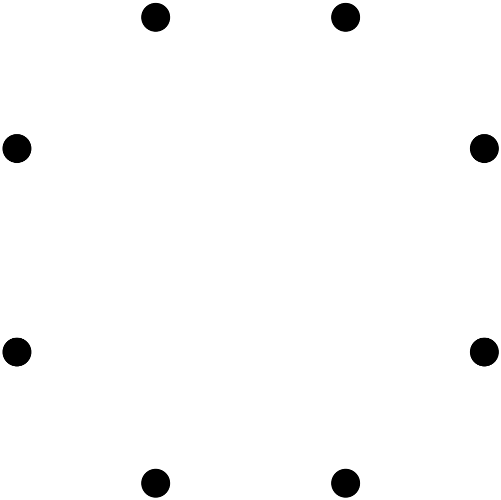
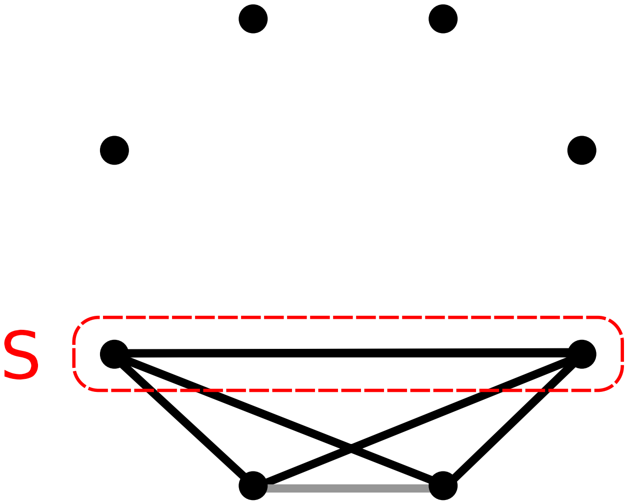
B2: Set $x_{i,j} = 1$ for all $x_{i,j} \in \mathcal{O}$.

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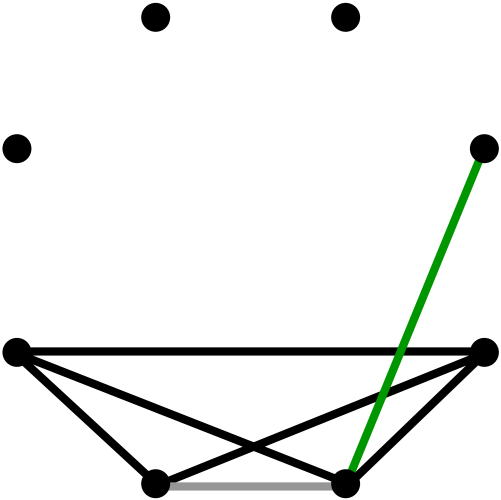

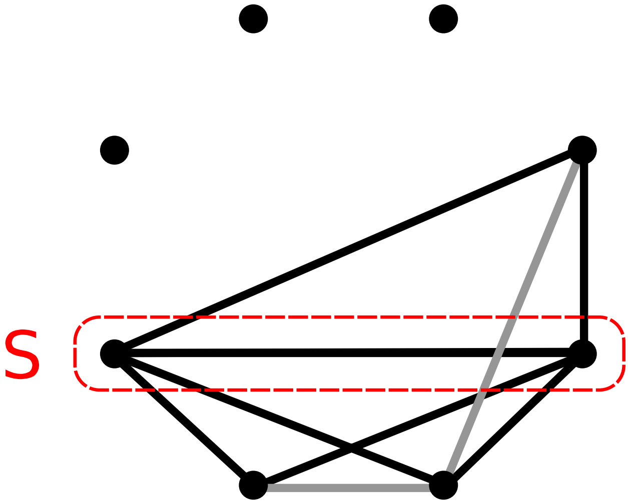
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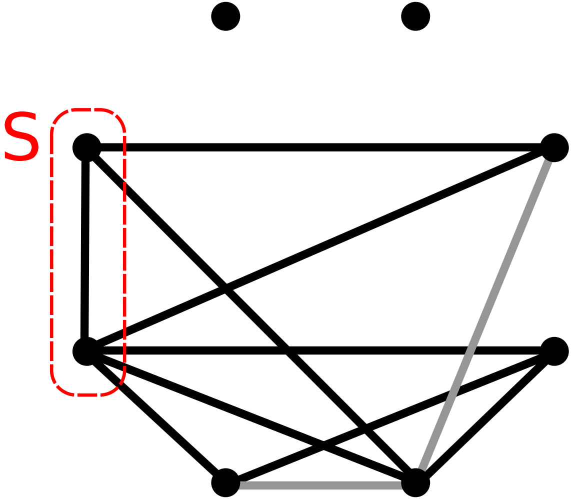
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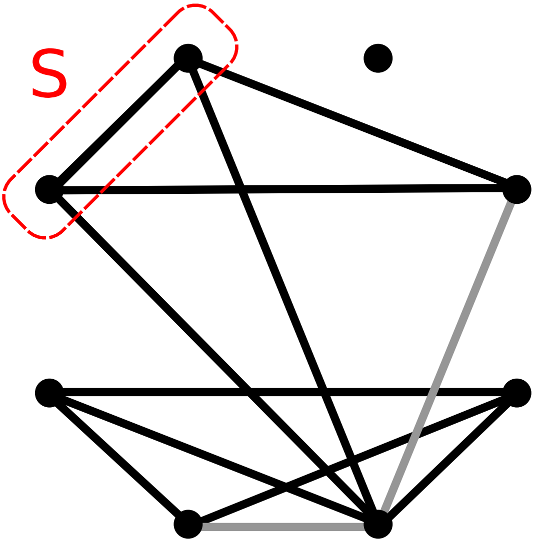
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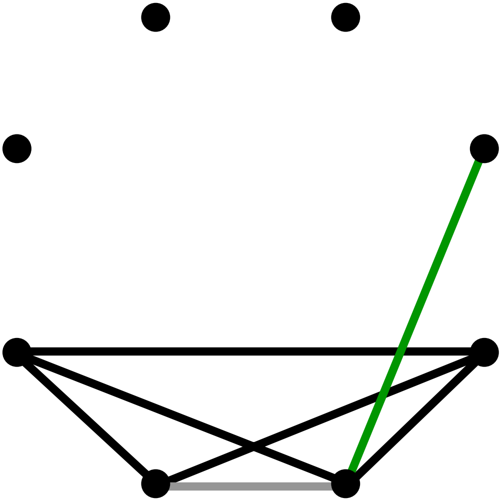

