Computational Combinatorics Blog

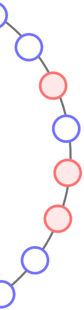
http://computationalcombinatorics.wordpress.com/

An online resource for how to use and extend computational methods in combinatorics, including discussions on the following topics:

- Using software as black box.
- Isomorph-free generation.
- Canonical labelings, orbit calculations.
- Orbital branching.
- Flag Algebras. (on the way)
- Local search techniques (on the way)
- More...

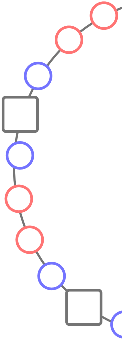
Guest authors are requested!

Distinguishing Extension Number

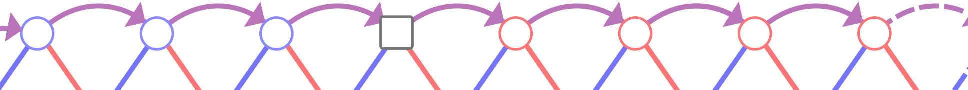


Michael Ferrara Ellen Gethner Stephen G. Hartke
Derrick Stolee* Paul S. Wenger

University of Illinois
stolee@illinois.edu
http://www.math.illinois.edu/~stolee/

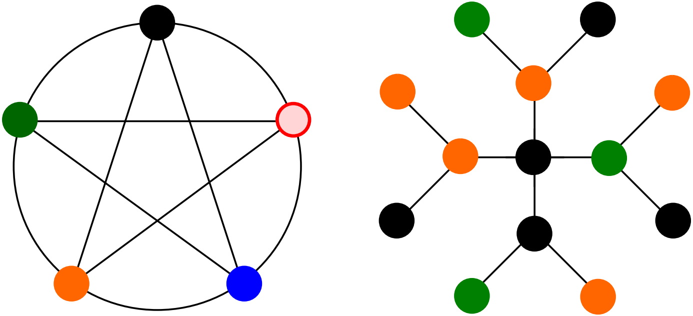


October 20, 2012



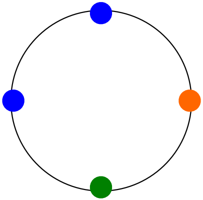
Distinguishing Colorings

A k-coloring of $V(G)$ **distinguishes** G if the only color-preserving automorphism of G is the identity function.



The minimum k such that G has a distinguishing k-coloring is the **distinguishing number,** $D(G)$.

Coloring Cycles



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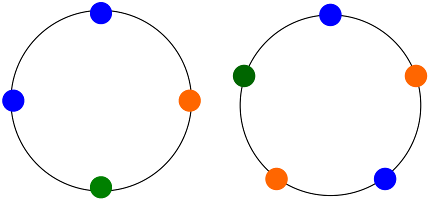

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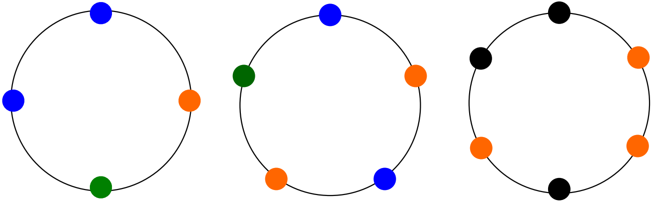

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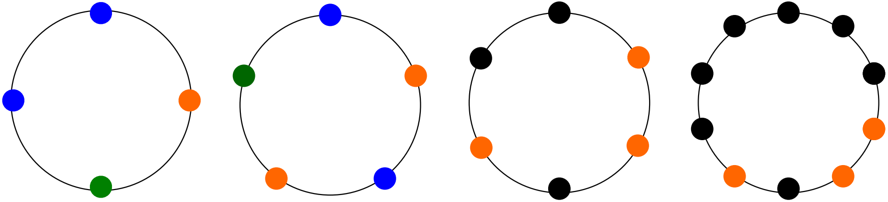

$$D(C_n) = \begin{cases} 3 & \text{if } n \leq 5 \\ 2 & \text{if } n \geq 6 \end{cases}.$$

Distinguishing Extension Number

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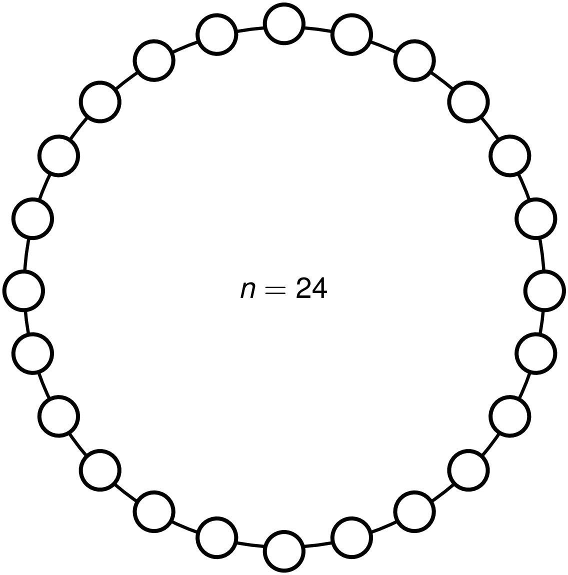
We restrict the Rascal from selecting W as a **fixing set.**

(the point-wise stabilizer of W in G is trivial.)

