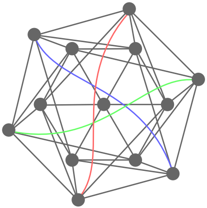
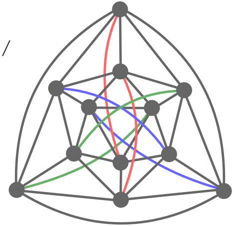
Combinatorics Using Computational Methods

Derrick Stolee

University of Nebraska–Lincoln
s-dstolee1@math.unl.edu
http://www.math.unl.edu/∼s-dstolee1/





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Dissertation Defense

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Advisors and Committee

Stephen Hartke
Mathematics

Vinod Variyam
Computer Science and Engineering

Jamie Radcliffe
Mathematics

Stephen Scott
CSE

Christina Falci
Sociology

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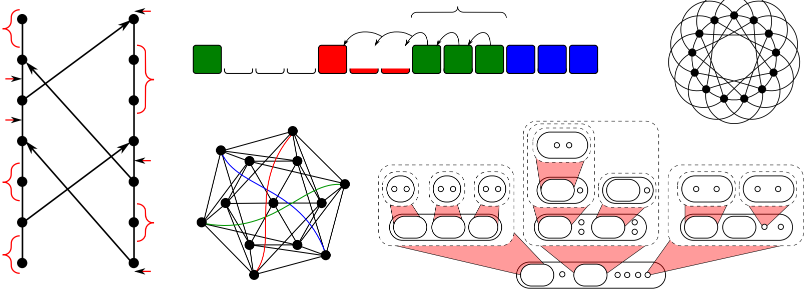
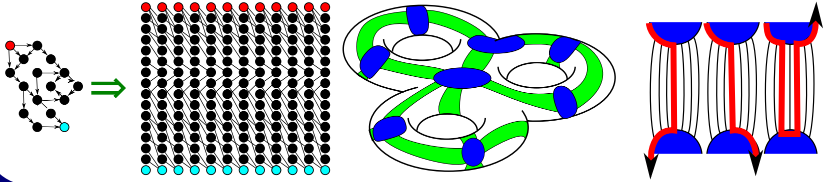


Computational Complexity



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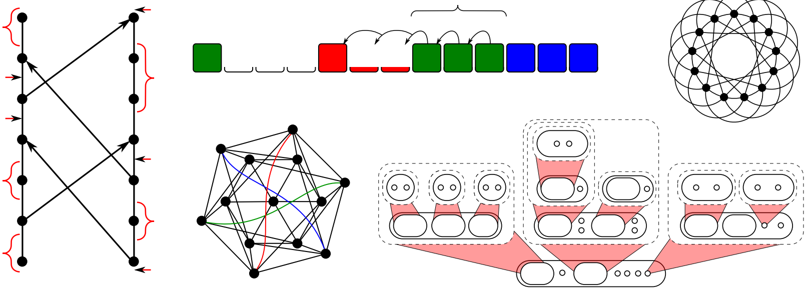

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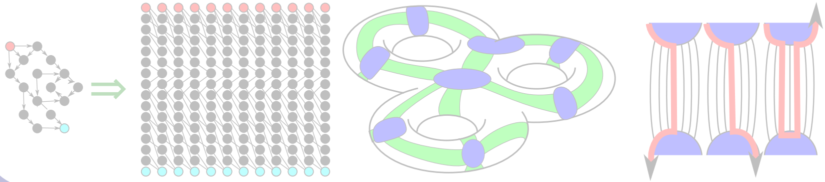

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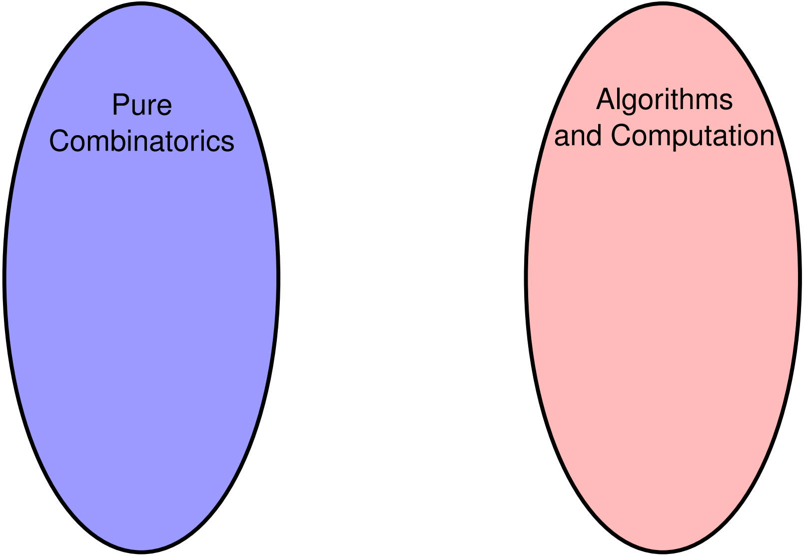

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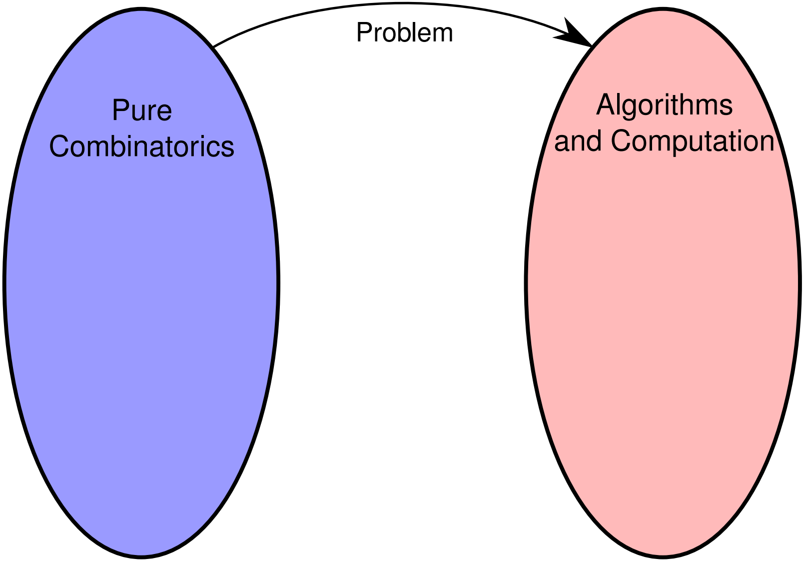

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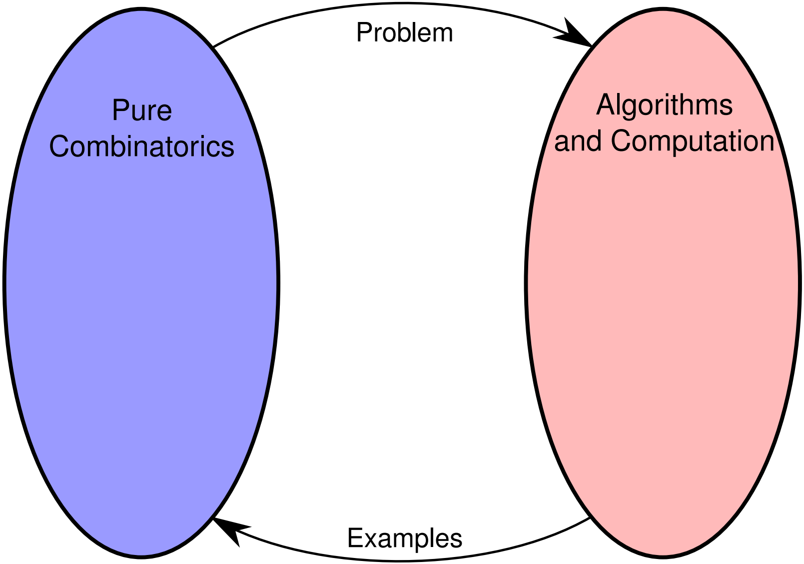

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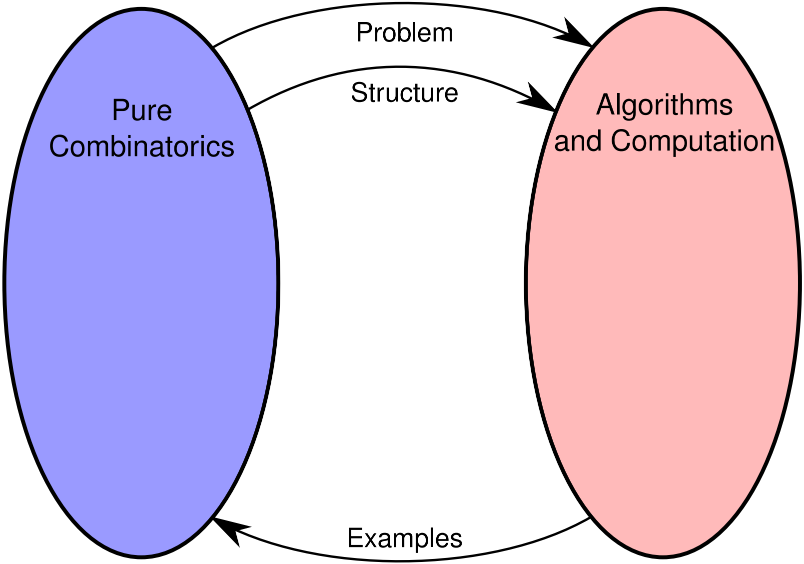

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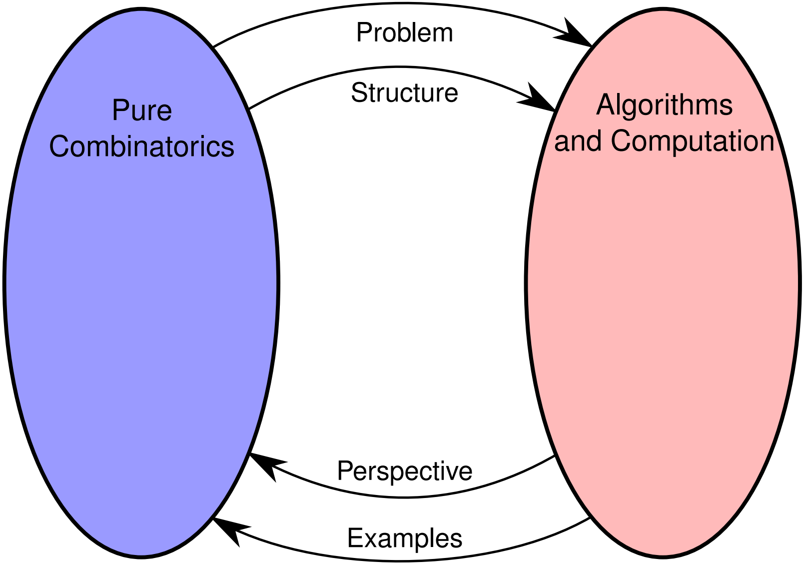

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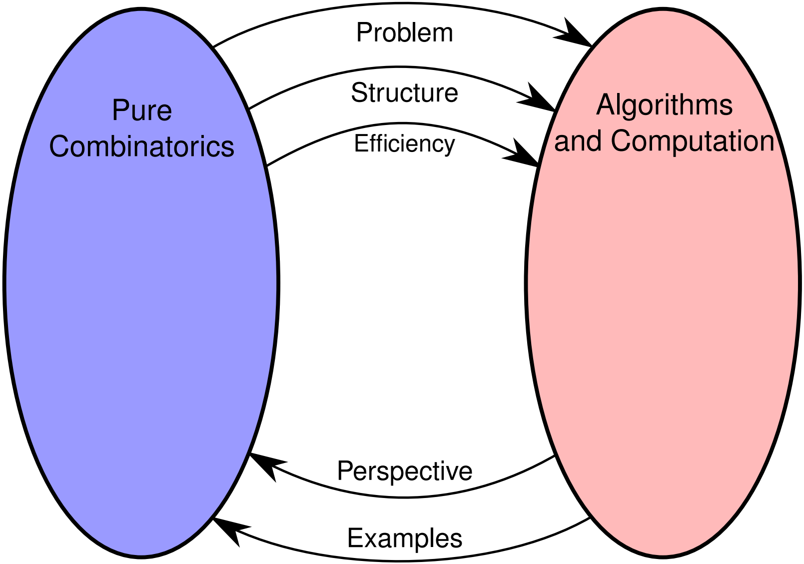