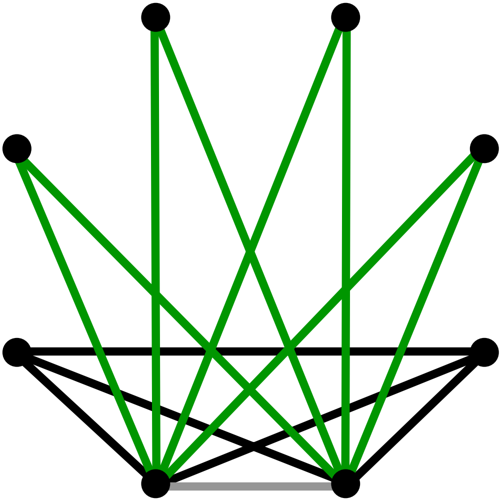

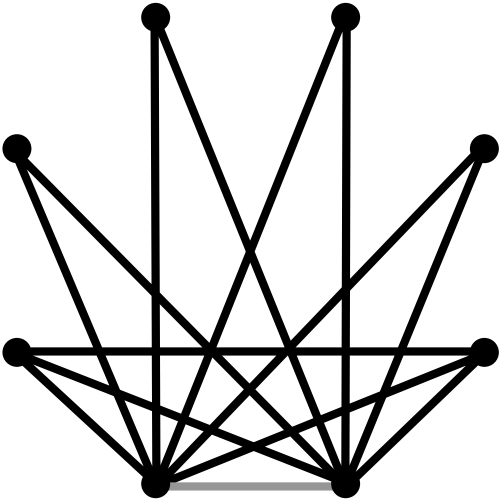

Search Times

n	$r = 4$	$r = 5$	$r = 6$	$r = 7$	$r = 8$
10	0.10 s	0.37 s	0.13 s	0.01 s	0.01 s
11	0.68 s	5.25 s	1.91 s	0.28 s	0.09 s
12	4.58 s	1.60 m	25.39 s	1.97 s	1.12 s
13	34.66 s	34.54 m	6.53 m	59.94 s	20.03 s
14	4.93 m	10.39 h	5.13 h	20.66 m	2.71 m
15	40.59 m	23.49 d	10.08 d	12.28 h	1.22 h
16	6.34 h	1.58 y	1.74 y	34.53 d	1.88 d
17	3.44 d			8.76 y	115.69 d
18	53.01 d				
19	2.01 y				
20	45.11 y				

Total CPU times using Open Science Grid.

Strongly Regular Graphs

Custom Augmentations

An (n, k, λ, μ) **strongly regular graph** is a k-regular graph G on n vertices where every vertex pair $u, v \in V(G)$ has

- If uv is an edge, $|N(u) \cap N(v)| = \lambda$.
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We use the λ and μ constraints for custom augmentations.

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Custom Augmentations

λ-Augmentation

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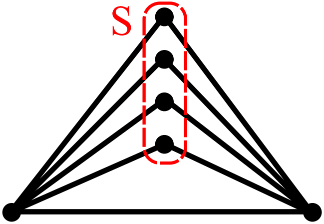

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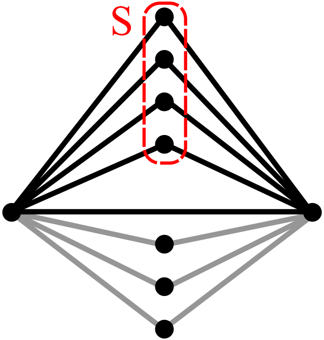

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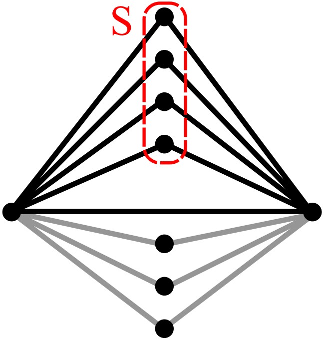

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μ-Augmentation

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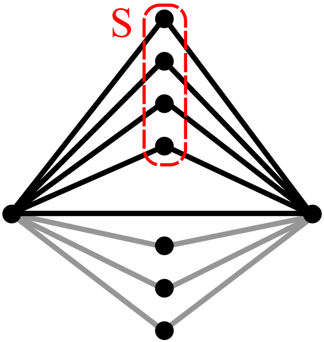

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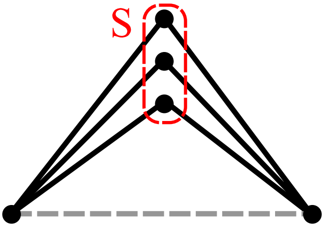

μ-Augmentation

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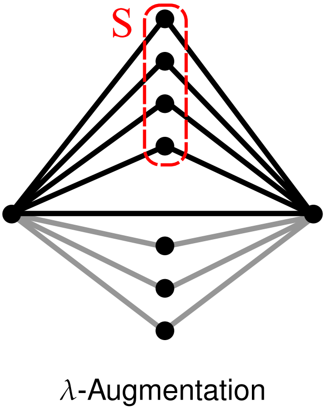
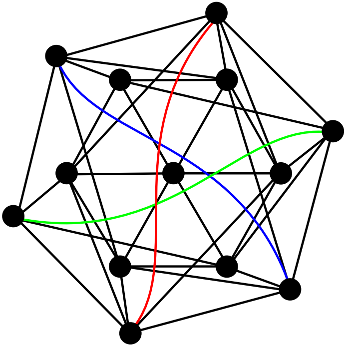
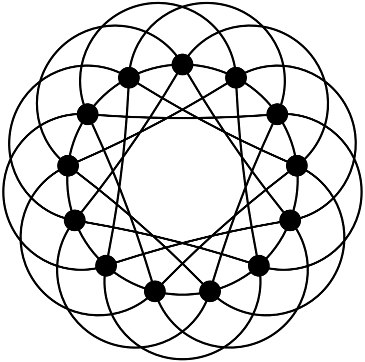


4-Primitive Graphs

$n = 13$



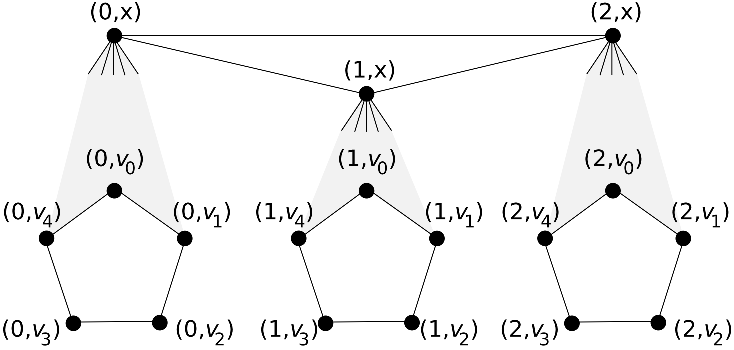
$G_{13}^{(A)}$



Payley(13)

