A linear programming approach to the Manickam-Miklós-Singhi Conjecture

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The Question

Q: For a nonnegative sum $\sum_{i=1}^{n} x_i \geq 0$, how many partial sums are nonnegative?

The Answer

Theorem (Bier–Manickam, '87) If $\sum_{i=1}^{n} x_i \geq 0$, then there are at least 2^{n-1} nonnegative partial sums $\sum_{i \in S} x_i \geq 0$.

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S has nonnegative sum if and only if it contains 1.

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Theorem. If $\mathcal{F} \subseteq 2^{[n]}$ is an intersecting family, then $|\mathcal{F}| \leq 2^{n-1}$.

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Conjecture (Manickam–Miklós–Singhi, '88) If $n \geq 4k$ and $\sum_{i=1}^{n} x_i \geq 0$, then there are at least $\binom{n-1}{k-1}$ nonnegative partial ***k*-sums** $\sum_{i \in S} x_i \geq 0$, where $|S| = k$.

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Example: $x_1 = n$, $x_2 = \cdots = x_n = -1$.
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Theorem (Erdős–Ko–Rado, '61) If $n \geq 2k$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is an intersecting family, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

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Good Question!

For $k \geq 2$ and $n = 3k + 1$, we have $\binom{n-3}{k} < \binom{n-1}{k-1}$, and

$$x_1 = \cdots = x_{n-3} = 3, \qquad x_{n-2} = x_{n-1} = x_n = -(n-3)$$

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A: 4 is the next integer.

It works eventually!

Definition Let $g(n,k)$ be the minimum number of nonnegative k-sums in a nonnegative sum $\sum_{i=1}^{n} x_i \geq 0$.

Theorem (Bier–Manickam, '87) There exists a minimum integer $f(k)$ such that $g(n,k) = \binom{n-1}{k-1}$ for all $n \geq f(k)$.

...eventually...

Bier–Manickam, ’87: $f(k) \leq k(k-1)^k(k-2)^k + k(k-1)^2(k-2) + k^2$

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Pokrovskiy '13: $f(k) \leq 10^{46} k$

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Our Results

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$$f(5) = 17$$

$$f(6) = 20$$

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$$f(k) = 3k + 2 \text{ for } 2 \leq k \leq 7.$$

Our Method

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Lemma (Chowdhury, '12) If $g(n,k) = \binom{n-1}{k-1}$, then $g(n+k,k) = \binom{n+k-1}{k-1}$.

The Endgame

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Theorem (Bier–Manickam, '87) If k divides n, then $g(n,k) = \binom{n-1}{k-1}$.

Multiples of k

Theorem (Bier–Manickam, '87) If k divides n, then $g(n,k) = \binom{n-1}{k-1}$.

Theorem (Baranyai, '75) If k divides n, then K_n^k decomposes into $\binom{n-1}{k-1}$ perfect matchings $M_1, \dots, M_{\binom{n-1}{k-1}}$.

$$\sum_{S \in M_j} \sum_{i \in S} x_i = \sum_{i=1}^{n} x_i \geq 0.$$

Our Method (Again)

$$x_1 \geq x_2 \geq x_3 \geq x_4 \geq x_5 \geq x_6 \geq x_7 \geq x_8$$

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Define $S \succeq T$ (S is to the left of T) if