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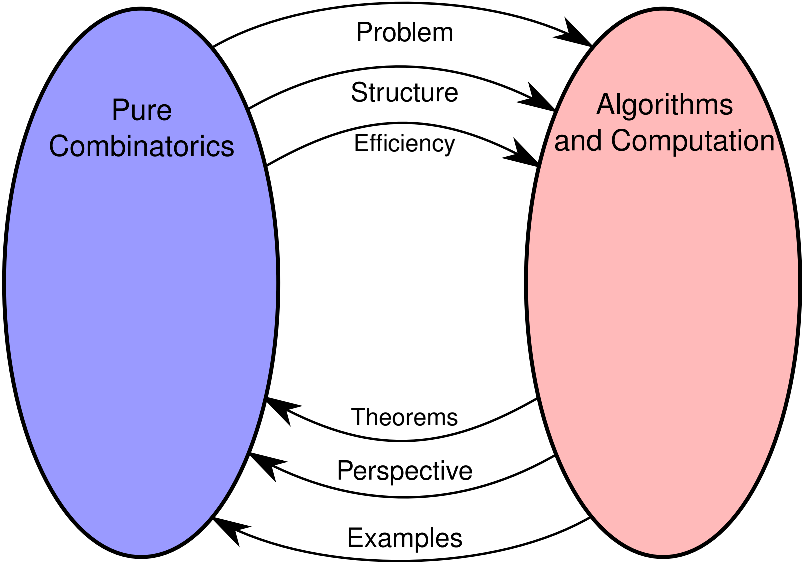

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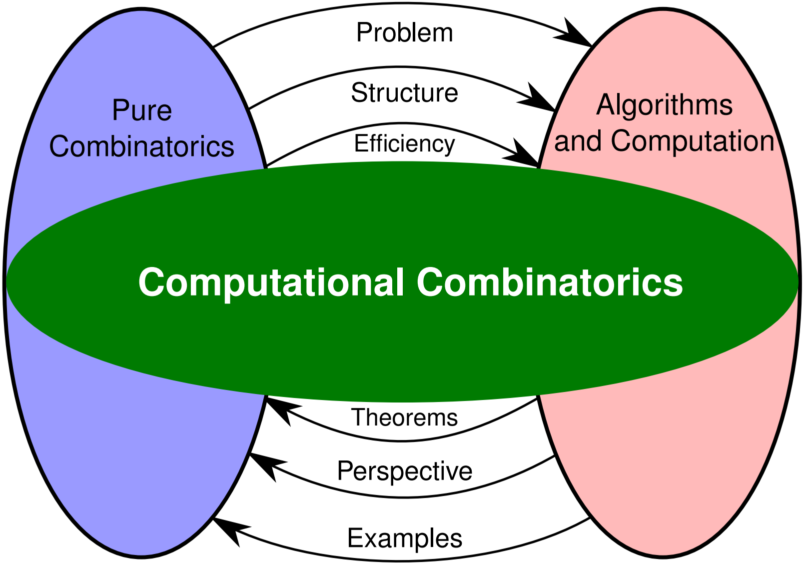

The Goal

Determine if certain **combinatorial objects** exist with given **structural** or **extremal** properties.

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Examples:

1. Is there a **projective plane** of order 10?
2. When do **strongly regular graphs** exist?
3. How many **Steiner triple systems** are there of order 19?

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Examples:

1. Is there a **projective plane** of order 10?
   (Lam, Thiel, Swiercz, 1989)
2. When do **strongly regular graphs** exist?
   (Spence 2000, Coolsaet, Degraer, Spence 2006, many others)
3. How many **Steiner triple systems** are there of order 19?
   (Kaski, Östergård, 2004)

Problems Tackled in This Thesis

1. Which numbers are representable as the number of chains in a width-two poset?
2. Which colorings of $\{1, \ldots, n\}$ avoid monochromatic progressions?
3. How many edges can exist in a graph with p perfect matchings?
4. What graphs are uniquely K_r-saturated?

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Main Technique: Combinatorial Search

Goal: Determine if certain combinatorial objects exist with given structural or extremal properties.

Idea: Build objects *piece-by-piece* from *base examples* to enumerate all desired examples of a given order.

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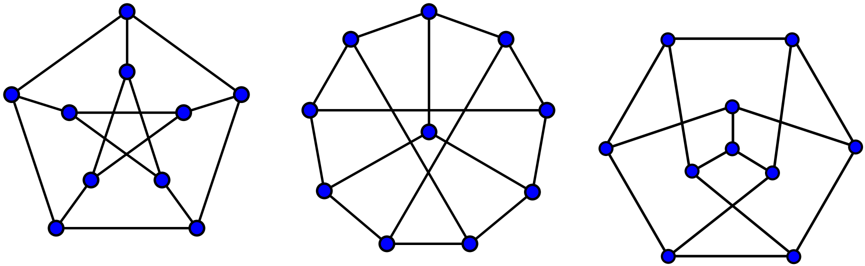
Most interesting properties are invariant under **isomorphism***.*

Combinatorial Object: Graphs

A **graph** G of **order** n is composed of a set $V(G)$ of n vertices and a set $E(G)$ of edges, where the edges are unordered pairs of vertices.



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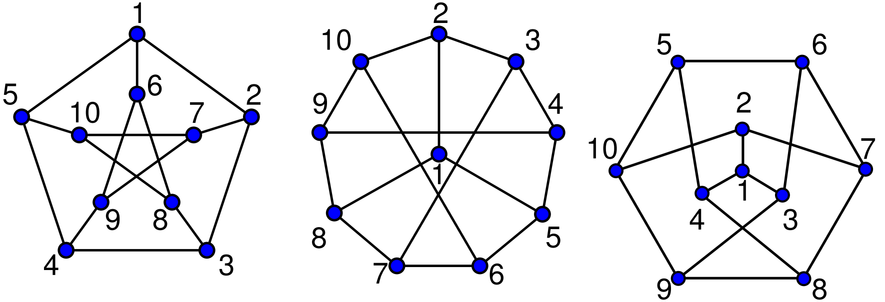
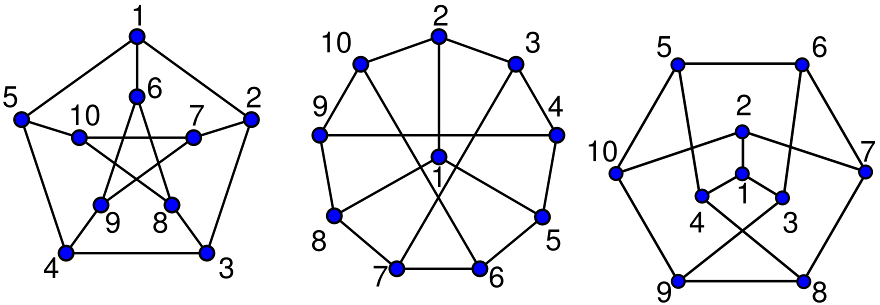


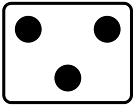
Combinatorial Object: Graphs

An **isomorphism** between G_1 and G_2 is a bijection from $V(G_1)$ to $V(G_2)$ that induces a bijection from $E(G_1)$ to $E(G_2)$.



Example: Generating Graphs by Edges

We can build graphs starting at $\overline{K_n}$ by adding edges.



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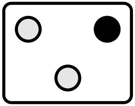

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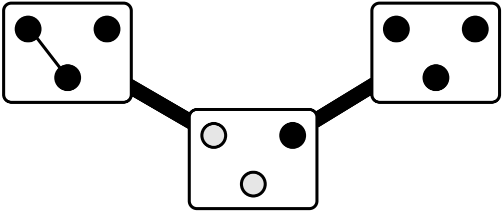

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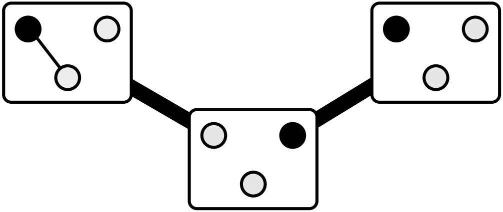

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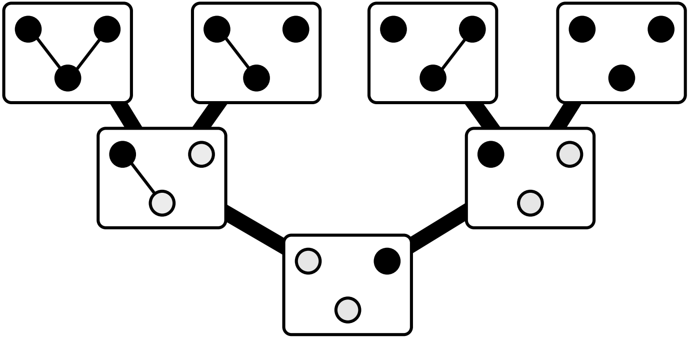

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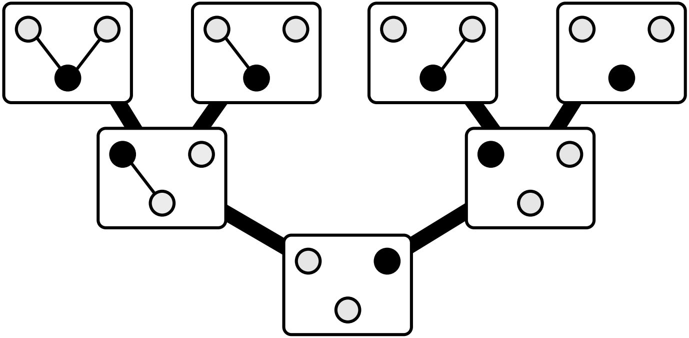

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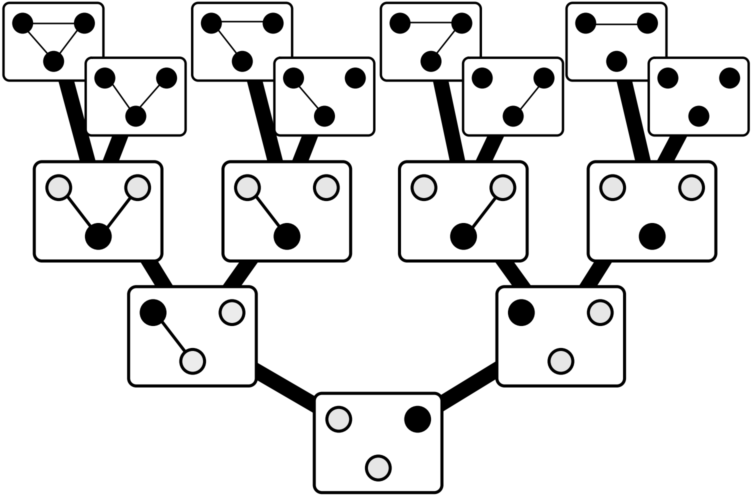

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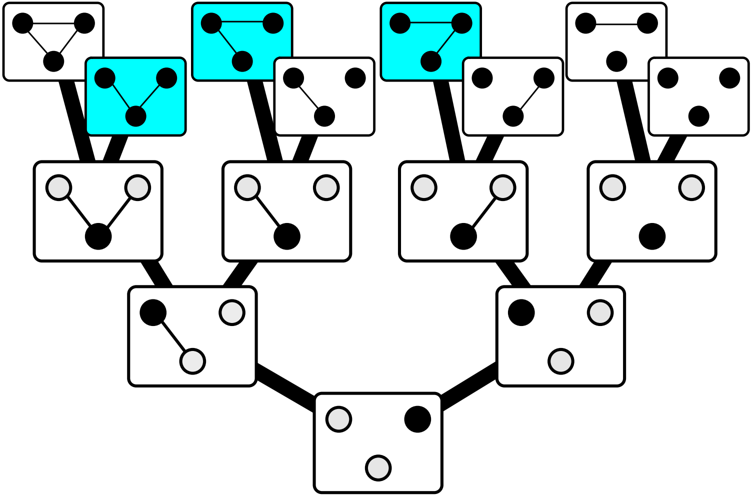

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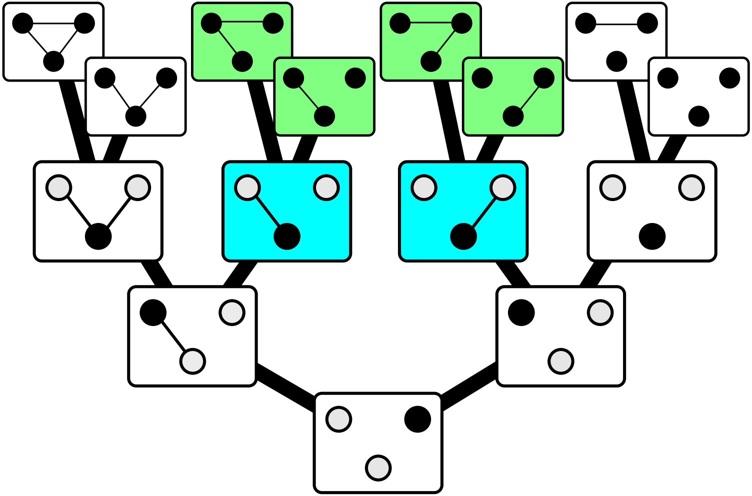

Two Techniques for Isomorphs

1. Canonical Deletion (McKay 1998)
   - Removes all isomorphs.
   - Not known how to integrate with constraint propagation.
   - High cost per object.