Extension Number on Cycles



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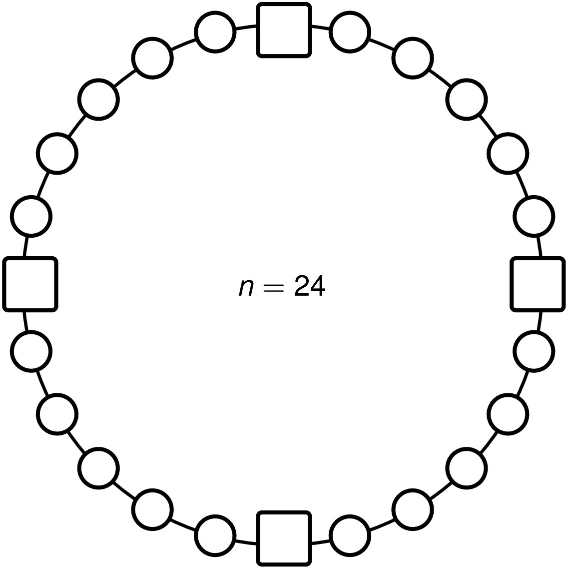

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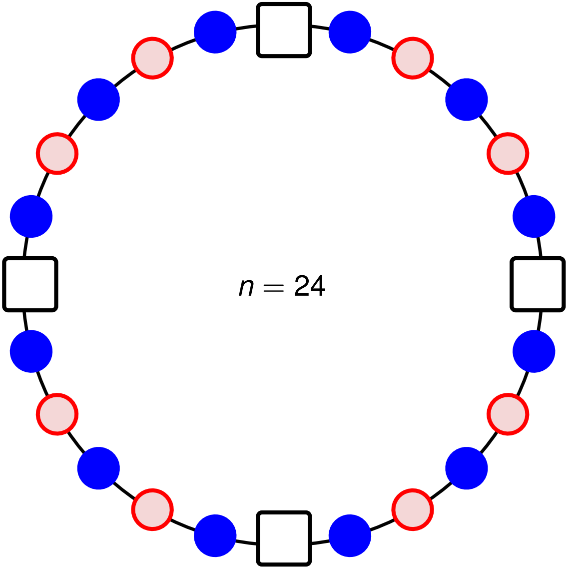

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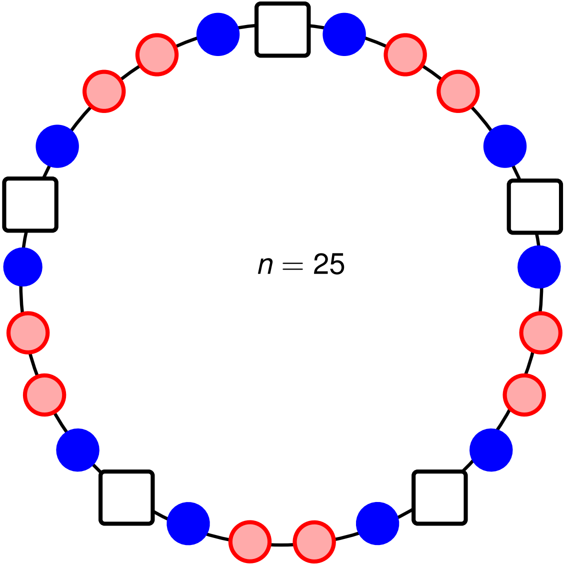

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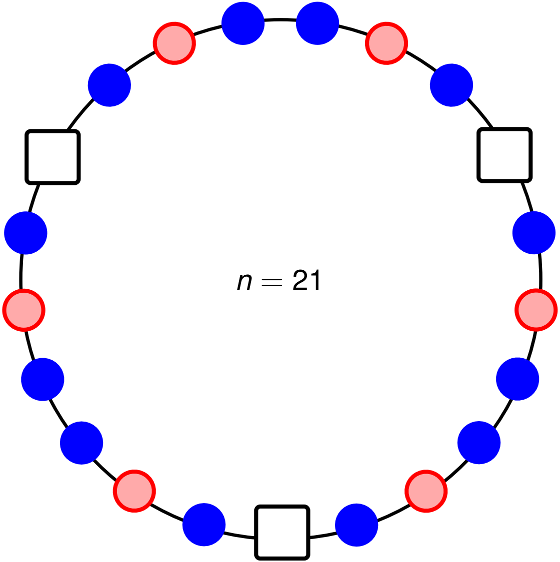

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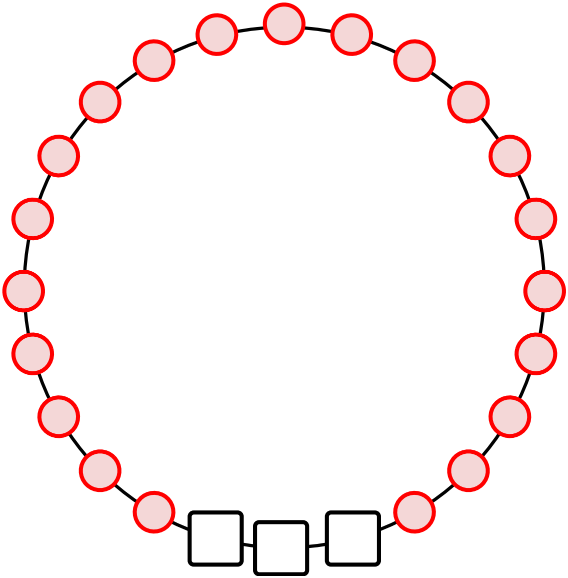