4-Primitive Graphs

$n = 18 : G_{18}^{(A)}$



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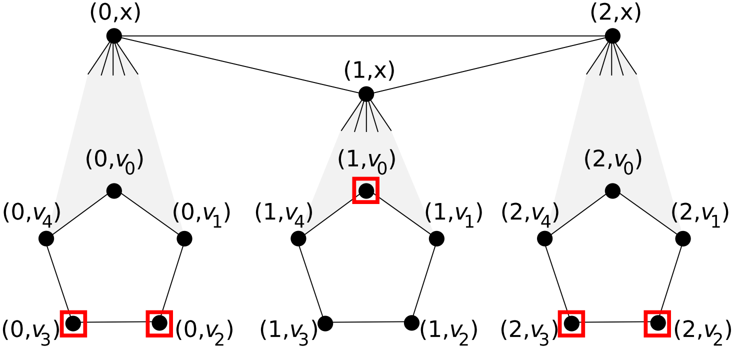

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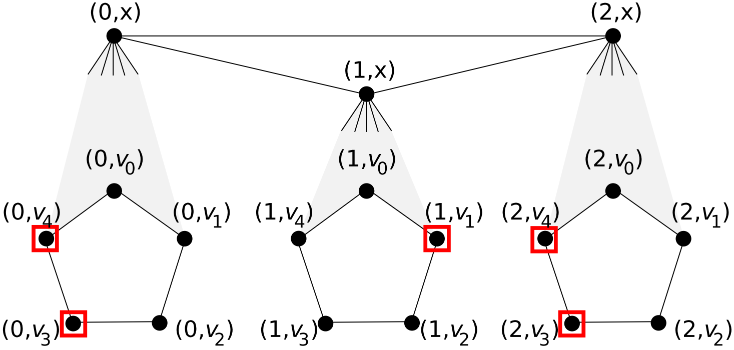

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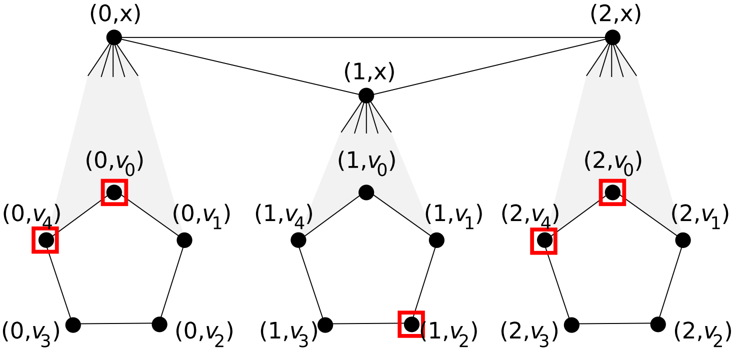

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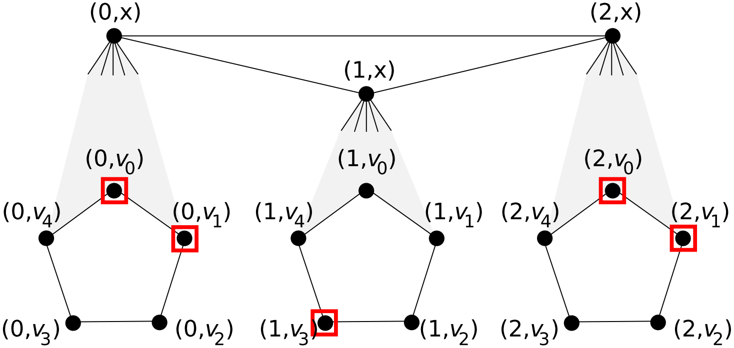

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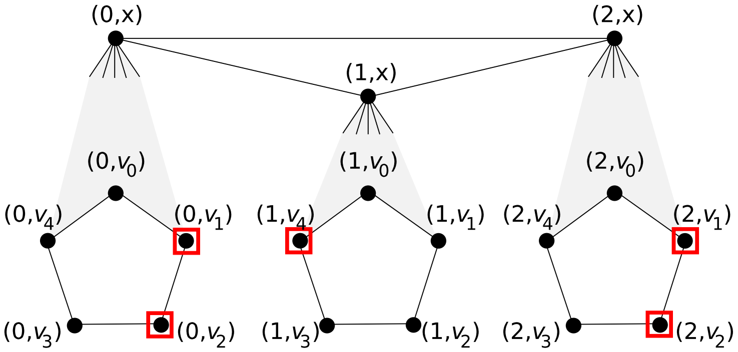

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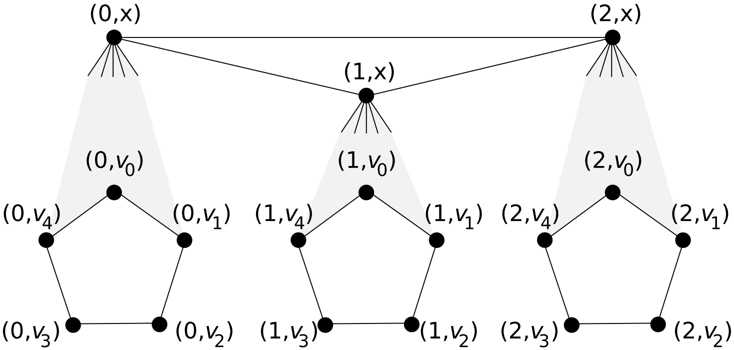

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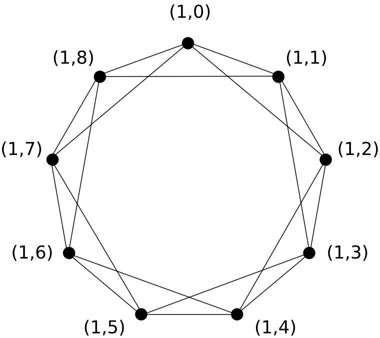

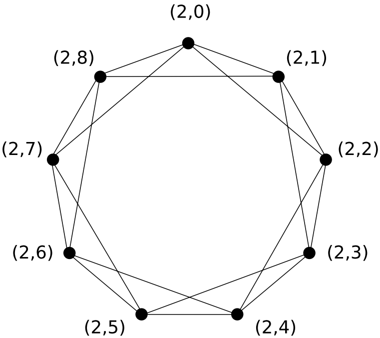

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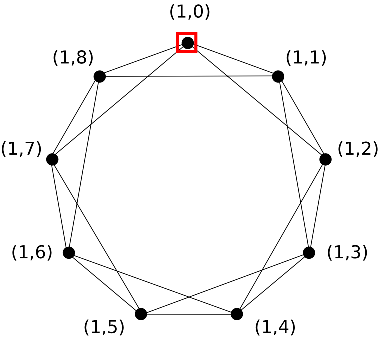