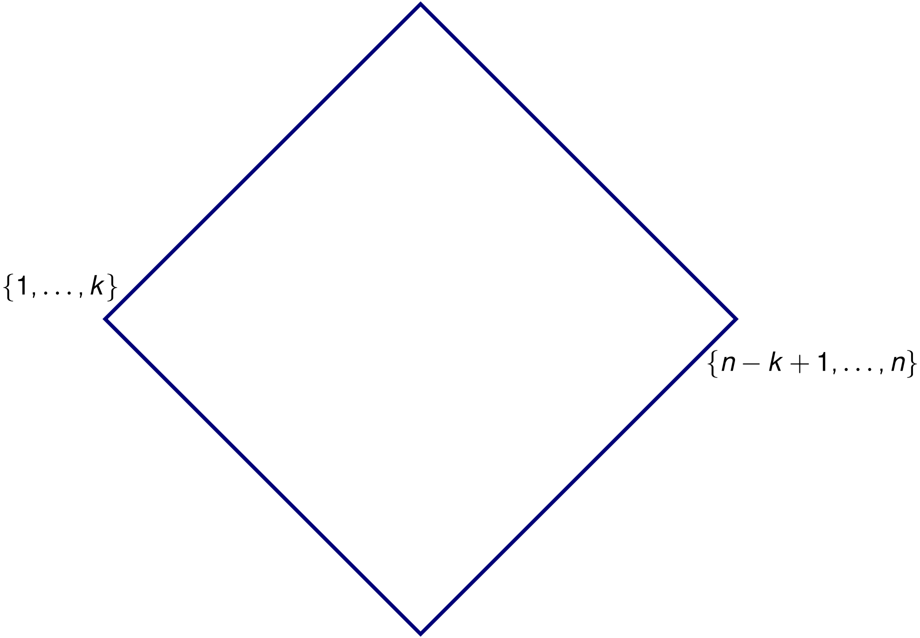
$$S = \{i_1 \leq i_2 \leq \cdots \leq i_k\}, T = \{j_1 \leq j_2 \leq \cdots \leq j_k\},$$

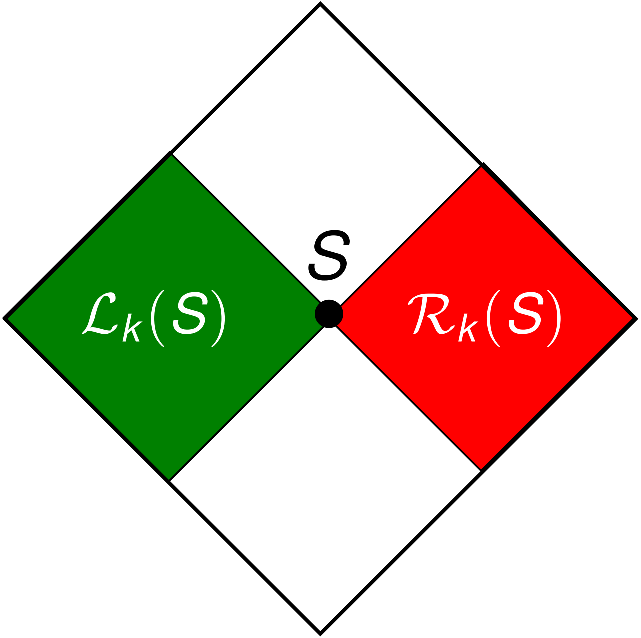
and

$$i_\ell \leq j_\ell \text{ for all } \ell \in \{1, \ldots, k\}.$$

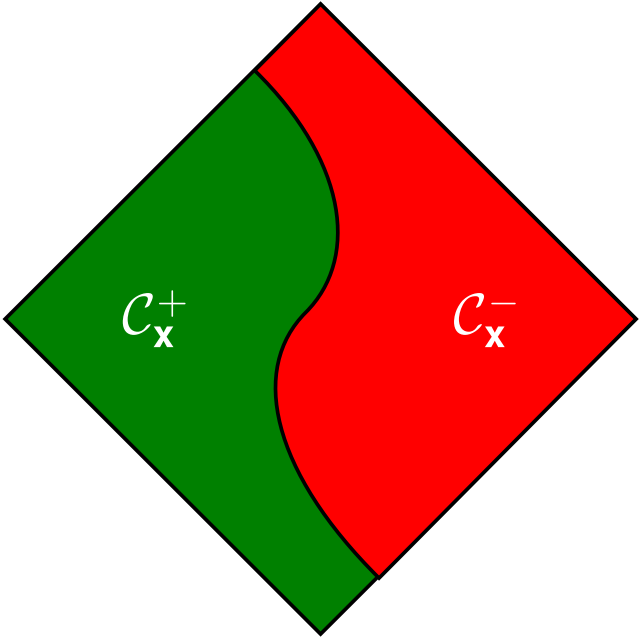
Equivalently:

$$x_{i_\ell} \geq x_{j_\ell} \text{ for all } \ell \in \{1, \ldots, k\} \text{ and all } \mathbf{x} \in F_n.$$