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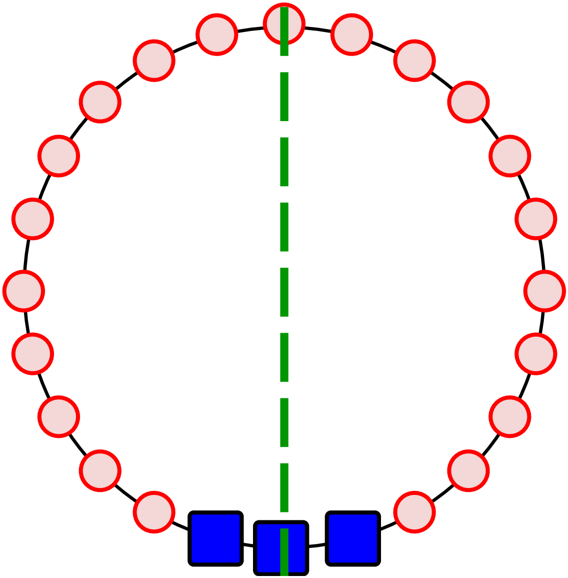

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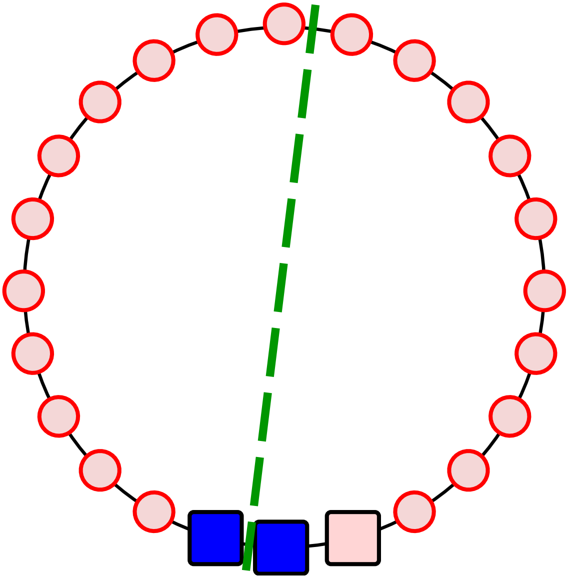

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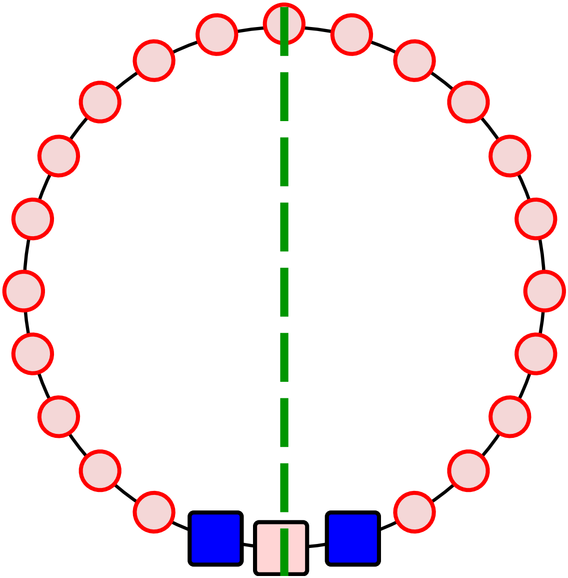

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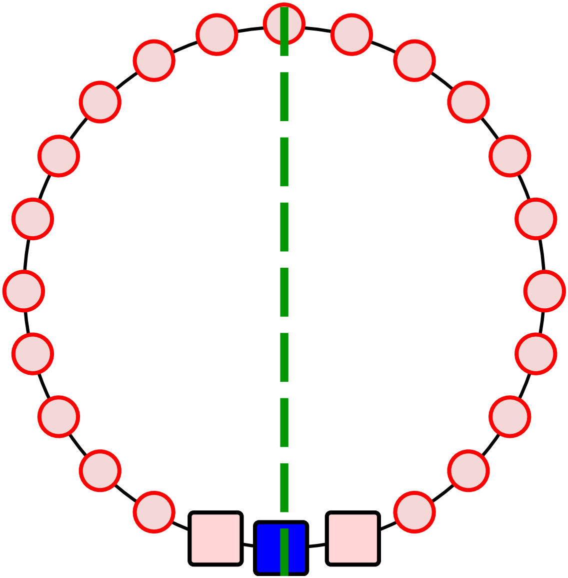

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(We will use $k = 2$.)

Conjecture. For $n \geq 6$, $D_e(C_n) = \begin{cases} 6 & 5 \text{ divides } n \\ 5 & 4 \text{ divides } n \\ 4 & \text{otherwise} \end{cases}$.

Theorem. If the Rascal uses only one color, then the Gentleman wins with the proper number of blanks.

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Theorem. The Gentleman wins if there are five blanks in **general position**.

Theorem. For all $n \geq 6$, $D_e(C_n) \leq 21$.

Extension Number on Prime Cycles

Theorem. If $p \geq 13$ is prime, then $D_e(C_p) = 4$.

$\text{Aut}(C_p)$ is the dihedral group of order $2p$.

1. All rotations are primitive.
2. Two reflections make a rotation.

Extending in Prime Cycles

Step 1: Monochromatic Colorings

There are three blanks where the pairwise distances are distinct.

Color them with the other color.

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Color them with the other color.

(We now assume the coloring is not monochromatic.)

Extending in Prime Cycles

Step 2: Position of W

Let W be a set of blanks.

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There exists $w_0 \in W$ such that the reflection about w_0 sends $W - w_0$ to elements not in W.

Let $W = \{w_0, w_1, w_2, w_3\}$.

Step 3: Restrictions on Colorings

Since p is prime, any rotation is **primitive**, and hence has one orbit of vertices.

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All extensions must allow only one **reflection**

There is at most one coloring c_{forb} which allows reflection about w_0.