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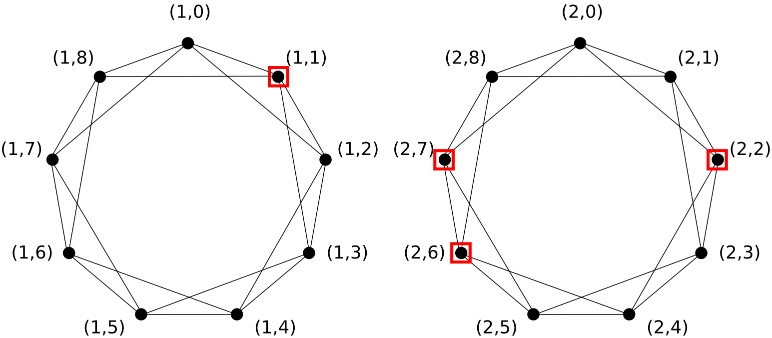

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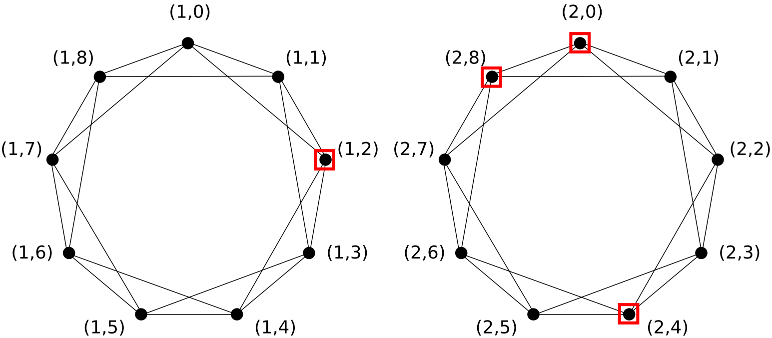

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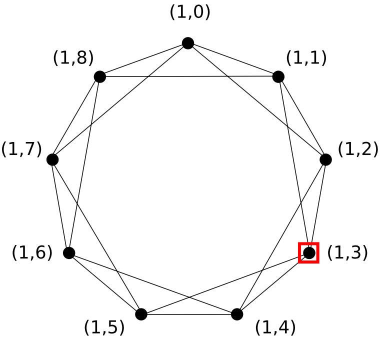

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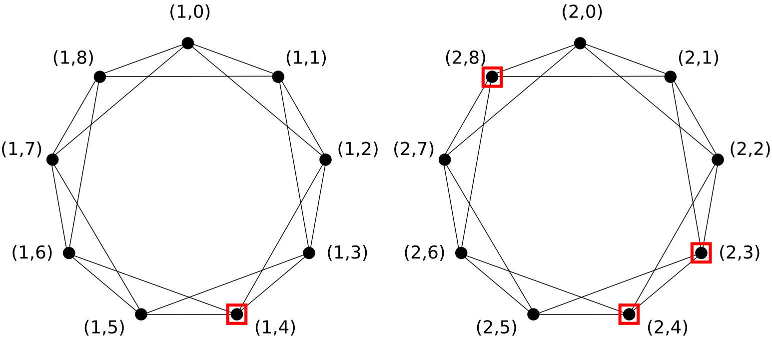

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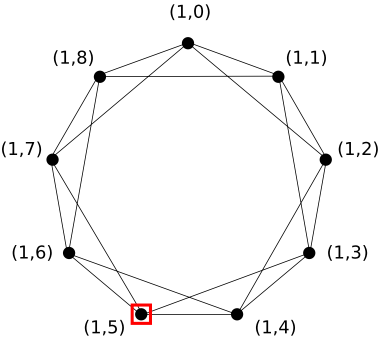

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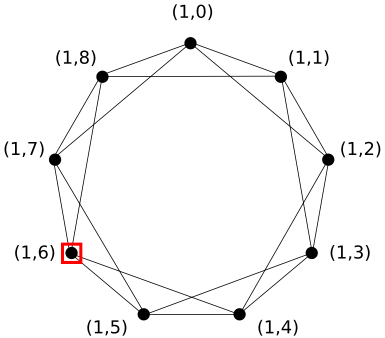

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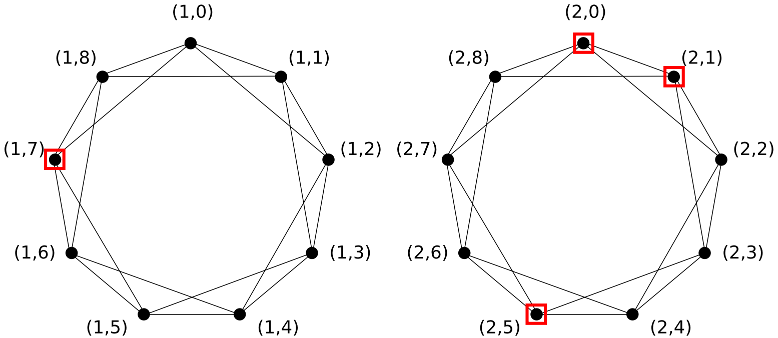

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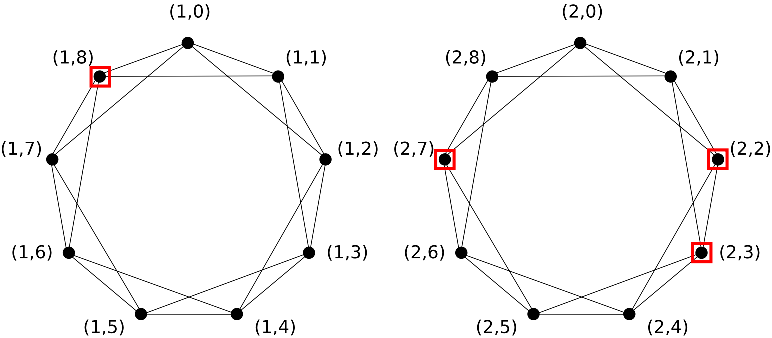