$\{1,\ldots,k\}$
$\{n-k+1,\ldots,n\}$



S
$\mathcal{L}_k(S)$
$\mathcal{R}_k(S)$



$\mathcal{C}_{\mathbf{x}}^{+}$
$\mathcal{C}_{\mathbf{x}}^{-}$

Branch-and-Cut Strategy

MMSSearch(n,k,t, $\mathcal{A}^+$, $\mathcal{A}^-$):
Determine if there is an $\mathbf{x} \in F_n$ *with* $\mathcal{C}^+_{\mathbf{x}} \supseteq \mathcal{A}^+, \mathcal{C}^-_{\mathbf{x}} \supseteq \mathcal{A}^-$,
and fewer than t *nonnegative* k*-sums*

if $|\mathcal{L}_k(\mathcal{A}^+)| \geq t$ **then**
return Null
end if

if $\mathcal{L}_k(\mathcal{A}^+) \cup \mathcal{R}_k(\mathcal{A}^-) = \binom{[n]}{k}$ **then**
output $(\mathcal{A}^+, \mathcal{A}^-)$
end if

Select $S \notin \mathcal{L}_k(\mathcal{A}^+) \cup \mathcal{R}_k(\mathcal{A}^-)$

call MMSSearch$(n, k, t, \mathcal{A}^+ \cup \{S\}, \mathcal{A}^-)$

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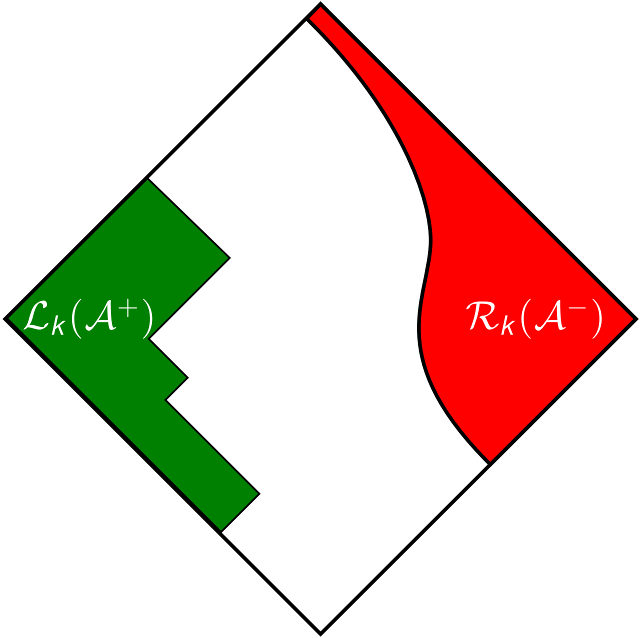
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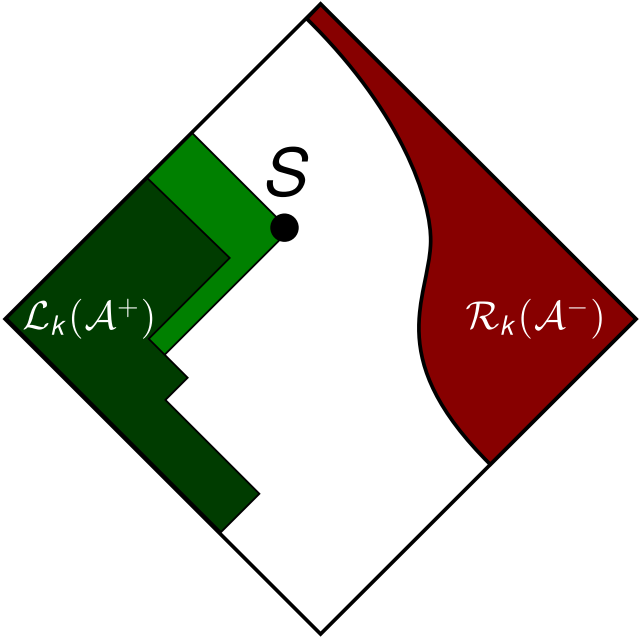