Extending in Prime Cycles

Step 4: Building a Rotation

Let $c_1 : \{w_1, w_2, w_3\} \to \{\text{Red}, \text{Blue}\}$ be a coloring on W which differs from c_{forb} in at least two positions.

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Let $\tau_{\text{R}}^{(1)}$ be the reflection given when assigning $c_1(w_0)$ to be **Red**.

Let $\tau_{\text{B}}^{(1)}$ be the reflection given when assigning $c_1(w_0)$ to be **Blue**.

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$\sigma_{(1)} = \tau_{\text{B}}^{(1)} \cdot \tau_{\text{R}}^{(1)}$ is the **rotation** given by performing $\tau_{\text{R}}^{(1)}$ then $\tau_{\text{B}}^{(1)}$.

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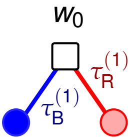

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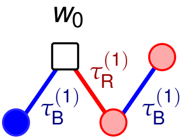

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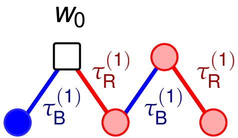

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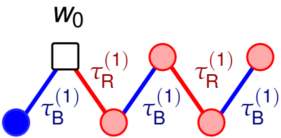

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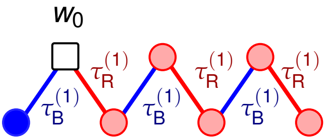

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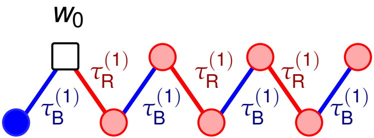

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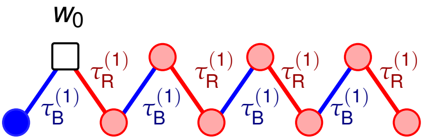

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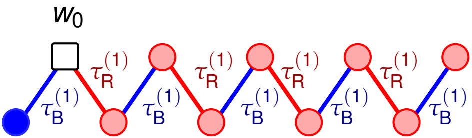

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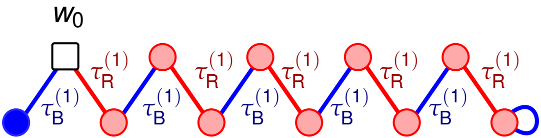

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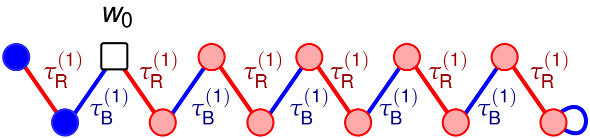

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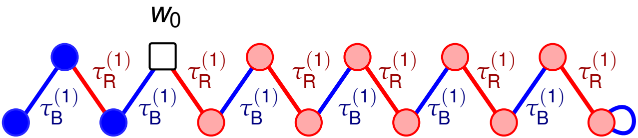

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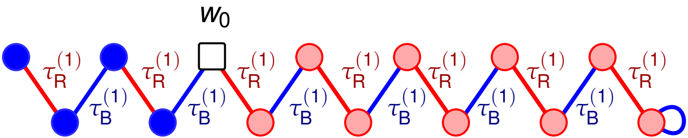

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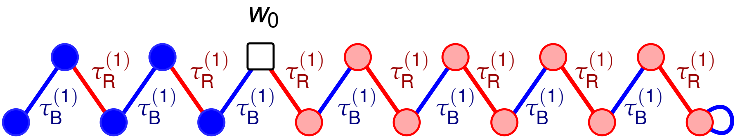

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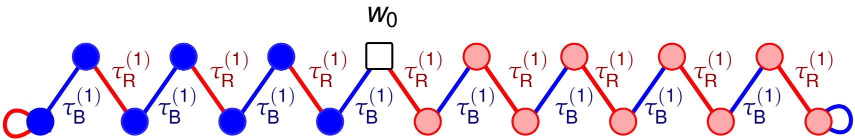

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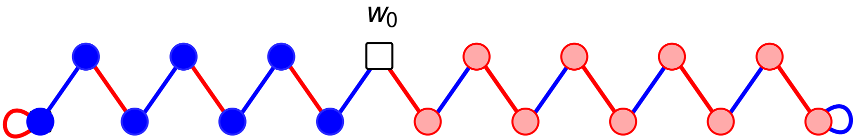

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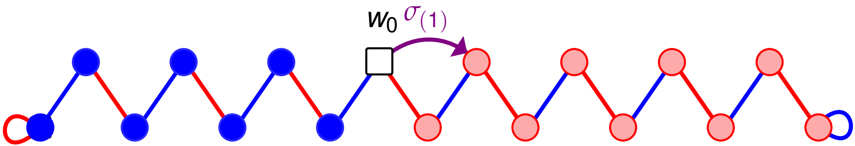

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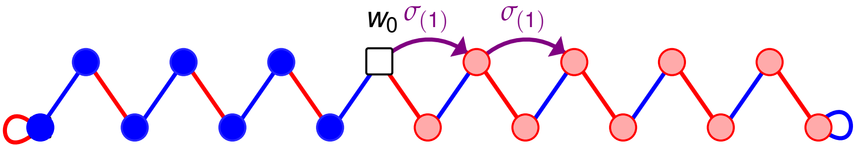

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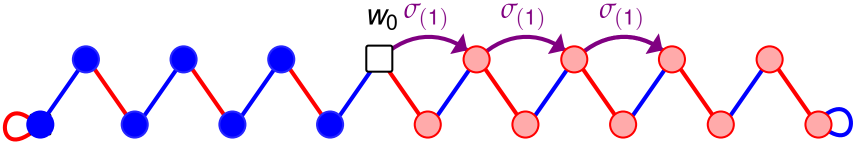