2. Orbital Branching (Ostrowski, Linderoth, Rossi, Smriglio 2007)
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   Overview in Chapter 6

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   Overview in Chapter 10

Search by Augmentations

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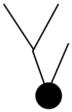

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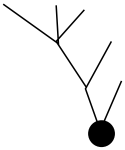

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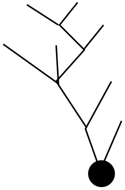

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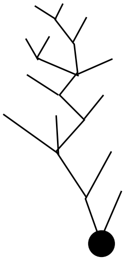

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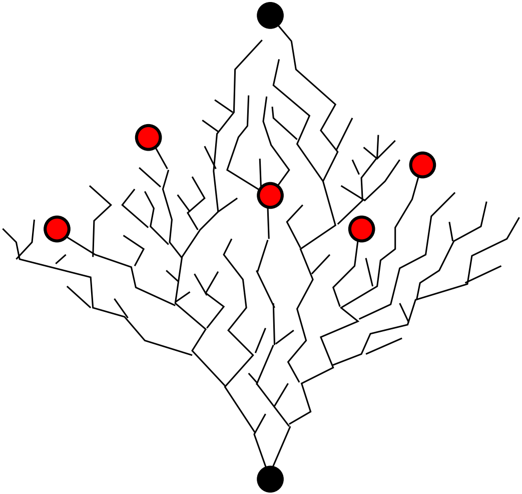

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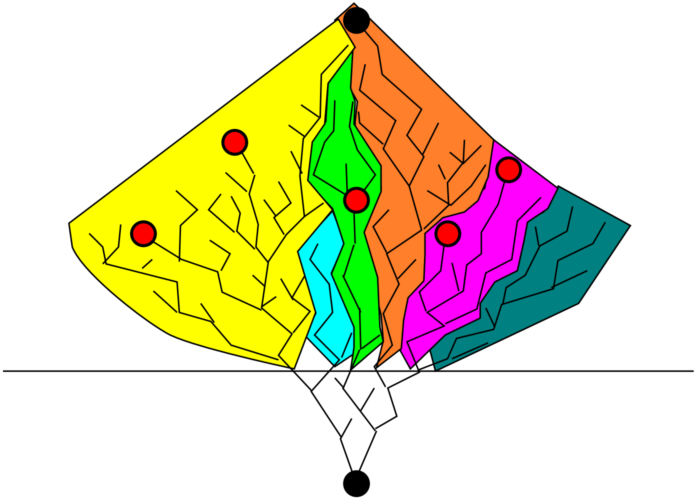

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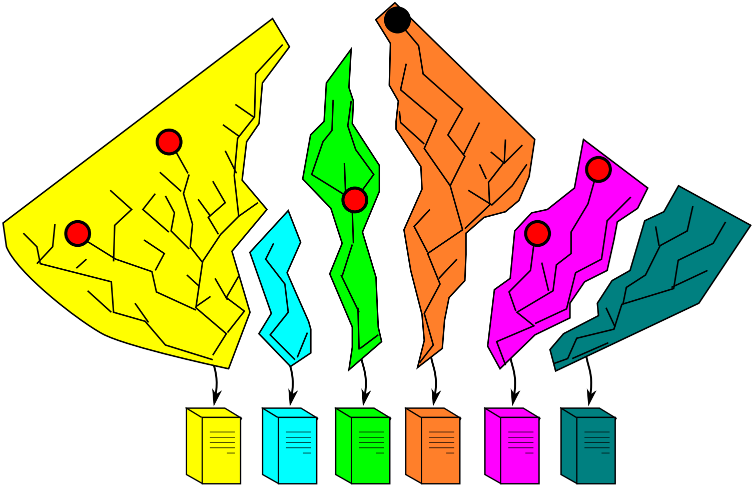

Implementation

My **TreeSearch** library enables parallelization in the Condor scheduler.

Executes on the **Open Science Grid**, a collection of supercomputers around the country.

Open Science Grid

Problems Tackled in This Thesis

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   (with Kupin, Reiniger) Chapter 4
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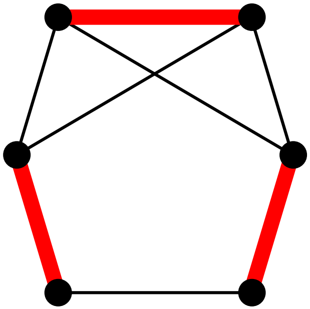
Perfect Matchings

A **perfect matching** is a set of edges which cover each vertex exactly once.



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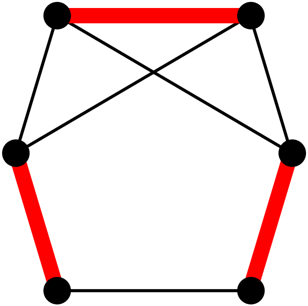
$\Phi(G)$ is the number of perfect matchings in the graph G.



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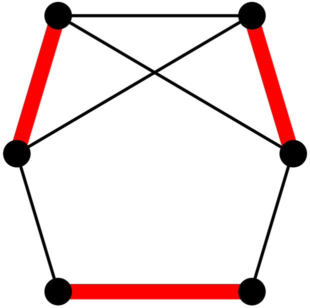

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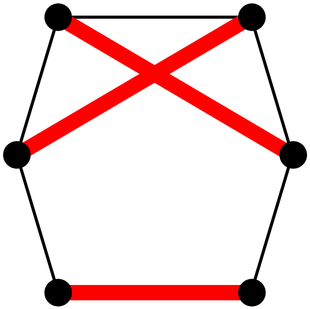

$\Phi(G) = 3$
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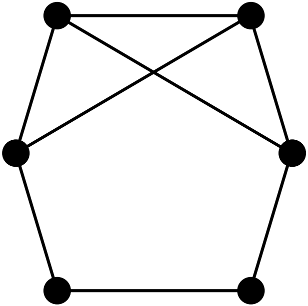

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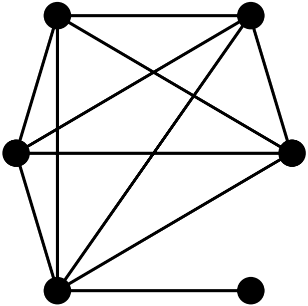

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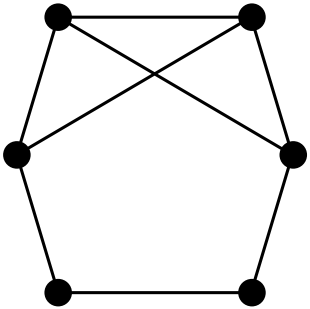

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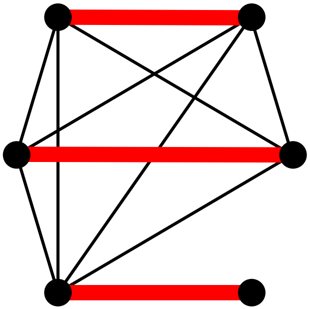

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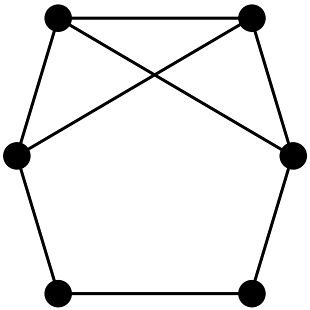

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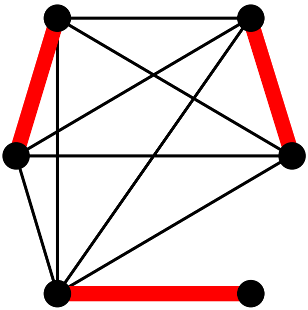

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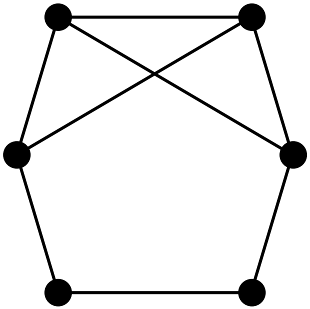

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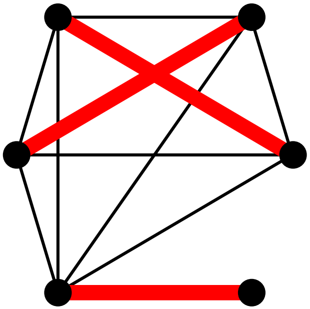

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Perfect Matchings

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Question (Dudek, Schmitt, 2010) What is the maximum number of edges in a graph with exactly n vertices and p perfect matchings?

Definition Let n be an even number and fix $p \geq 1$.

$$f(n,p) = \max\{|E(G)| : |V(G)| = n, \Phi(G) = p\}.$$

Graphs attaining this number of edges are ***p*-extremal**.

Hetyei's Theorem

Theorem (Hetyei's Theorem, 1986) For all even $n \geq 2$,

$$f(n, 1) = \frac{n^2}{4}.$$

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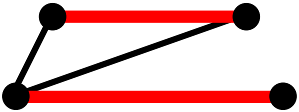

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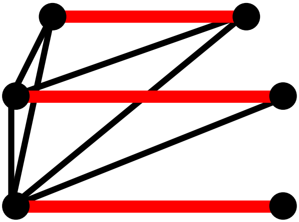

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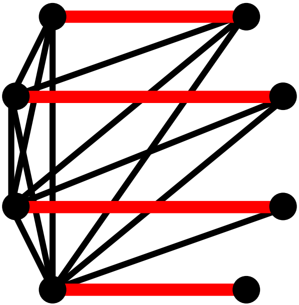

The Form of $f(n,p)$

Theorem (Dudek, Schmitt, 2010) For each p, there exist constants n_p, c_p so that for all $n \geq n_p$,

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p	1	2	3	4	5	6
c_p	0	1	2	2	2	3
	H	Dudek, Schmitt, 2010				

Structure Theorem

Theorem (Hartke, Stolee, West, Yancey, 2011) For a fixed p, every graph G with n vertices, p perfect matchings, and $f(n, p) = \frac{n^2}{4} + c_p$ edges is composed of a finite list of **fundamental graphs** combined in specified ways.