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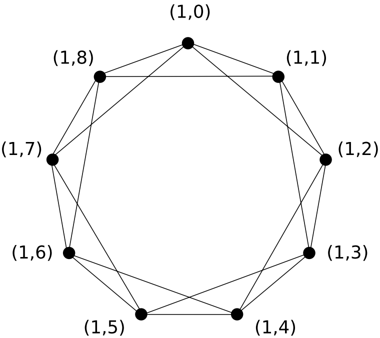
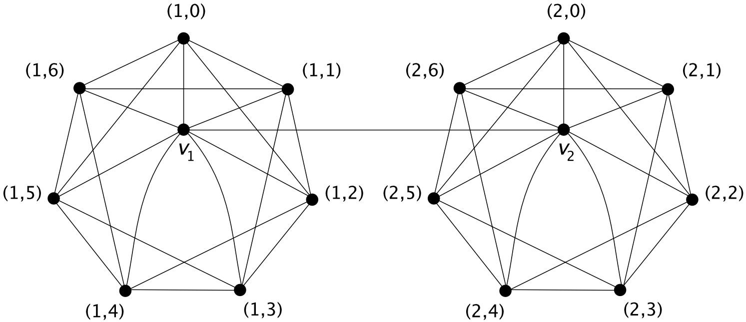


5-Primitive Graphs

$n = 16$: $G_{16}^{(A)}$ is irregular!



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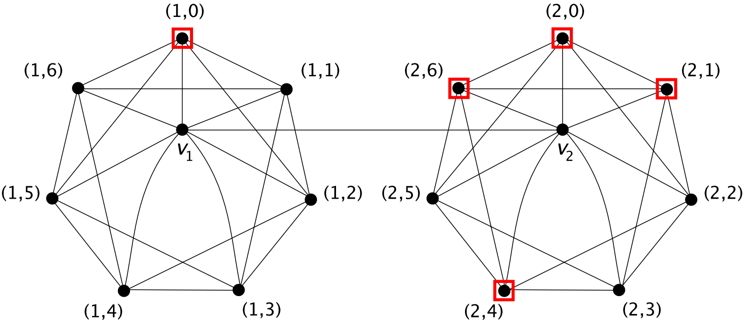

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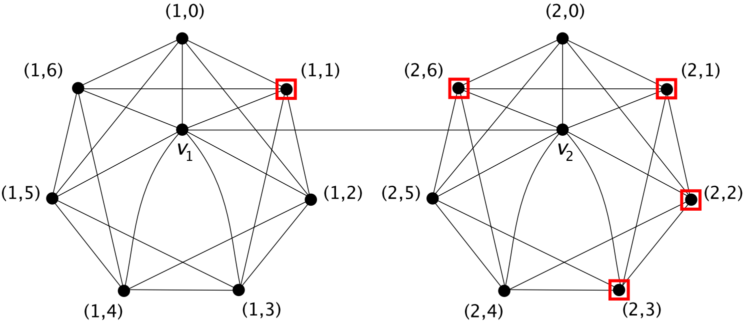

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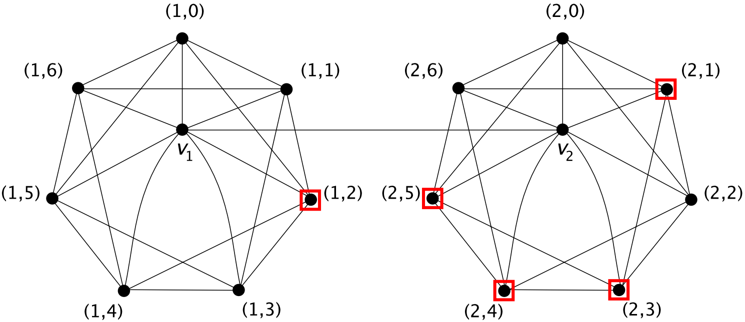

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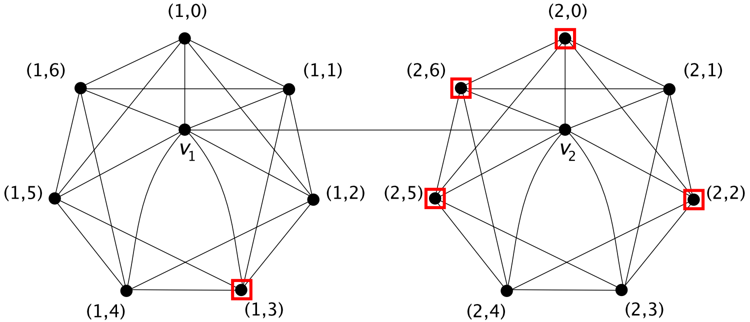

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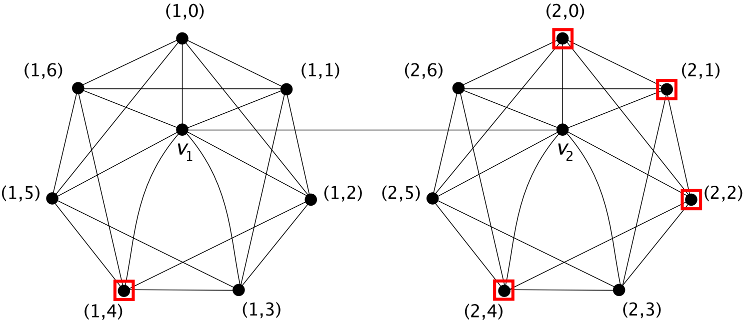

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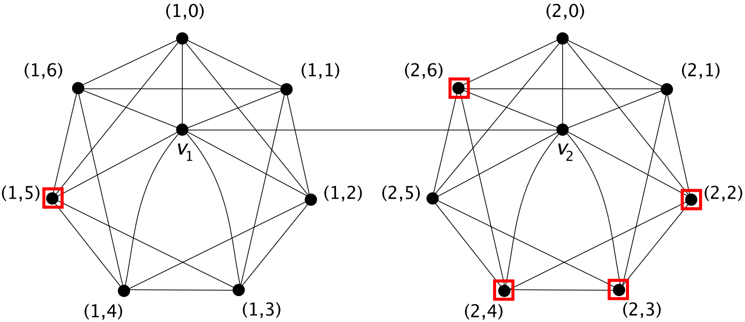

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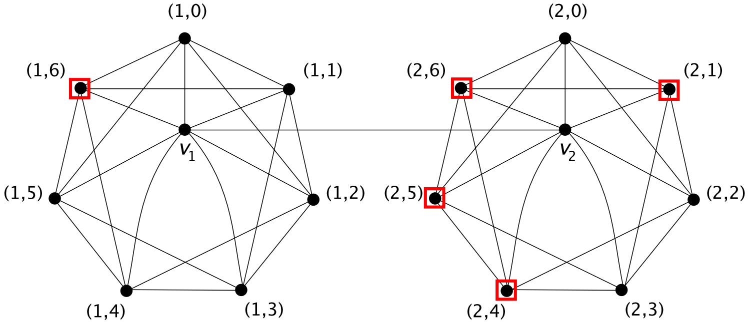