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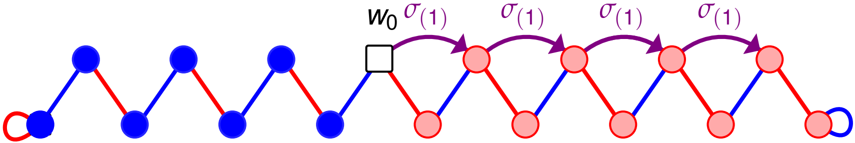

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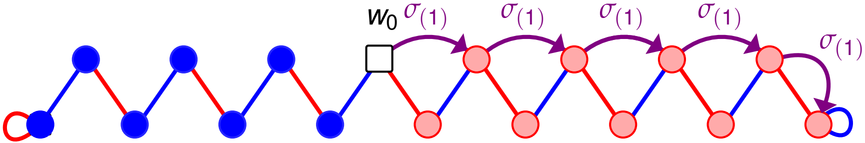

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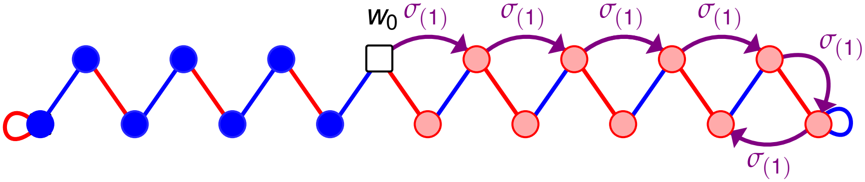

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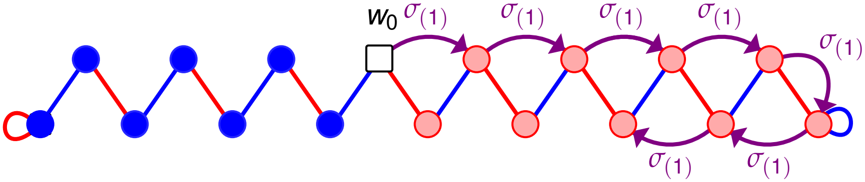

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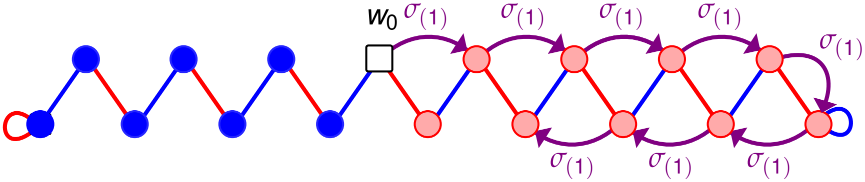

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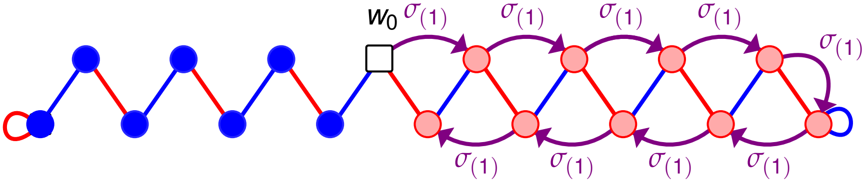

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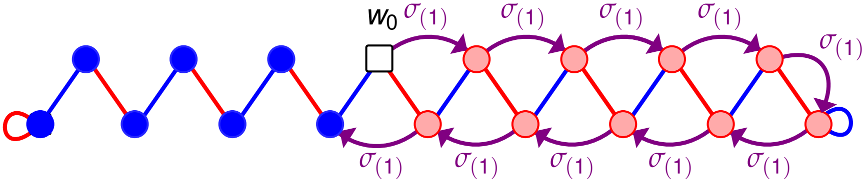

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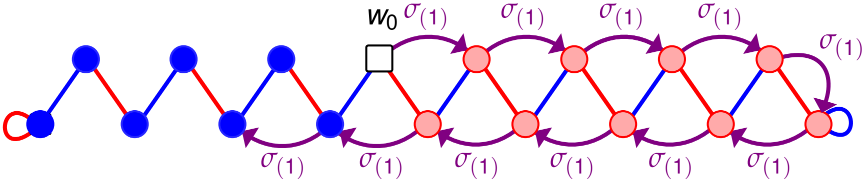

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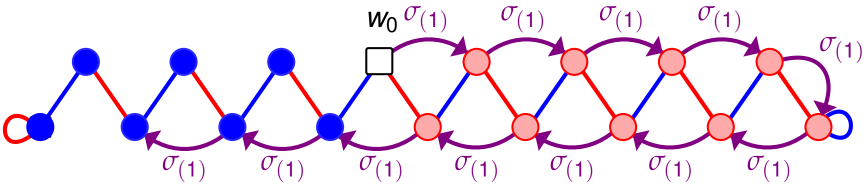

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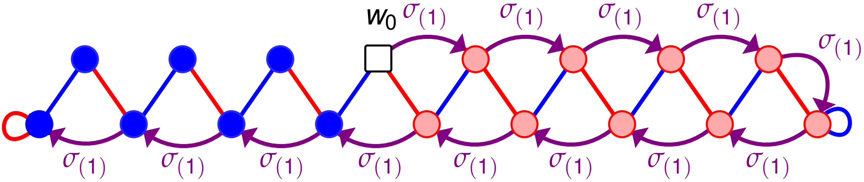