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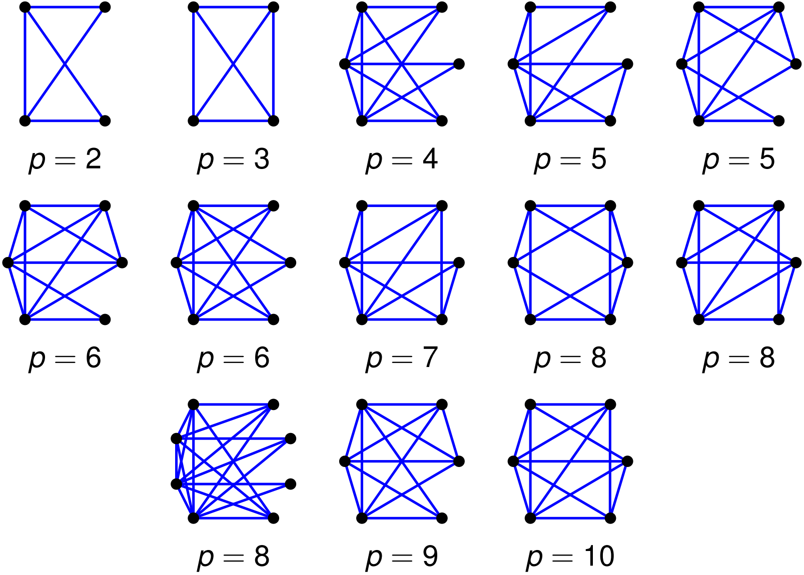
Proof involves several classic structure theorems from matching theory in an extremal setting.

For $p \leq 10$, the graphs have order at most 12.

Using standard software (McKay's *geng*) we found the graphs and computed c_p.

Fundamental Graphs for $2 \leq p \leq 10$



c_p for small p

p	1	2	3	4	5	6	7	8	9	10
c_p	0	1	2	2	2	3	3	3	4	4
	H	Dudek, Schmitt 2010					HSWY 2011			

c_p for small p

p	1	2	3	4	5	6	7	8	9	10
c_p	0	1	2	2	2	3	3	3	4	4
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Q: Is c_p monotone in p?

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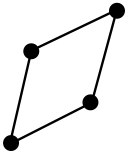

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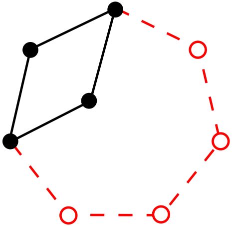

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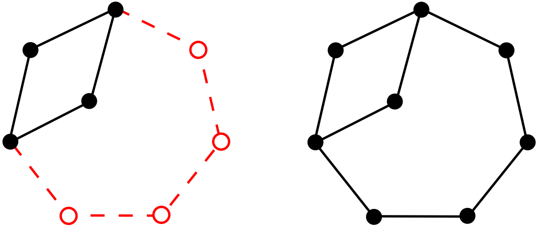

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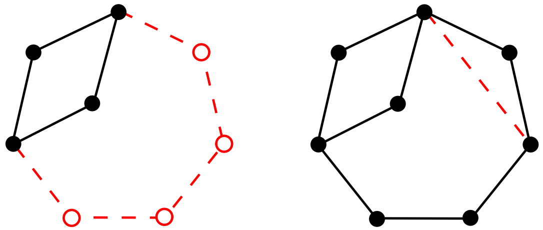

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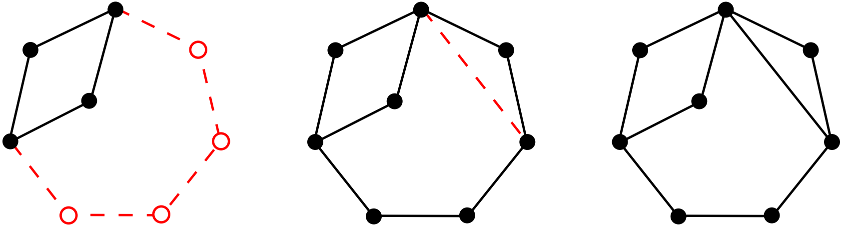

Computational Method

Developed a computational method from:

1. **Augmentations:** Lovász Two Ear Theorem.
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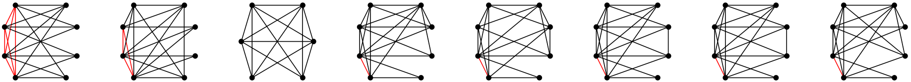
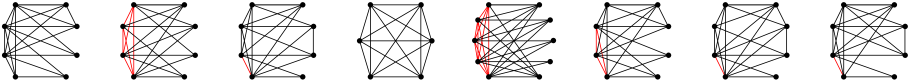
Before: Stuck at $p \leq 10$ when searching on most 12 vertices.

Now: Found graphs for all $p \leq 27$ on up to 22 vertices.

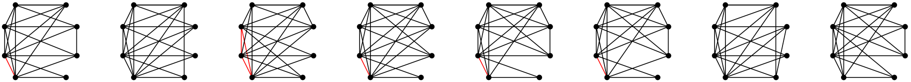
Fundamental Graphs for $11 \leq p \leq 27$



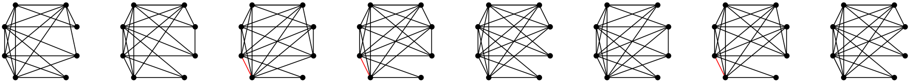
$p = 11$ $p = 11$ $p = 12$ $p = 13$ $p = 13$ $p = 13$ $p = 13$ $p = 13$



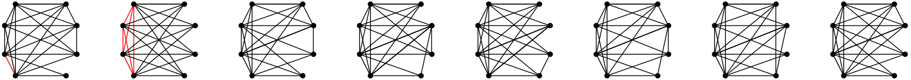
$p = 13$ $p = 14$ $p = 14$ $p = 15$ $p = 16$ $p = 16$ $p = 16$ $p = 16$



$p = 17$ $p = 17$ $p = 18$ $p = 18$ $p = 19$ $p = 19$ $p = 19$ $p = 19$



$p = 19$ $p = 19$ $p = 20$ $p = 21$ $p = 21$ $p = 21$ $p = 22$ $p = 23$



$p = 24$ $p = 24$ $p = 25$ $p = 25$ $p = 26$ $p = 26$ $p = 26$ $p = 27$

c_p for small p

p	1	2	3	4	5	6	7	8	9	10
c_p	0	1	2	2	2	3	3	3	4	4
	H	Dudek, Schmitt, 2010					HSWY, 2011			

p	11	12	13	14	15	16	17	18	19	20
c_p	3	5	3	4	6	4	4	5	4	5
	Stolee, 2011									

p	21	22	23	24	25	26	27
c_p	5	5	5	6	5	5	6
	Stolee, 2011						

c_p for small p

p	1	2	3	4	5	6	7	8	9	10
c_p	**0**	**1**	**2**	**2**	**2**	**3**	**3**	**3**	**4**	**4**
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p	11	12	13	14	15	16	17	18	19	20
c_p	3	**5**	3	4	**6**	4	4	5	4	5
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p	21	22	23	24	25	26	27
c_p	5	5	5	**6**	5	5	**6**
	Stolee, 2011						

c_p not monotonic in p !

Blue numbers match conjectured upper bound.

Problems Tackled in This Thesis

1. Which numbers are representable as the number of chains in a width-two poset?
   (with Kupin, Reiniger) Chapter 4
2. Which colorings of $\{1, \ldots, n\}$ avoid monochromatic progressions?
   (with Jobson, Kézdy) Chapter 5
3. How many edges can exist in a graph with p perfect matchings?
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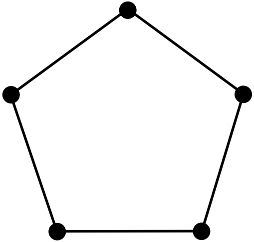
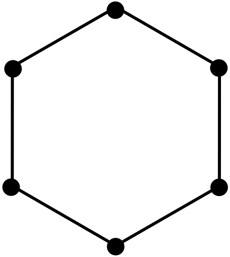
H-Saturated Graphs

Definition A graph G is ***H*-saturated** if

- G does not contain H as a subgraph. (***H*-free**)
- For every $e \in E(\overline{G})$, $G + e$ contains H as a subgraph.



5-cycle



6-cycle

Example: $H = K_3$ where K_r is the **complete graph** on r vertices.

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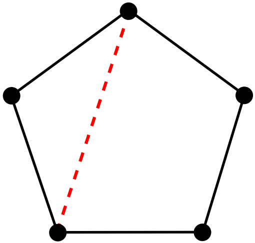

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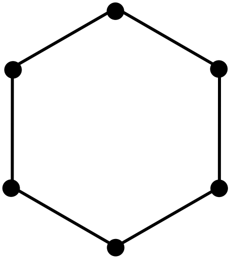

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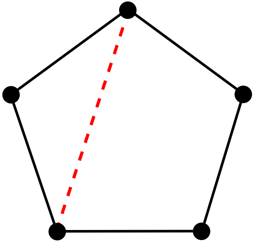

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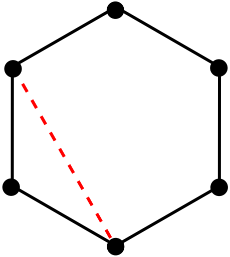

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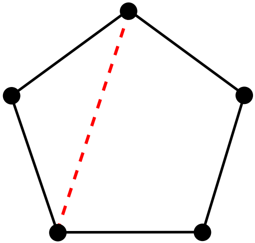

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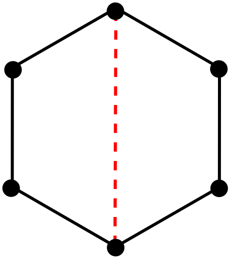

6-cycle
is **not** K_3-saturated

Example: $H = K_3$ where K_r is the **complete graph** on r vertices.

Turán's Theorem

Theorem (Turán, 1941) Let $r \geq 3$. If G is K_r-saturated on n vertices, then G has **at most** $\left(1 - \frac{1}{r-1}\right)\frac{n^2}{2}$ edges (asymptotically).

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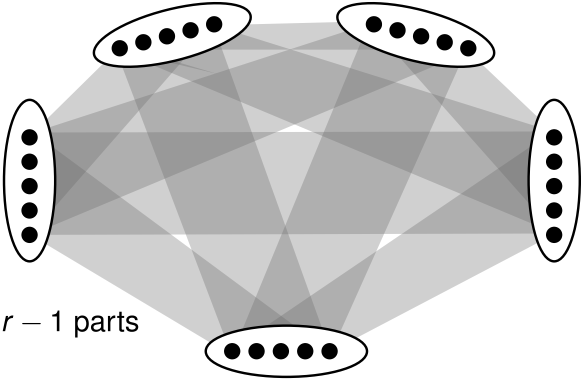

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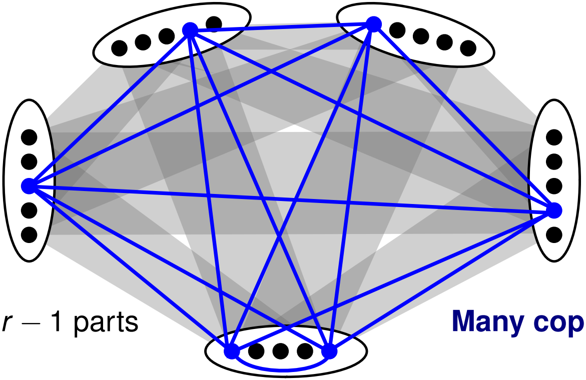

Erdős, Hajnal, and Moon

Theorem (Erdős, Hajnal, Moon, 1964) Let $r \geq 3$. If G is K_r-saturated on n vertices, then G has **at least** $\binom{r-2}{2} + (r-2)(n-r+2)$ edges.

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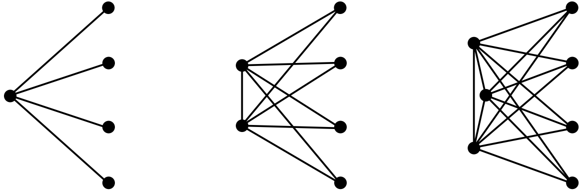

1-book   2-book   3-book

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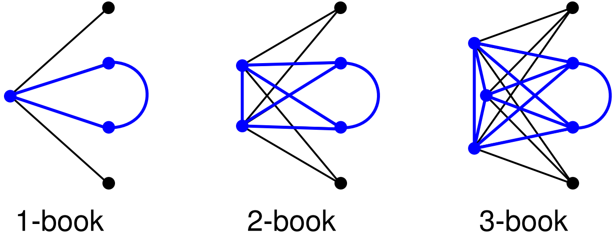

Exactly one copy of K_r!