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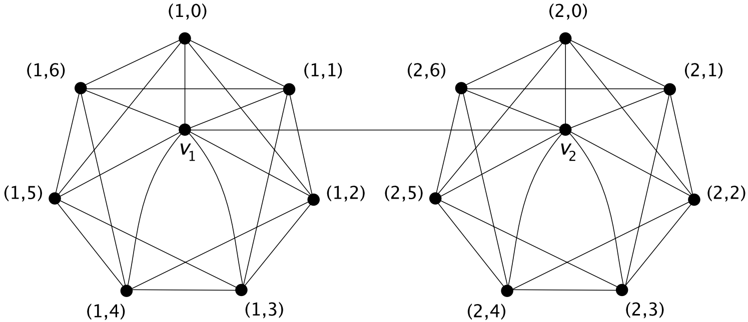

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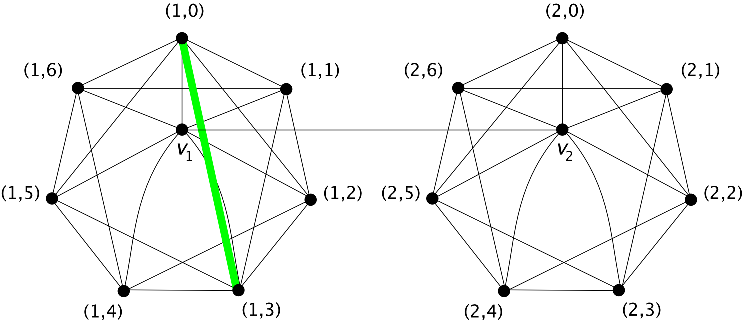

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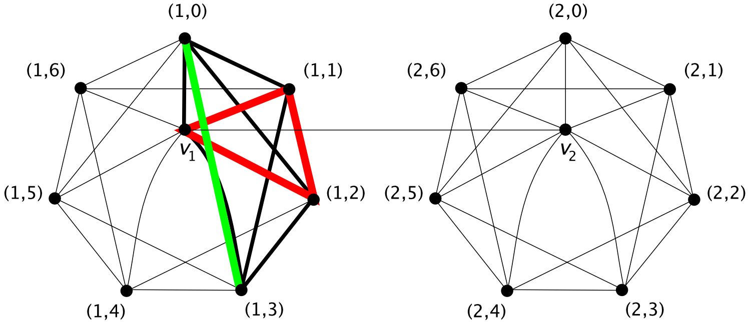

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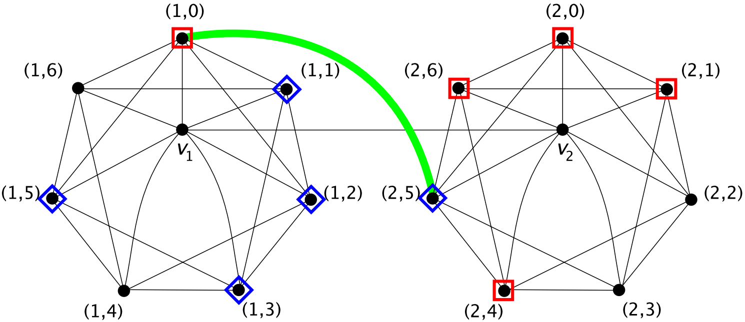

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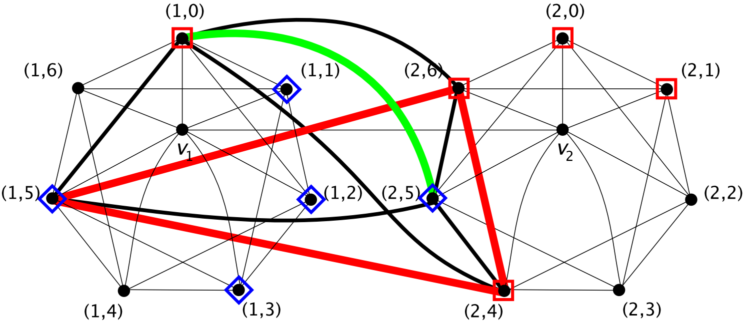

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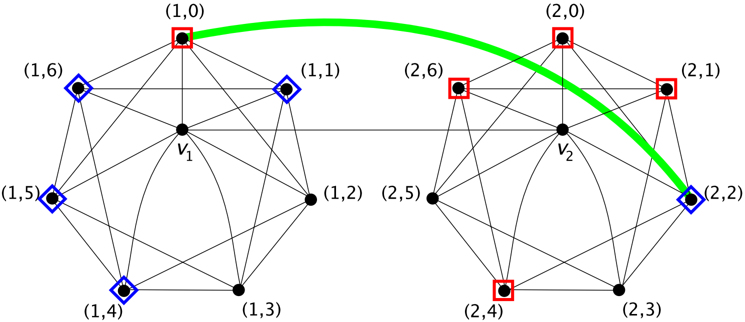

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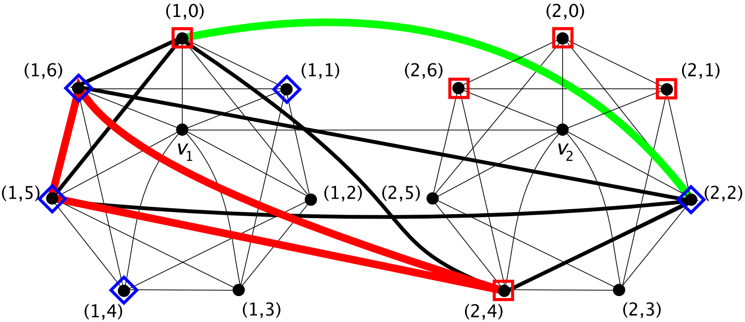

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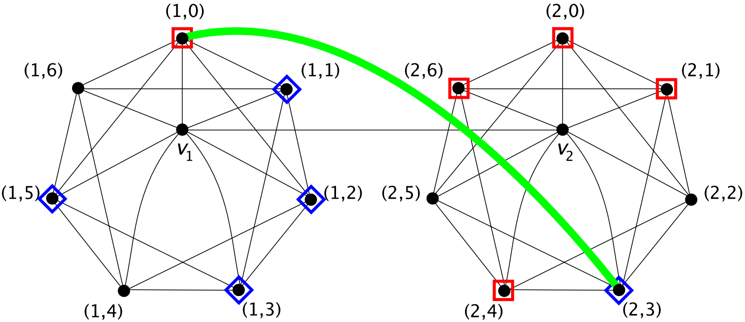