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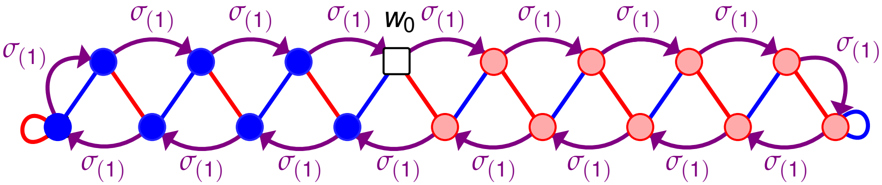

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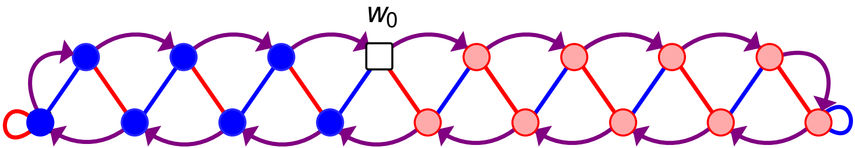

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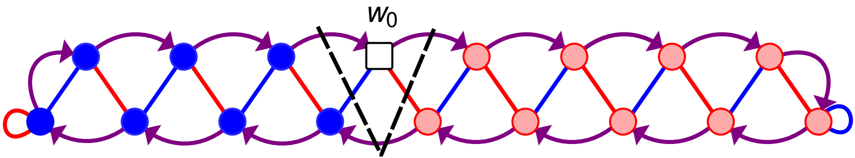

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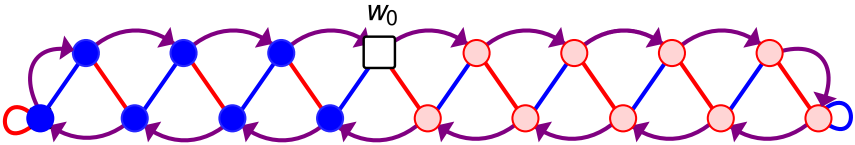

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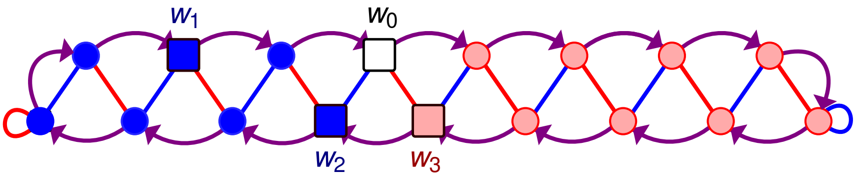
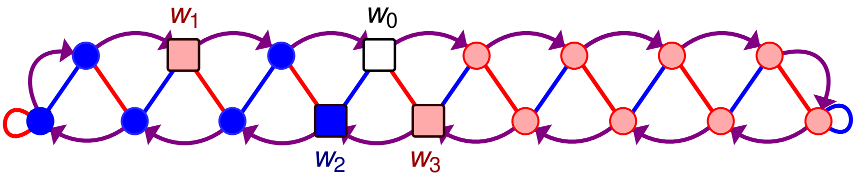


There are three other blanks w_1, w_2, w_3!

Some blank (say w_1) is not one of $\tau_R^{(1)}(w_0)$, $\tau_B^{(1)}(w_0)$.

Extending in Prime Cycles

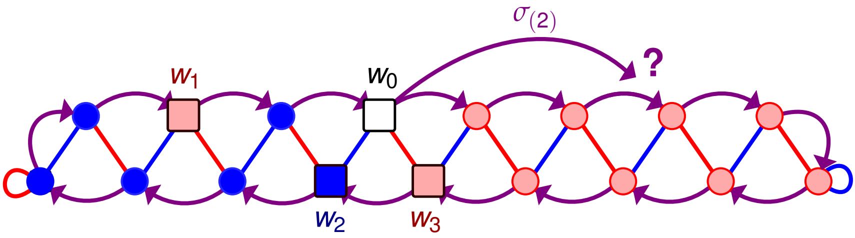


Change the color of w_1 from **Blue** to **Red**.

This creates a new coloring, c_2.

Since c_2 differs from c_1 by one element, $c_2 \neq c_{\text{forb}}$.

Extending in Prime Cycles



Therefore, the red/blue extensions of c_2 to w_0 form:

1. Red/blue reflections $\tau_R^{(2)}$ and $\tau_B^{(2)}$.

2. Rotation $\sigma_{(2)}$.

Extending in Prime Cycles

Wrapping Up

Since $p \geq 13$, $\sigma_{(1)}$ and $\sigma_{(2)}$ cannot coexist!

(Several details are omitted...)

Therefore, one of these extensions of c_1 or c_2 must have been distinguishing!

AND NOW FOR SOMETHING (not so) COMPLETELY DIFFERENT.

Extension Number on Circles and Lines

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($\mathbb{S}^1$ is the circle, $\mathbb{S}^2$ is the usual sphere.)

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We can play the Rascal/Gentleman game on points in $\mathbb{S}^d$ or $\mathbb{R}^k$.