Uniquely H-Saturated Graphs

The Turán graph has **many** copies of K_r when an edge is added.

The books have **exactly one** copy of K_r when an edge is added.

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The Turán graph has **many** copies of K_r when an edge is added.

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Definition A graph G is **uniquely H-saturated** if G does not contain H as a subgraph and for every edge $e \in \overline{G}$ admits **exactly one** copy of H in $G + e$.

We consider the case where $H = K_r$ (an **r-clique**).

Uniquely K_3-Saturated Graphs

Lemma (Cooper, Lenz, LeSaulnier, Wenger, West, 2011)

The uniquely K_3-saturated graphs are either **stars** or **Moore graphs** of diameter 2 and girth 5.

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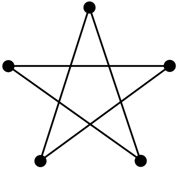
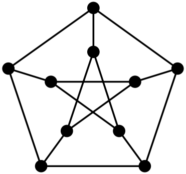
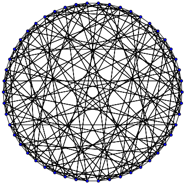


C_5



Petersen



Hoffman–Singleton

?

57-Regular
Order 3250

Dominating Vertices

Adding a dominating vertex to a uniquely K_r-saturated graph creates a uniquely K_{r+1}-saturated graph.

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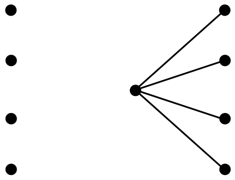

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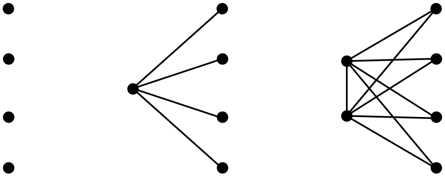

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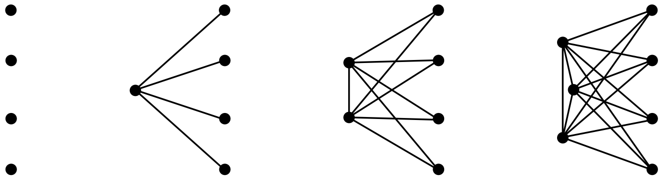

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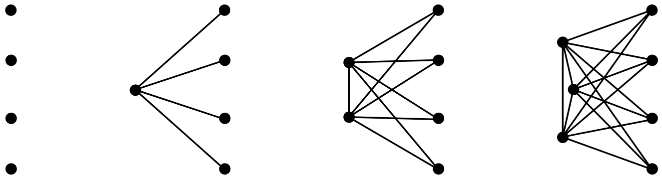

Call uniquely K_r-saturated graphs without a dominating vertex

***r*-primitive**.

r-Primitive Graphs

A uniquely K_r-saturated graph with no dominating vertex is ***r*-primitive**.

r-Primitive Graphs

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2**-primitive** graphs are **empty graphs**.

r-Primitive Graphs

A uniquely K_r-saturated graph with no dominating vertex is ***r*-primitive**.

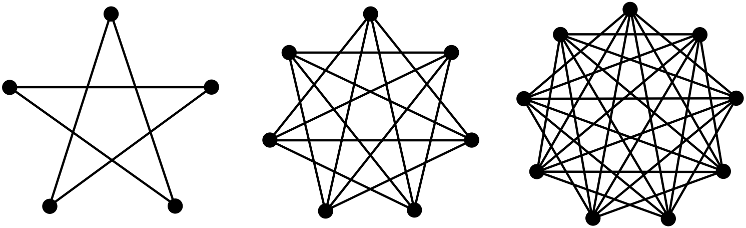
2**-primitive** graphs are **empty graphs**.

3**-primitive** graphs are **Moore graphs** of diameter 2 and girth 5.

r-Primitive Graphs

A uniquely K_r-saturated graph with no dominating vertex is **r-primitive**.

For $r \geq 1$, $\overline{C_{2r-1}}$ is r-primitive.



$\overline{C_5}$ $\overline{C_7}$ $\overline{C_9}$

(Collins, Cooper, Kay, Wenger, 2010)

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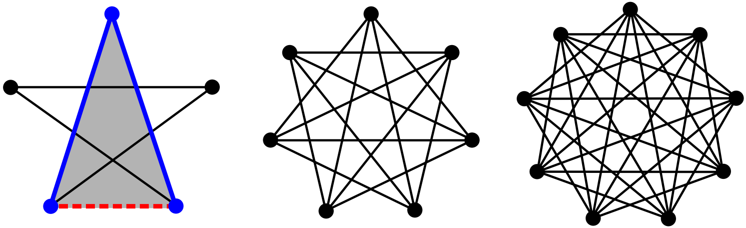

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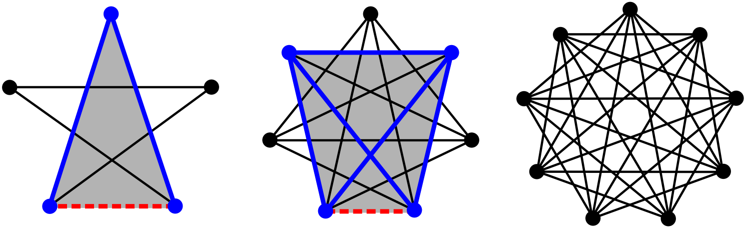

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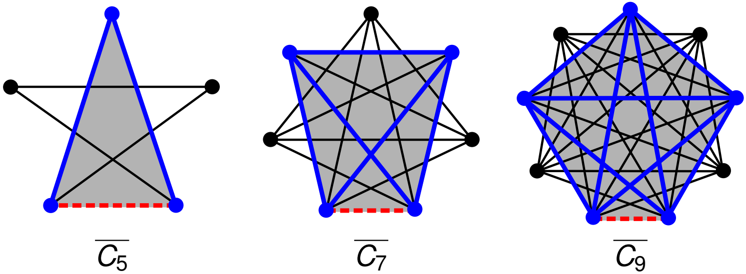

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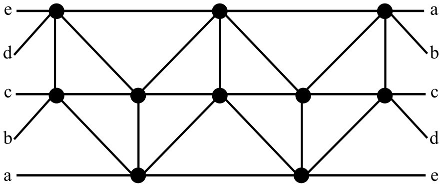
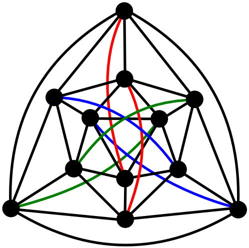
Uniquely K_4-Saturated Graphs



10 vertices



12 vertices

Previously known 4-primitive graphs (Collins, Cooper, Kay, 2010)

Two Questions

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1. Fix $r \geq 3$. Are there a **finite number** of r-primitive graphs?

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2. Is every r-primitive graph **regular**?

Edges and Non-Edges

Non-edges are crucial to the structure of r-primitive graphs.

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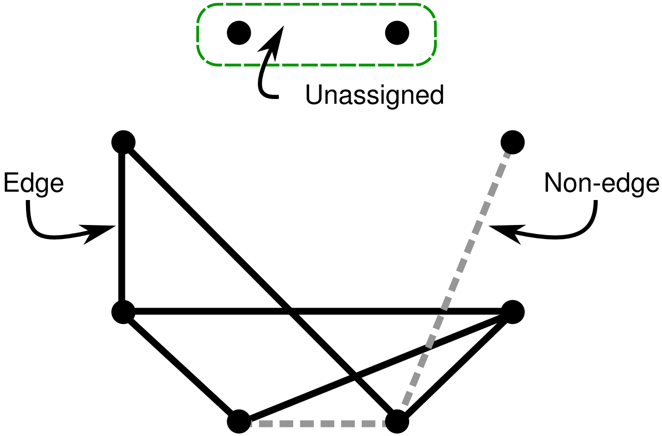
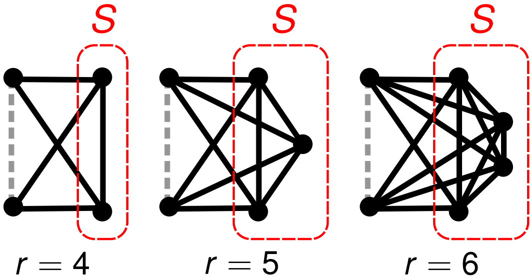


Trigraph

K_r-Completions

For every non-edge we add, we add a K_r-**completion**:

ij a non-edge **if and only if** there exists a set $S \subset [n]$, $|S| = r - 2$, so that ia, ja, and ab are edges for all $a, b \in S$.



Computational Method

Developed a computational method from:

1. **Augmentations:** K_r-Completions.
2. **Isomorphs:** Orbital Branching. Ostrowsky *et al.*
3. **Pruning:** Contains K_r or double-completion.

Exhaustive Search Times

n	$r = 4$	$r = 5$	$r = 6$	$r = 7$	$r = 8$
10	0.10 s	0.37 s	0.13 s	0.01 s	0.01 s
11	0.68 s	5.25 s	1.91 s	0.28 s	0.09 s
12	4.58 s	1.60 m	25.39 s	1.97 s	1.12 s
13	34.66 s	34.54 m	6.53 m	59.94 s	20.03 s
14	4.93 m	10.39 h	5.13 h	20.66 m	2.71 m
15	40.59 m	23.49 d	10.08 d	12.28 h	1.22 h
16	6.34 h	1.58 y	1.74 y	34.53 d	1.88 d
17	3.44 d			8.76 y	115.69 d
18	53.01 d				
19	2.01 y				
20	45.11 y				

Total CPU times using Open Science Grid.

⟵ clique size ⟶

⟵ vertices ⟶

$n \setminus r$		2	3	4	5	6	7	8	
2									
3									
4									
5									
6									
7									
8									
9									
10									
11									
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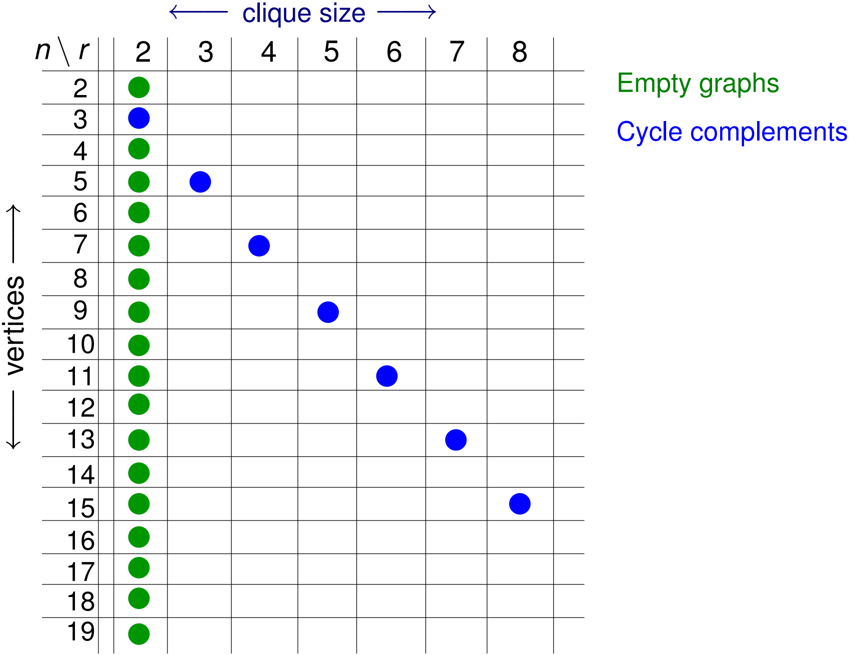
13									
14									
15									
16									
17									
18									
19									