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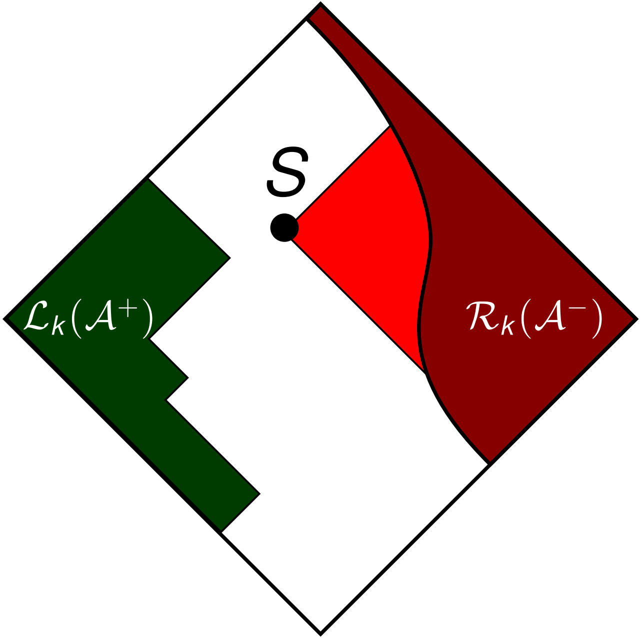

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Refining the Algorithm

Of course, just because $\mathcal{L}_k(\mathcal{A}^+) \cup \mathcal{R}_k(\mathcal{A}^-)$ is a partition of the k-sets **does not necessarily imply** there exists an $\mathbf{x} \in F_n$ with $\mathcal{C}^+_{\mathbf{x}} = \mathcal{L}_k(\mathcal{A}^+)$.

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We need a connection between the **discrete** and **continuous**!

The Linear Program

$$
\begin{aligned}
&\mathcal{P}(k, n, \mathcal{A}^+, \mathcal{A}^-) : \\
&\quad \text{minimize} \quad x_1 \\
&\quad \text{subject to} \sum_{i=1}^{n} x_i \geq 0 \\
&\qquad\qquad x_i - x_{i+1} \geq 0 \qquad \forall i \in \{1, \ldots, n-1\} \\
&\qquad\qquad \sum_{i \in S} x_i \geq 0 \qquad \forall S \in \mathcal{A}^+ \\
&\qquad\qquad \sum_{i \in T} x_i \leq -1 \quad \forall T \in \mathcal{A}^- \\
&\qquad\qquad x_1, \ldots, x_n \in \mathbb{R}
\end{aligned}
$$

Revised Algorithm

MMSSearch(n,k,t, $\mathcal{A}^+$, $\mathcal{A}^-$):
Determine if there is an $\mathbf{x} \in F_n$ *with* $\mathcal{C}_\mathbf{x}^+ \supseteq \mathcal{A}^+, \mathcal{C}_\mathbf{x}^- \supseteq \mathcal{A}^-$,
and fewer than t *nonnegative* k*-sums*

if $|\mathcal{L}_k(\mathcal{A}^+)| \geq t$ **or** $\mathcal{P}(n, k, \mathcal{A}^+, \mathcal{A}^-)$ is infeasible **then**
  return Null
end if

if solution to $\mathcal{P}(n, k\mathcal{A}^+, \mathcal{A}^-)$ has fewer than t nonnegative k-sums **then**
  output solution to $\mathcal{P}(n, k, \mathcal{A}^+, \mathcal{A}^-)$
end if

Select $S \notin \mathcal{L}_k(\mathcal{A}^+) \cup \mathcal{R}_k(\mathcal{A}^-)$

call MMSSearch$(n, k, t, \mathcal{A}^+ \cup \{S\}, \mathcal{A}^-)$

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Lemma. If $|\mathcal{L}_k(S)| \geq t$, then $\sum_{i \in S} x_i < 0$.

Lemma. If $t \leq \binom{n-1}{k-1}$, $1 \in S$, and $|\mathcal{L}_k(S)| + g(n-k, k) \geq t$, then $\sum_{i \in S} x_i < 0$.

Proof.

Let $T = \{1, n-k+2, \ldots, n\}$.

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Thus, if $\sum_{i \in S} x_i \geq 0$, then all sets in $\mathcal{L}_k(S)$ have nonnegative sum.