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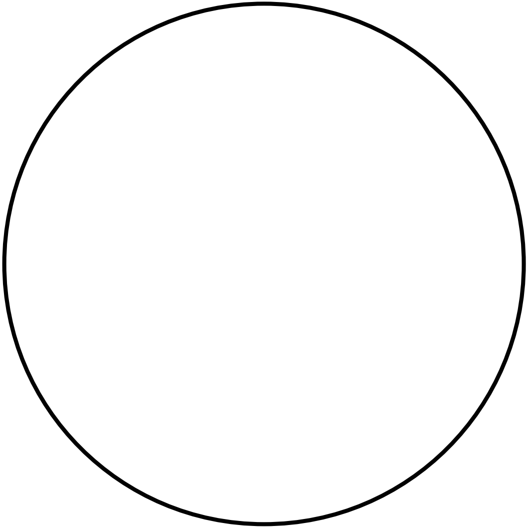
Conjecture. (Space!) $D_e(\mathbb{R}^3) = 10.$

Embedding Cycles on the Circle



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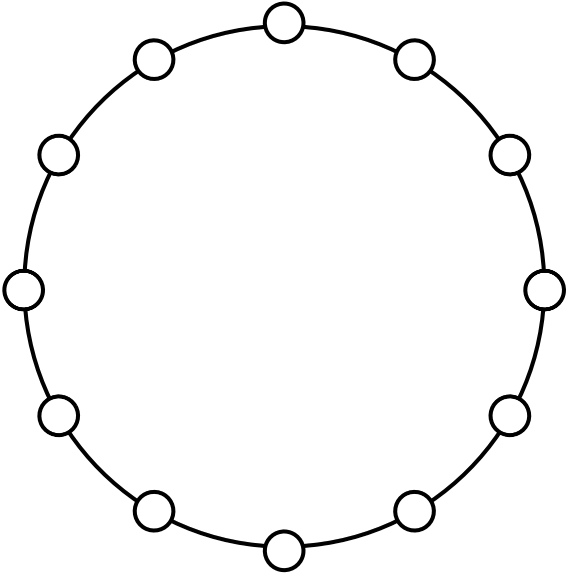
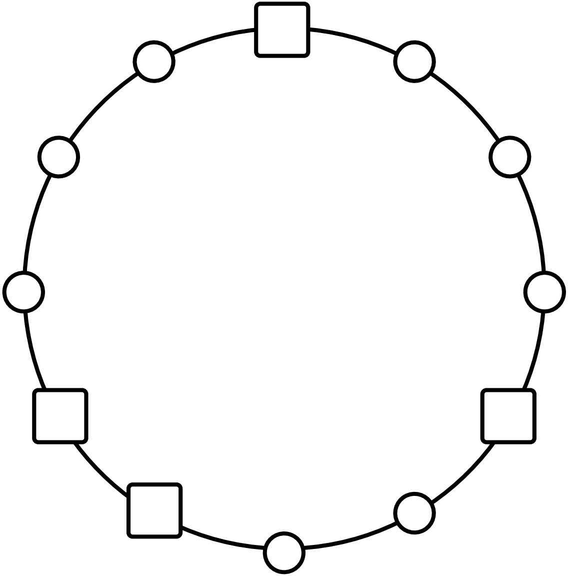


Embedding Cycles on the Circle



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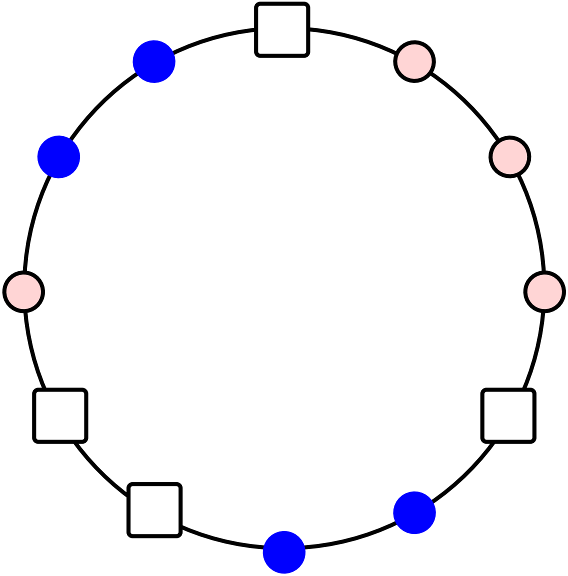

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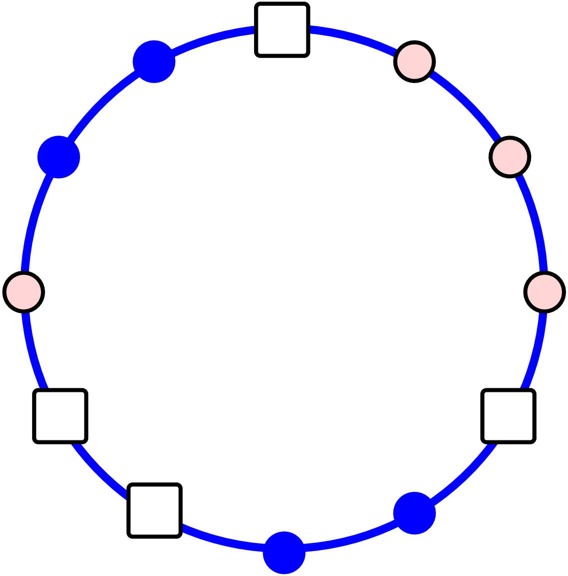

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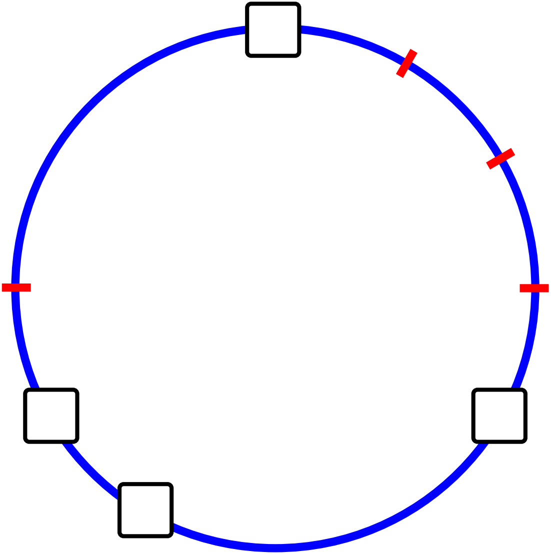