← clique size →

Empty graphs

vertices

$n \setminus r$	2	3	4	5	6	7	8
2	●						
3	●						
4	●						
5	●						
6	●						
7	●						
8	●						
9	●						
10	●						
11	●						
12	●						
13	●						
14	●						
15	●						
16	●						
17	●						
18	●						
19	●						


clique size
n \ r
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4
5
6
7
8
2
3
4
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15
16
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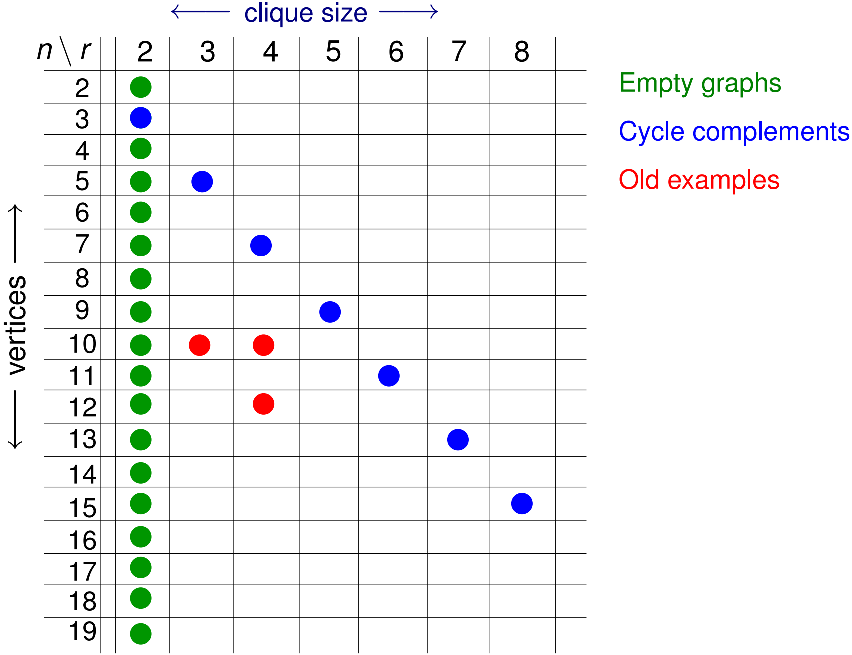
18
19
vertices
Empty graphs
Cycle complements



clique size
n \ r
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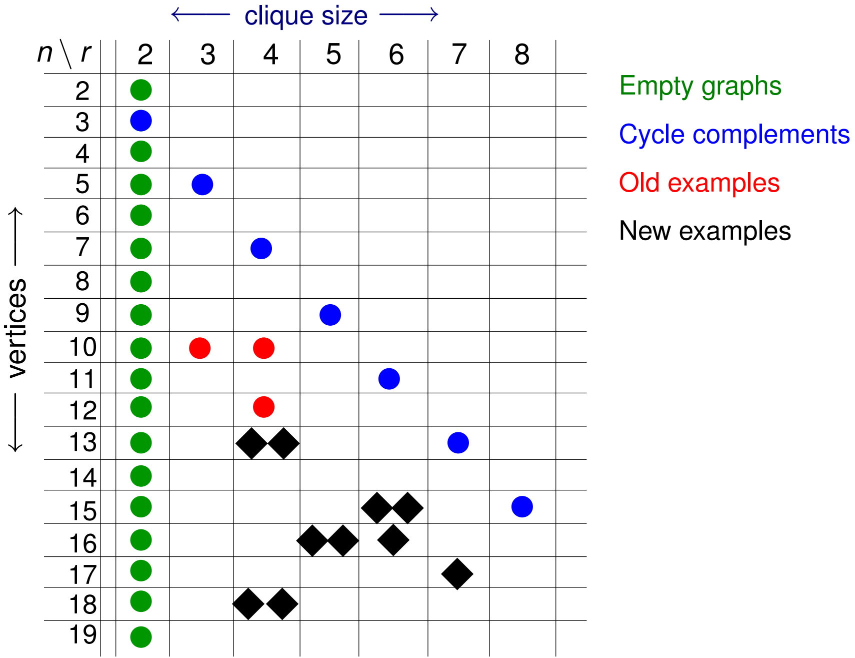
Old examples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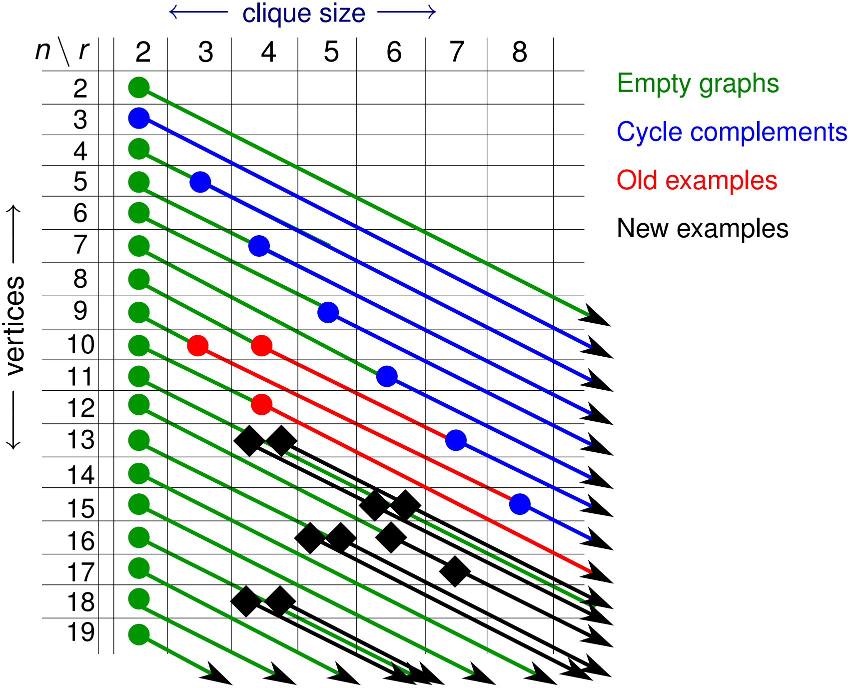

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n \ r
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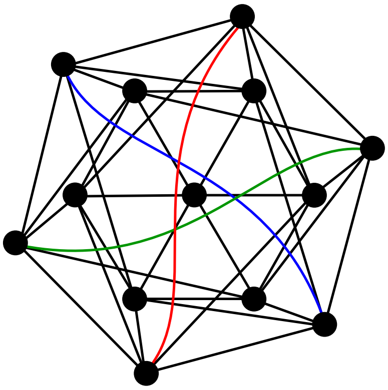
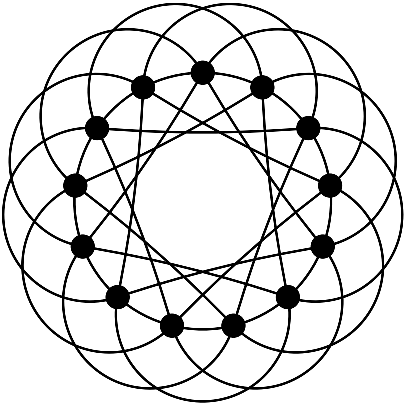
4-Primitive Graphs

$n = 13$



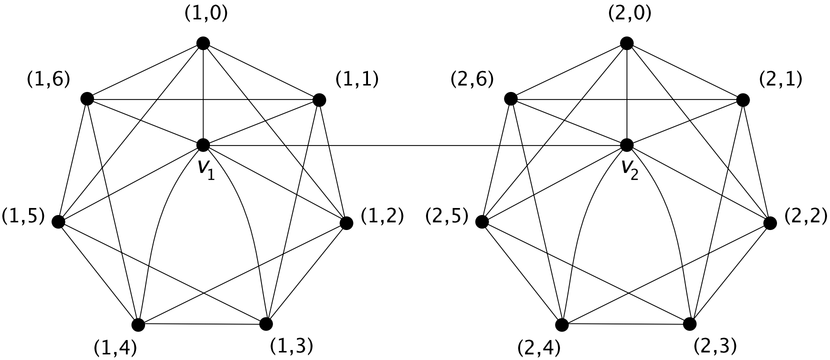
$G_{13}^{(A)}$



Paley(13)

5-Primitive Graph

$n = 16 : G_{16}^{(A)}$



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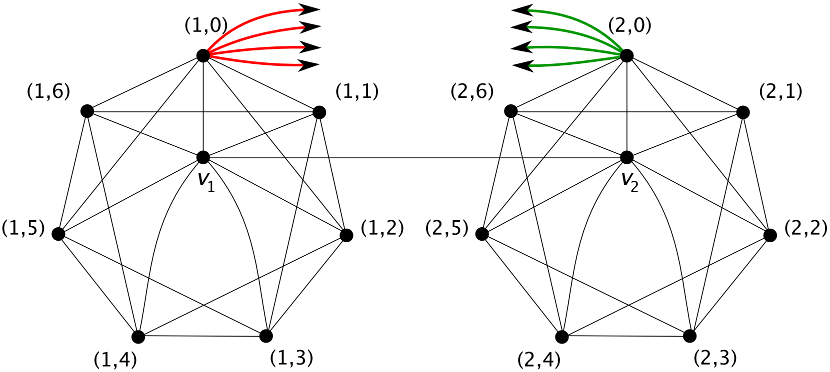

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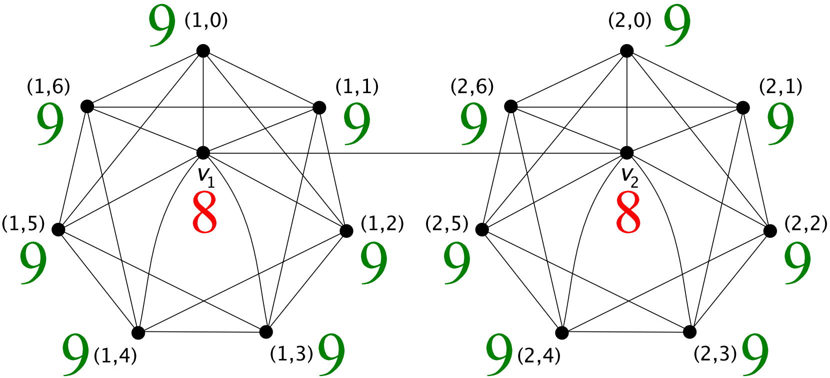

Not all *r*-primitive graphs are regular!

7-Primitive Graph

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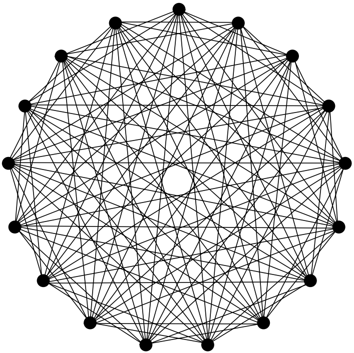

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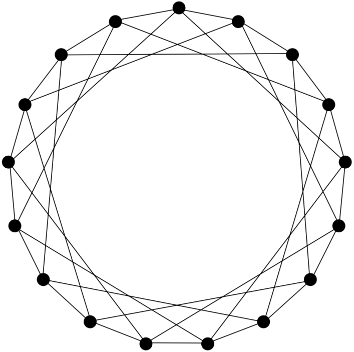

Let Γ be a group and $S \subseteq \Gamma$ a set of generators.

The undirected **Cayley graph** $C(\Gamma, S)$ has vertex set Γ and for all $a \in \Gamma$ and $b \in S$, there is an edge between a and ab.

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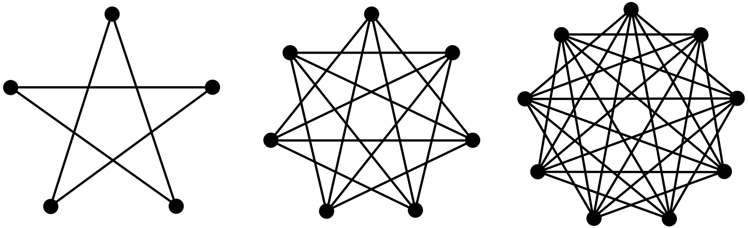
For $r \geq 1$, $\overline{C}(\mathbb{Z}_{2r-1}, \{1\}) \cong \overline{C_{2r-1}}$ is r-primitive.



$\overline{C_5}$ $\overline{C_7}$ $\overline{C_9}$

Two or Three Generators

S	r	n
$\{1,4\}$	7	17
$\{1,6\}$	16	37
$\{1,8\}$	29	65
$\{1,10\}$	46	101
$\{1,12\}$	67	145

$g = 2$

S	r	n
$\{1,5,6\}$	9	31
$\{1,8,9\}$	22	73
$\{1,11,12\}$	41	133
$\{1,14,15\}$	66	211
$\{1,17,18\}$	97	307

$g = 3$

Infinite Families

Conjecture (Hartke, Stolee, 2012) Let $t \geq 1$,

$$n = 4t^2 + 1, \qquad \text{and} \qquad r = 2t^2 - t + 1.$$

The Cayley complement $\overline{C}(\mathbb{Z}_n, \{1, 2t\})$ is r-primitive.