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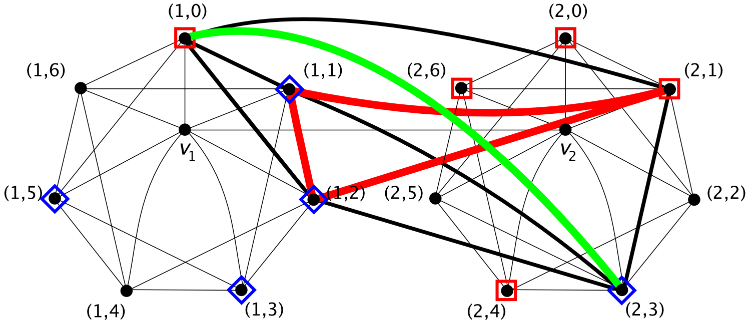
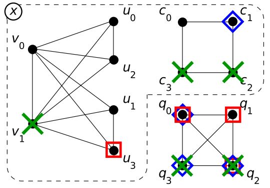
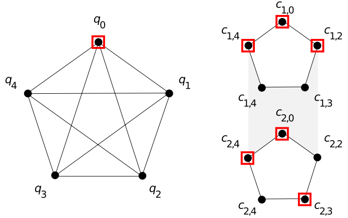


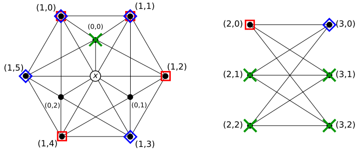
Other r-Primitive Graphs



$G_{15}^{(A)}$



$G_{15}^{(B)}$



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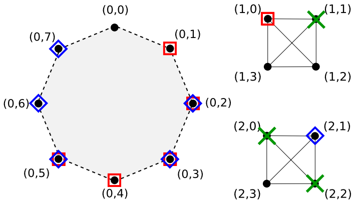
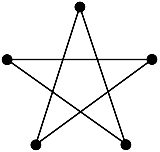


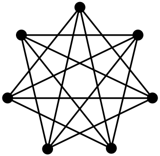
$G_{16}^{(C)}$

Infinite Families

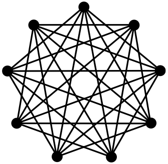
Recall: For $r \geq 1$, $\overline{C_{2r-1}}$ is r-primitive.



$\overline{C_5}$



$\overline{C_7}$



$\overline{C_9}$

Infinite Families

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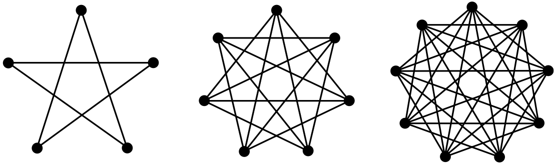

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Let n be an integer and $S \subseteq \mathbb{Z}_n$. The **Cayley complement** $\overline{C}(\mathbb{Z}_n, S)$ is the complement of the Cayley graph for $\mathbb{Z}_n$ with generator set S.

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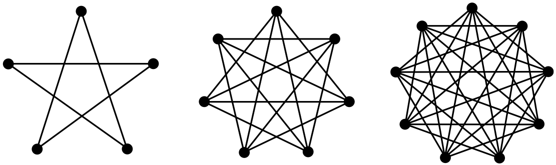

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$\overline{C}(\mathbb{Z}_{2r-1}, \{1\}) \cong \overline{C_{2r-1}}$ is r-primitive.

Two Generators

Theorem

Let $t \geq 1$, $n = 4t^2 + 1$, and $r = 2t^2 - t + 1$. The Cayley complement $\overline{C}(\mathbb{Z}_n, \{1, 2t\})$ is r-primitive.

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For $t = 1$, $r = 2$, and $\overline{C}(\mathbb{Z}_n, \{1, 2\}) \cong \overline{K_5}$.

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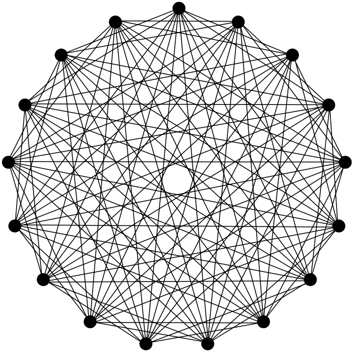

$t = 2, n = 17, r = 7$

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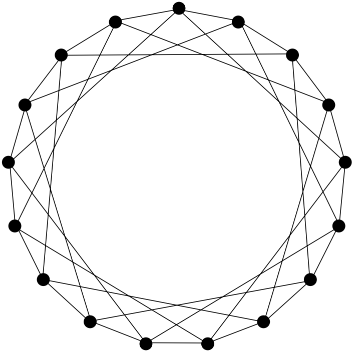

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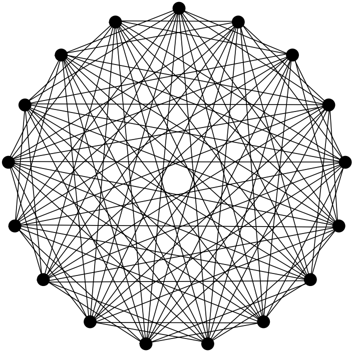

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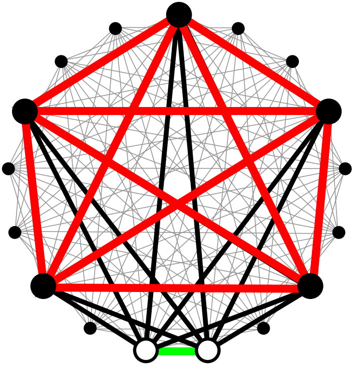

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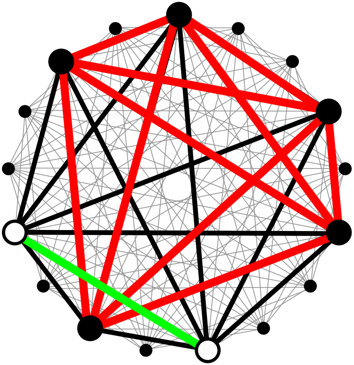

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Conjecture

Let $S \subseteq \mathbb{Z}_n$ have $|S| = 2$. The Cayley complement $\overline{C}(\mathbb{Z}_n, S)$ is r-primitive if and only if $\exists t \geq 1$, $n = 4t^2 + 1$, $r = 2t^2 - t + 1$, and $\overline{C}(\mathbb{Z}_n, S) \cong \overline{C}(\mathbb{Z}_n, \{1, 2t\})$.

Three Generators

We have a similar conjecture for $\overline{C}(\mathbb{Z}_n, S)$ when $|S| = 3$.

Verified for $1 \leq t \leq 6$.

When $t = 6$, we have $r = 97$, $n = 304$.