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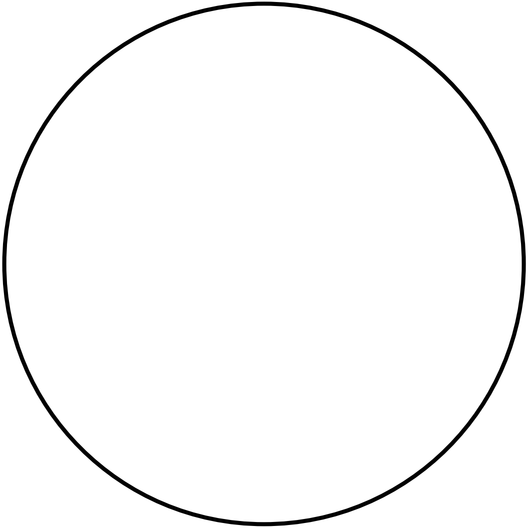

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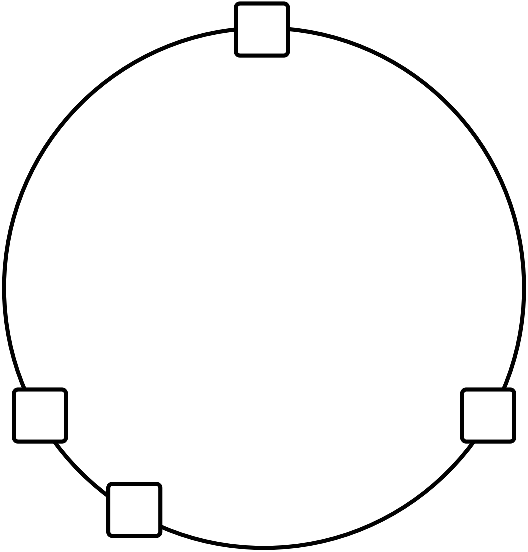

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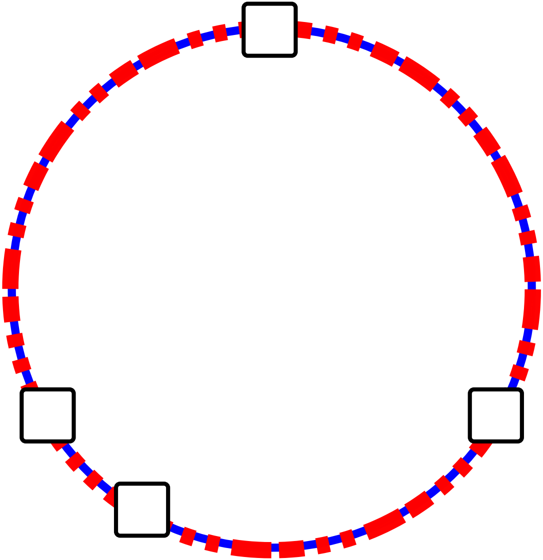

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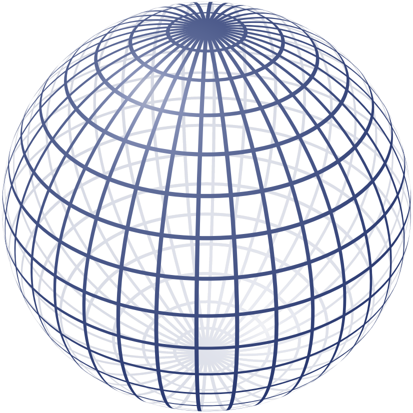

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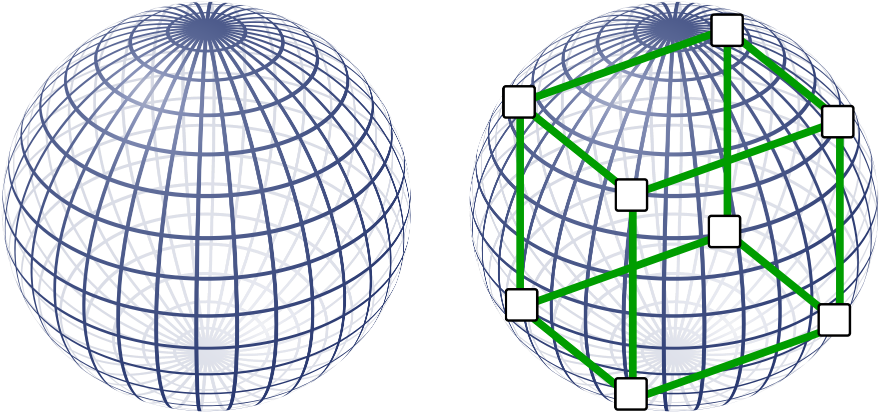

Sphere image from `http://en.wikipedia.org/wiki/Sphere`

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Conjecture. $D_e(\mathbb{R}^{d+1}) = D_e(\mathbb{S}^d) + 1.$

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Conjecture. $D_e(\mathbb{R}^3) = 10.$

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Corollary. (The Circle) $D_e(\mathbb{S}^1) \leq 21$.

Distinguishing Extension Number

Michael Ferrara Ellen Gethner Stephen G. Hartke
Derrick Stolee* Paul S. Wenger

University of Illinois
stolee@illinois.edu
http://www.math.illinois.edu/∼stolee/

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