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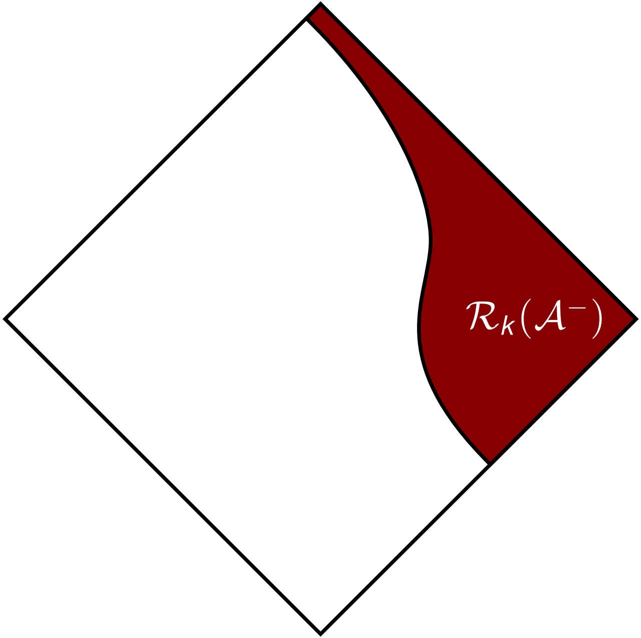
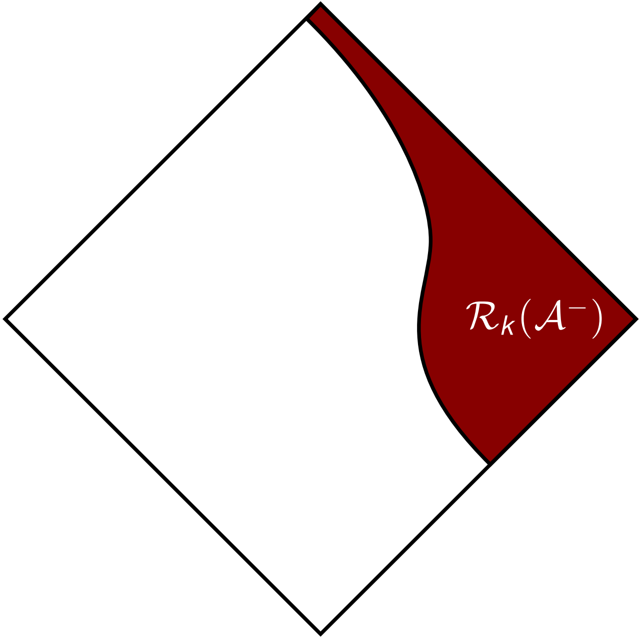
Thus, if $\sum_{i \in S} x_i \geq 0$, then all sets in $\mathcal{L}_k(S)$ have nonnegative sum.With those nonnegative k-sums in $\{2, \ldots, n-k+1\}$, we have at least $|\mathcal{L}_k(S)| + g(n-k,k) \geq t$ nonnegative k-sums! □


$\mathcal{R}_K(\mathcal{A}^-)$

Learning more about $\mathcal{A}^-$

Define $L^*(S) = |\mathcal{L}_k(S) \setminus \mathcal{L}_k(\mathcal{A}^+)|$.

Lemma. If $L^*(S) + |\mathcal{L}_k(\mathcal{A}^+)| \geq t$, then $\sum_{i \in S} x_i < 0$ for all $\mathbf{x} \in F_n$ with $\mathcal{C}_{\mathbf{x}}^+ \supseteq \mathcal{A}^+$.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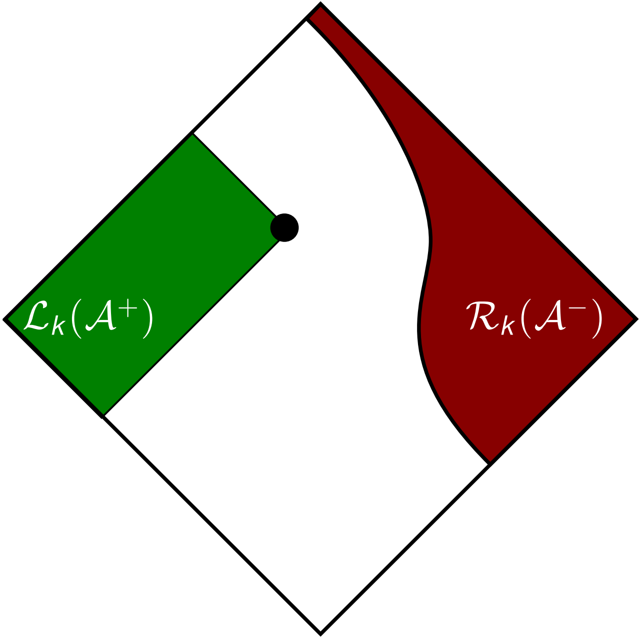