Three Generators

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When $t = 6$, we have $r = 97$, $n = 304$.

Pattern does not extend to $|S| \geq 4$!

More Generators

g	Generators	n	r
4	$\{1, 5, 8, 34\}$ $\{1, 11, 18, 34\}$	89	28
5	$\{1, 5, 14, 17, 25\}$	71	19
5	$\{1, 6, 14, 17, 36\}$	101	27
6	$\{1, 6, 16, 22, 35, 36\}$	97	21
7	$\{1, 20, 23, 26, 30, 32, 34\}$	71	15

Searching for uniquely saturated and strongly regular graphs with coupled augmentations[1]

Stephen G. Hartke   Derrick Stolee[2]
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s-dstolee1@math.unl.edu
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September 25, 2011

[1] Supported by NSF grant DMS-0914815.
[2] Supported by an AMS travel grant.

Two Generators

Theorem

Let $t \geq 1$, $n = 4t^2 + 1$, and $r = 2t^2 - t + 1$. The Cayley complement $G = \overline{C}(\mathbb{Z}_n, \{1, 2t\})$ is r-primitive.

Two Generators

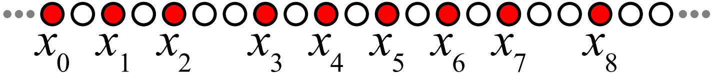
$n = 4t^2 + 1$, $r = 2t^2 - t + 1$, $G = \overline{C}(\mathbb{Z}_n, \{1, 2t\})$

Suppose $X \subseteq \mathbb{Z}_n$ is an r-clique in G.

Two Generators

$n = 4t^2 + 1$, $r = 2t^2 - t + 1$, $G = \overline{C}(\mathbb{Z}_n, \{1, 2t\})$

Elements are labeled $x_0, x_1, \ldots, x_i, \ldots$ (i modulo r).



Two Generators

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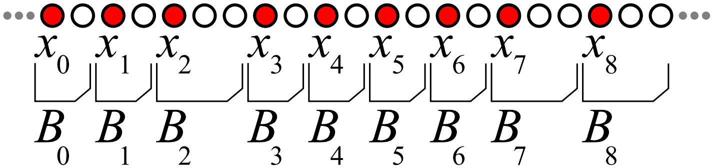
Blocks are sets $B_k = \{x_k, x_k + 1, \ldots, x_{k+1} - 1\}$ (k modulo r).
("Intervals" closed on element x_k and open on x_{k+1})



Two Generators

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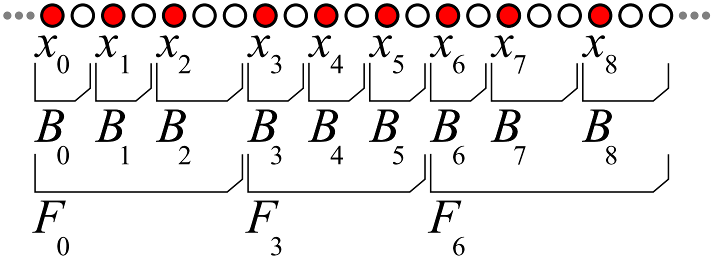
Frames are collections $F_j = \{B_j, B_{j+1}, \ldots, B_{j+t-1}\}$ (j modulo r).
(There are t blocks in each frame.)



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(a) Every block B_k has $|B_k| \geq 2$.

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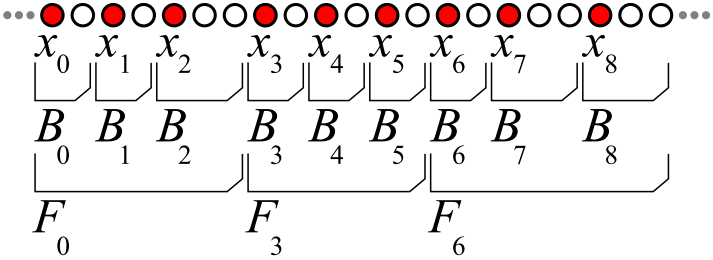
$2t$ is a generator, so $x_{j+t} \neq x_j + 2t$.



Two Generators

$n = 4t^2 + 1$, $r = 2t^2 - t + 1$, $G = \overline{C}(\mathbb{Z}_n, \{1, 2t\})$

So, $\sigma(F_j) := \sum_{B_k \in F_j} |B_k| = d_{\mathbb{Z}_n}(x_j, x_{j+t}) \geq 2t + 1$.

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Contradiction! $\therefore \omega(G) < r$.

Two Generators

$n = 4t^2 + 1$, $r = 2t^2 - t + 1$, $G = \overline{C}(\mathbb{Z}_n, \{1, 2t\})$

G is vertex-transitive and there is an automorphism of G ($x \mapsto -2tx$) that maps $\{0, 2t\}$ to $\{0, 1\}$.

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For unique saturation, we only need to check $G + \{0, 1\}$.

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Suppose X is an r-clique in $G + \{0, 1\}$.

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$$X = \{x_0 = 0, x_1 = 1, x_2, \ldots, x_{r-1}\}.$$

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Consider **frame family** $\mathcal{F}$

$$\mathcal{F} = \{F_1, F_{t+1}, F_{2t+1}, \ldots, F_{r-t}\}, \qquad |\mathcal{F}| = 2t - 1.$$

Two Generators

$n = 4t^2 + 1$, $r = 2t^2 - t + 1$, $G = \overline{C}(\mathbb{Z}_n, \{1, 2t\})$

$$n - 1 = \sum_{F_j \in \mathcal{F}} \sigma(F_j) \geq (2t - 1)(2t + 1) = n - 2.$$

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$F_{j'}$ has $(t-2)$ 2-blocks and two 3-blocks ($4 = 2 + 2$).

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$$n - 1 = \sum_{F_j \in \mathcal{F}} \sigma(F_j) \geq (2t - 1)(2t + 1) = n - 2.$$

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$F_{j'}$ has $(t - 2)$ 2-blocks and two 3-blocks ($4 = 2 + 2$).

All blocks of of X (except B_0) have size 2 or 3.

There are exactly $(2t + 1)$ 3-blocks.

Two Generators

$n = 4t^2 + 1$, $r = 2t^2 - t + 1$, $G = \overline{C}(\mathbb{Z}_n, \{1, 2t\})$

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(c) If $B_{k_0}, B_{k_1}, \ldots, B_{k_{2t}}$ be the 3-blocks.

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A unique solution for $k_0, \ldots, k_{2t}$: $k_{j+1} = k_j + t - 2$.

Defines X which is an r-clique. $\square$

Searching for uniquely saturated and strongly regular graphs with coupled augmentations[1]

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