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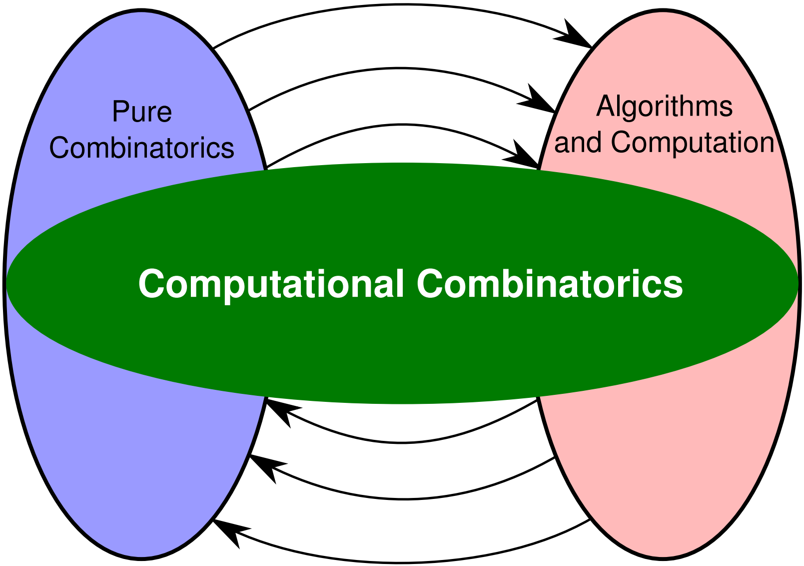
Computational Combinatorics



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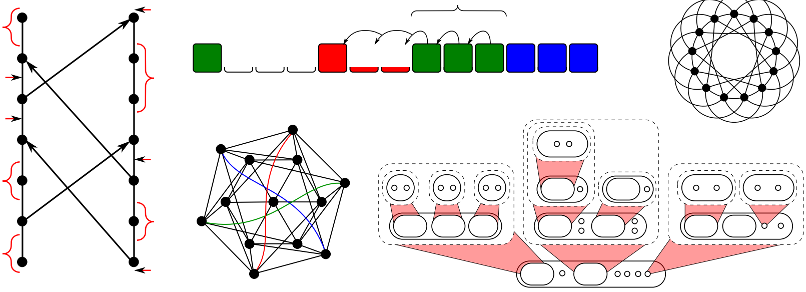
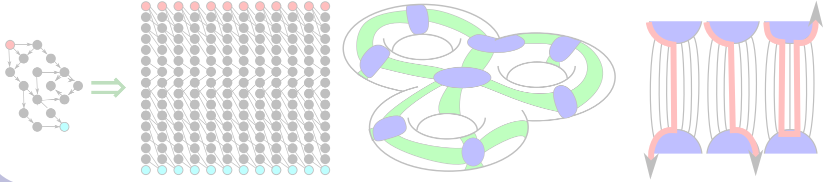


Computational Complexity



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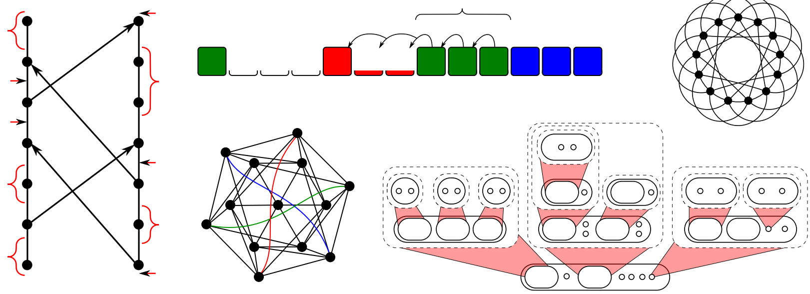

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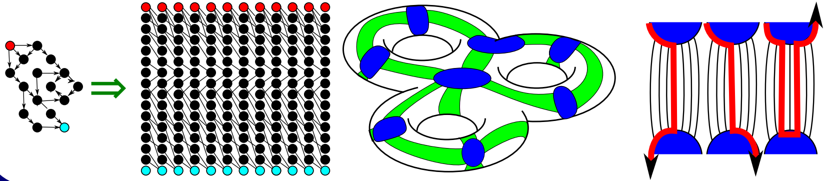

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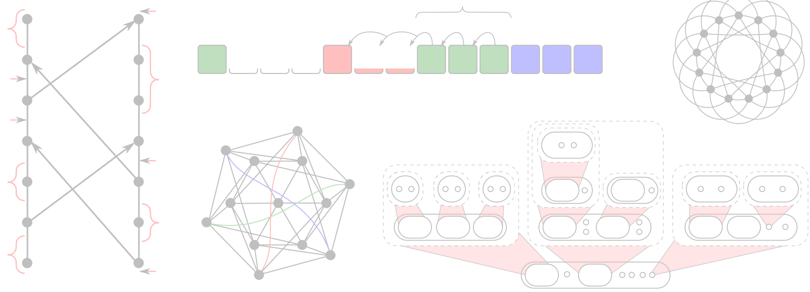

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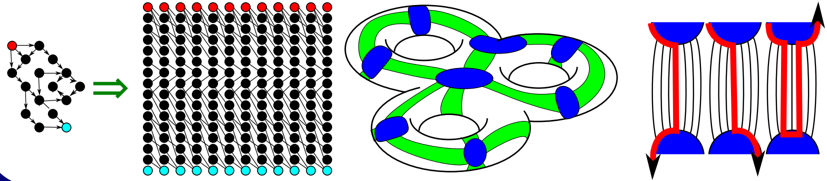

Complexity Results in This Thesis

1. ReachFewL $=$ ReachUL.
   (with Garvin, Tewari, Vinodchandran) Chapter 13
2. Reachability in surface-embedded acyclic digraphs.
   (with Vinodchandran) Chapter 14

Space-Bounded Complexity

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$$\mathsf{L} \subseteq \mathsf{NL}$$

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$G_{M,\mathbf{x}}$ has poly-size and can be written using log-space.

Meta-Theory of Space-Bounded Complexity

Every space-bounded complexity problem can be reduced to some form of the **reachability problem** in digraphs.

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Every space-bounded complexity problem can be reduced to some form of the **reachability problem** in digraphs.

$$\text{Reach} = \{\langle G, s, t\rangle : G \text{ is a directed graph with a path from } s \text{ to } t\}$$

$$\mathsf{L} \subseteq \mathsf{NL} \subseteq \mathsf{P}$$

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Log-space Classes and Reachability

L
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Complete:
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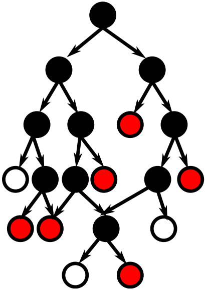

Complete:
Directed Reach

Log-space Classes and Reachability

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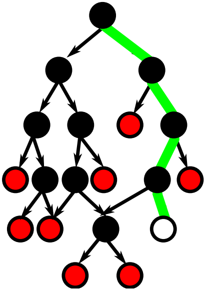
UL
Unambiguous



Contains:
Dir. Planar Reach
(Bourke, Tewari, Vinodchandran 09)

NL
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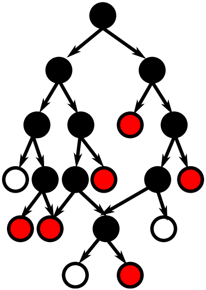
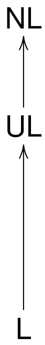


Complete:
Directed Reach

Other Perspectives



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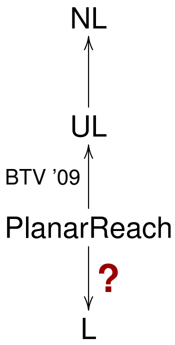

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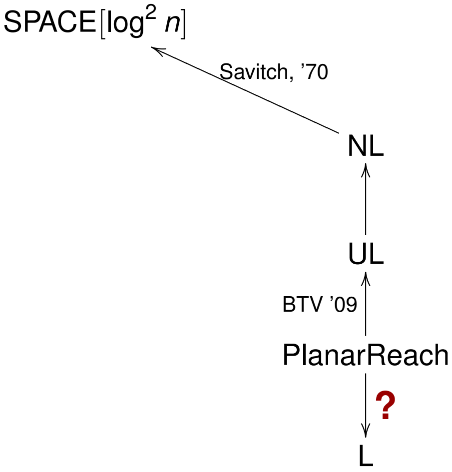

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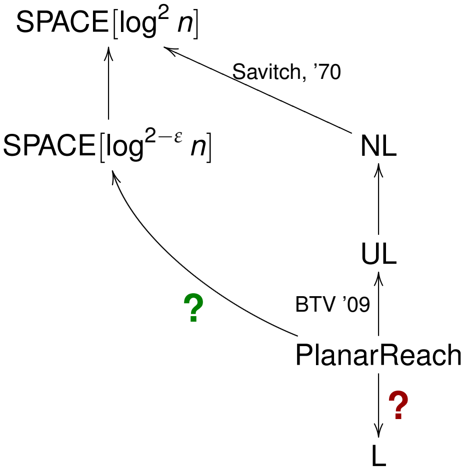

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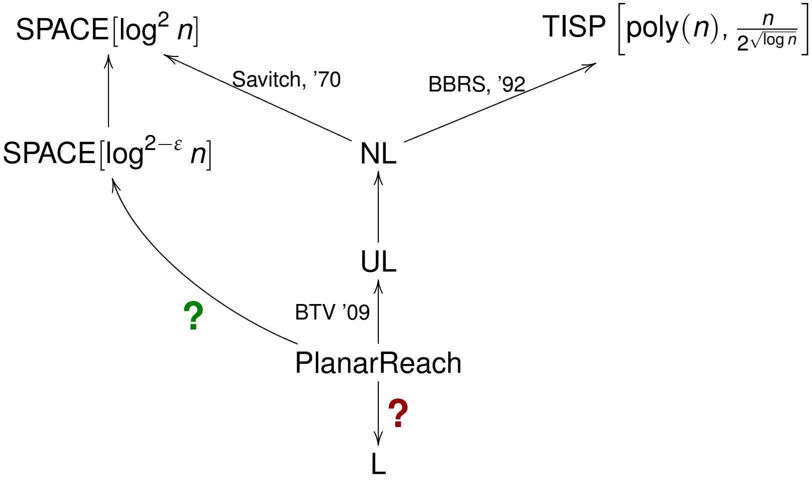

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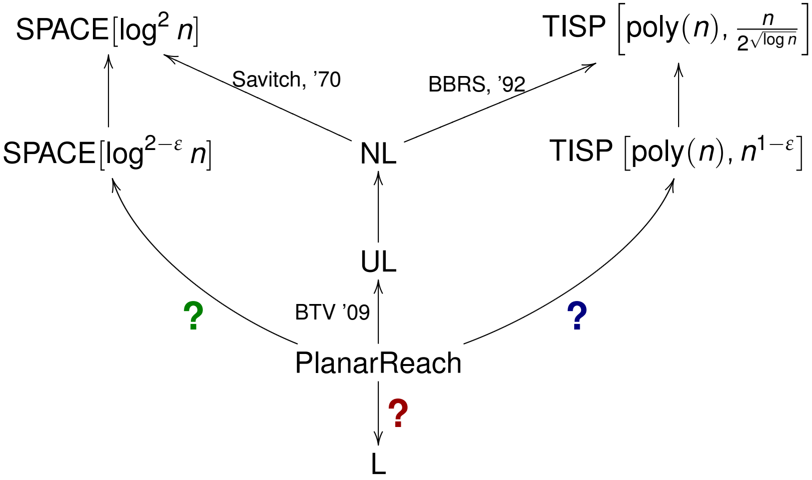

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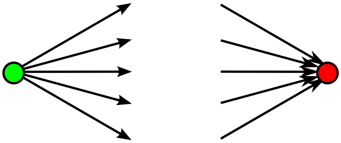
We also bound number of



sources **sinks**

Planar + Acyclic Reachability in Log-Space

1. Series-parallel graphs
   (Jakoby, Liśkiewicz, Reischuk, Tantau, ’06/’07)

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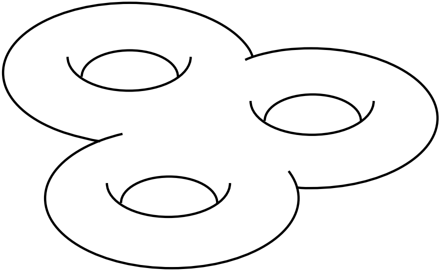
4. Log-source Multiple-Sink Planar DAGs
   (Stolee, Bourke, Vinodchandran, '10)

Surface-embedded graphs

We also extend to graphs embedded in *surfaces of low genus*.