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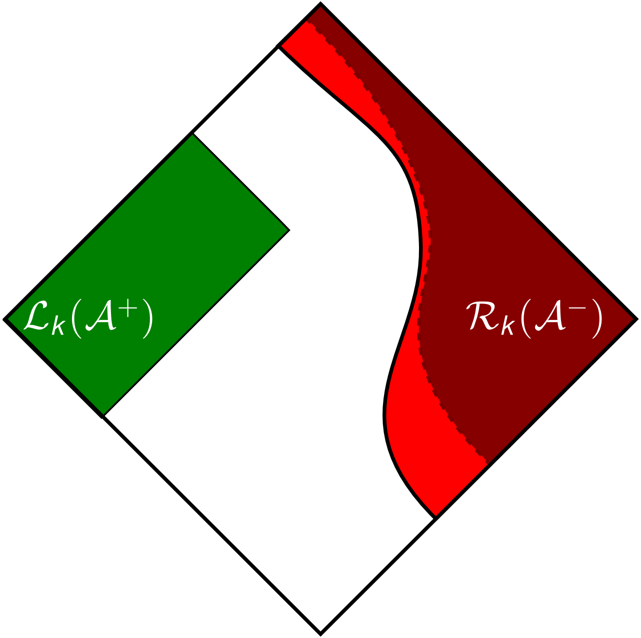

$\mathcal{L}_k(\mathcal{A}^+)$
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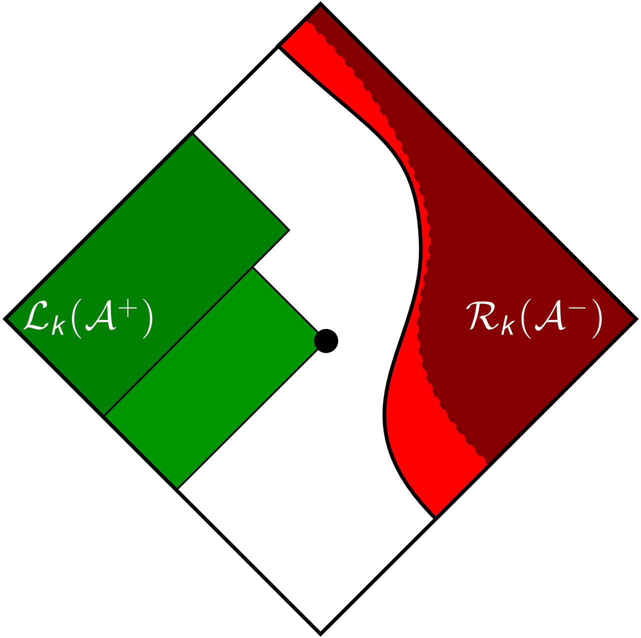
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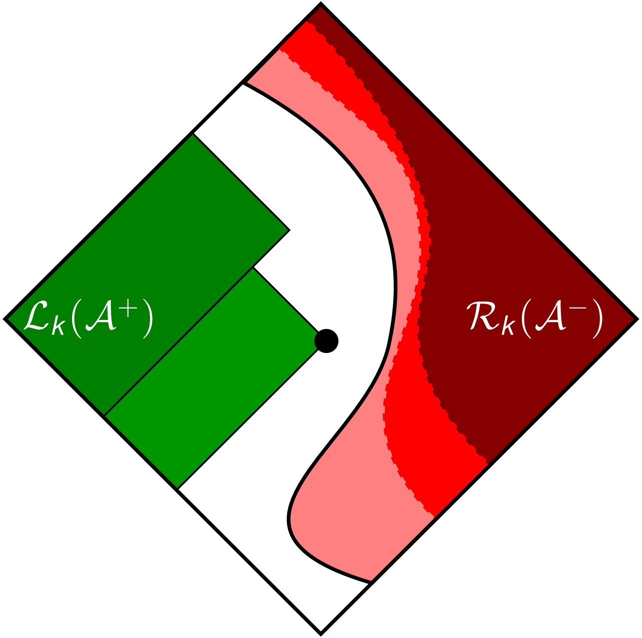

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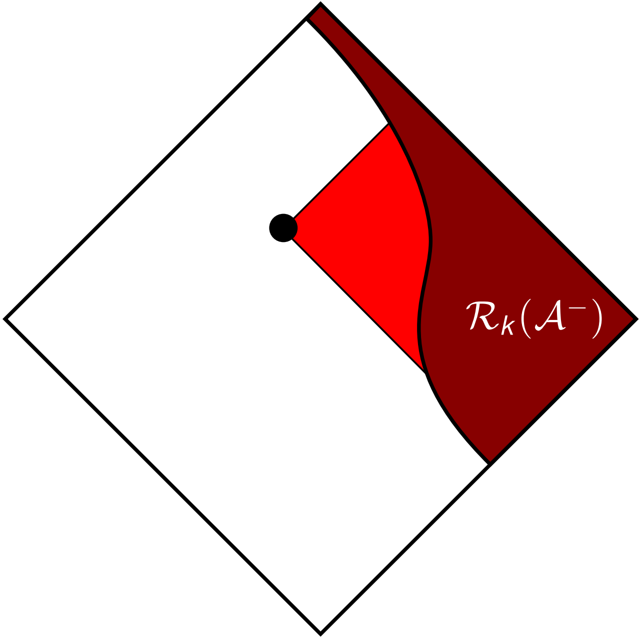
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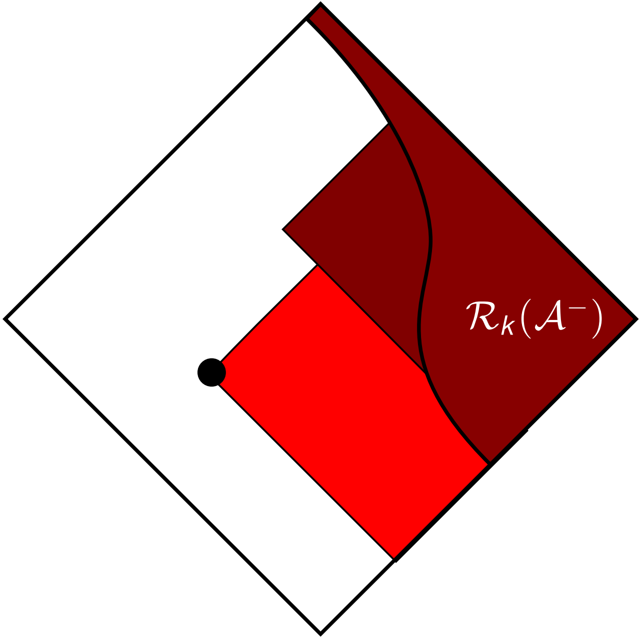
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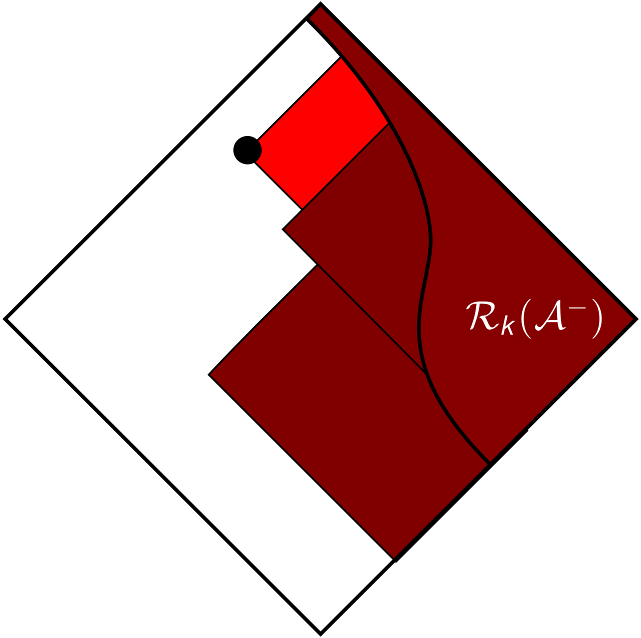
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Learning More About $\mathcal{A}^+$

If $\mathcal{P}(n, k, \mathcal{A}^+, \mathcal{A}^- \cup \{S\})$ is infeasible, then all vectors $\mathbf{x} \in F_n$ with $\mathcal{C}_{\mathbf{x}}^+ \supseteq \mathcal{A}^+$ and $\mathcal{C}^- \supseteq \mathcal{A}^-$ have $\sum_{i \in S} x_i < 0$.