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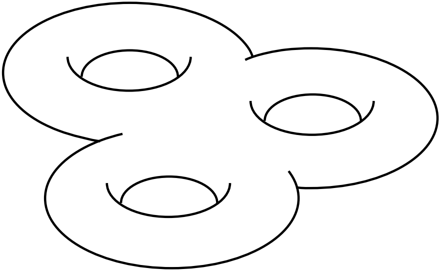


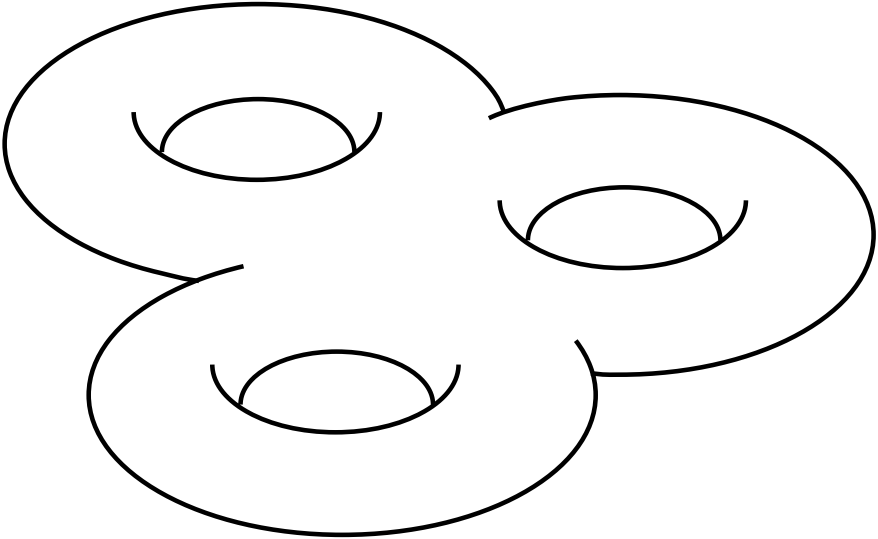
Let $\mathcal{G}(m, g)$ denote the **acyclic** digraphs with m **sources** and embedded in a **genus** g **surface**.

Reduction with Compression

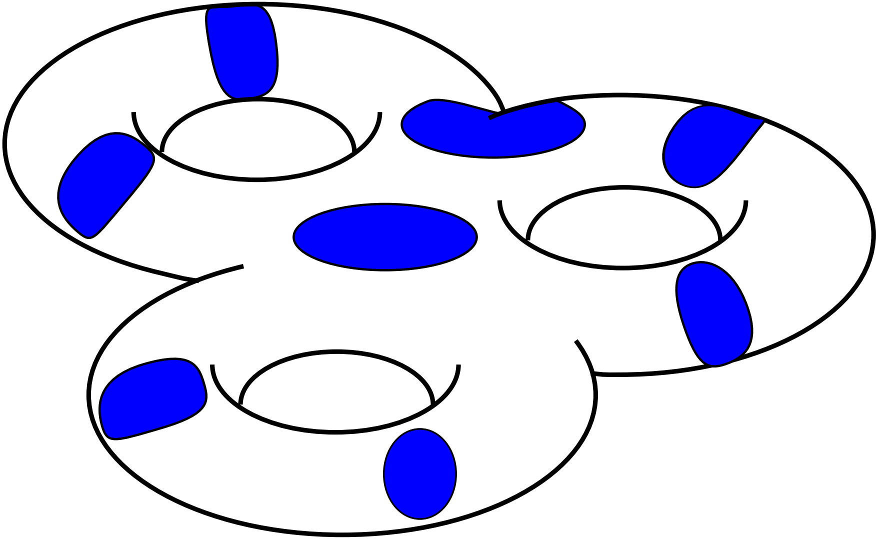
Theorem (Stolee, Vinodchandran, '12) Given a graph $G \in \mathcal{G}(m, g)$ and $s, t \in V(G)$, we can compute in **log-space** a graph G' with vertices s', t' so that

1. There is a path from s to t in G **if and only if** there is a path from s' to t' in G'.
2. G' has $O(m + g)$ vertices.

Topological Equivalence



Topological Equivalence



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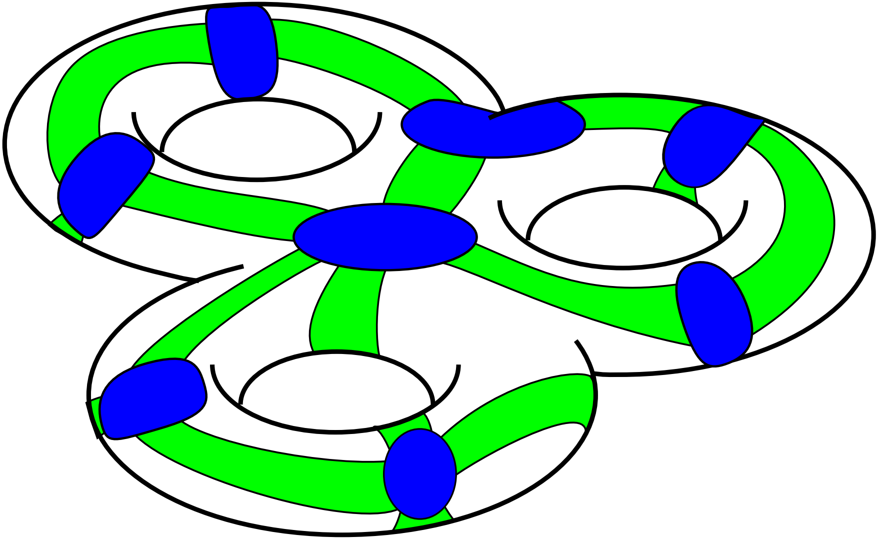

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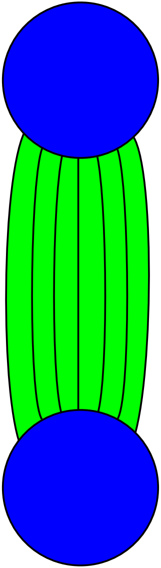

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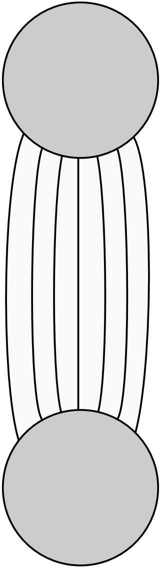

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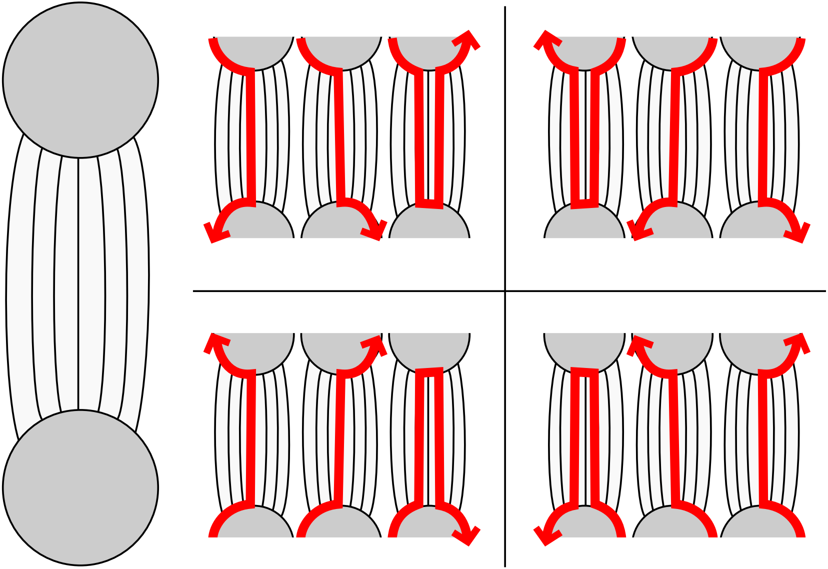

Our Results (Stolee, Vinodchandran, '12)

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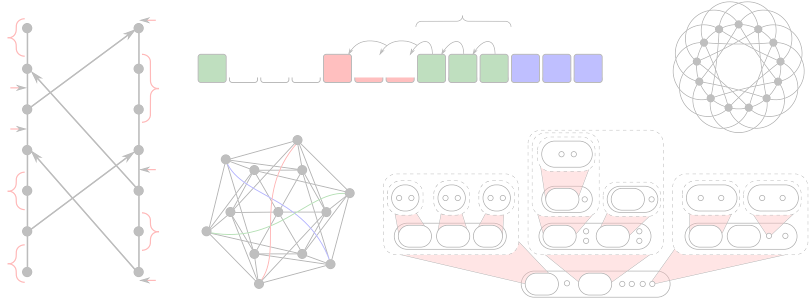
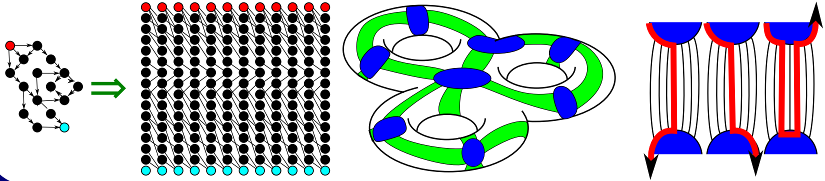
Computational Combinatorics



Computational Complexity



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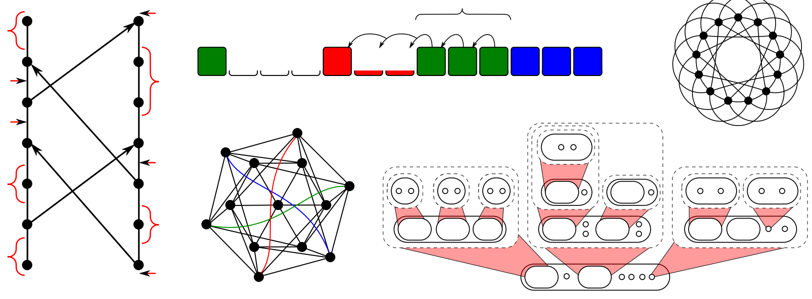

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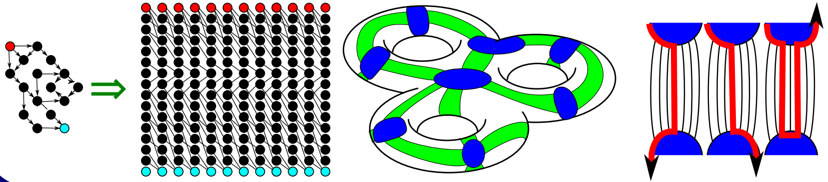
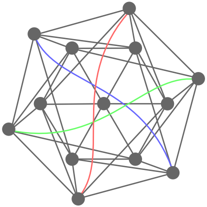
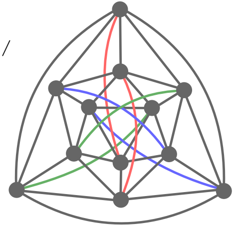


Combinatorics Using Computational Methods

Derrick Stolee

University of Nebraska–Lincoln
s-dstolee1@math.unl.edu
http://www.math.unl.edu/∼s-dstolee1/





March 13, 2012
Dissertation Defense

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and a University of Nebraska Presidential Fellowship.