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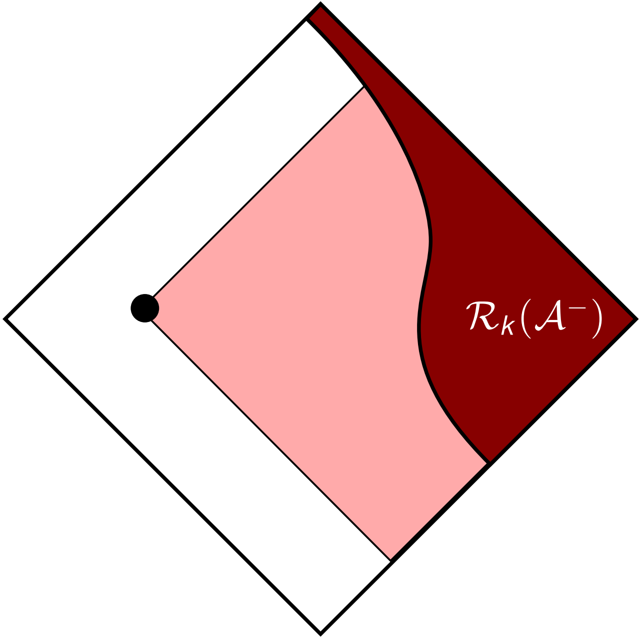
We **randomly sample** a set S to test.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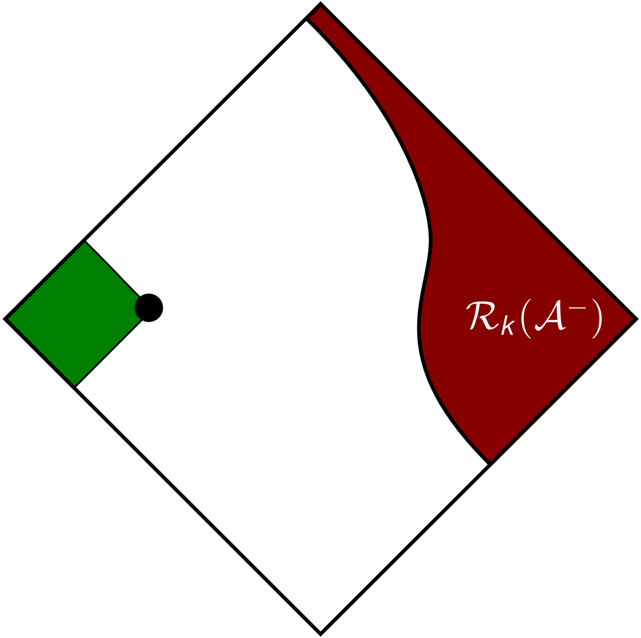
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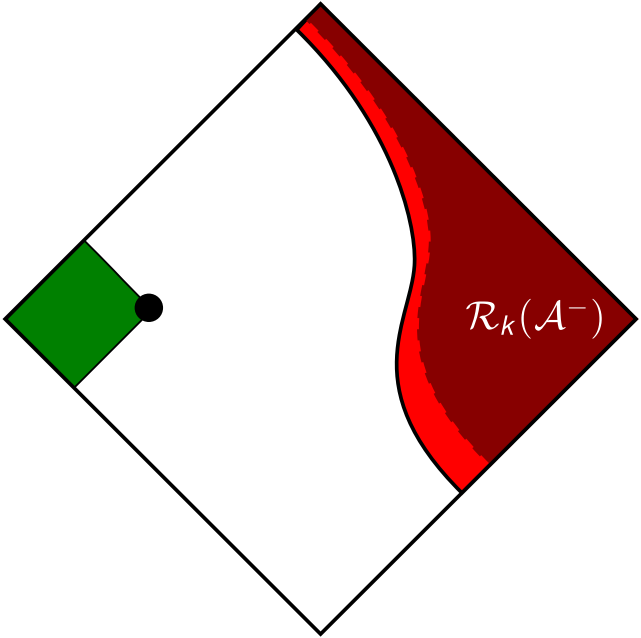
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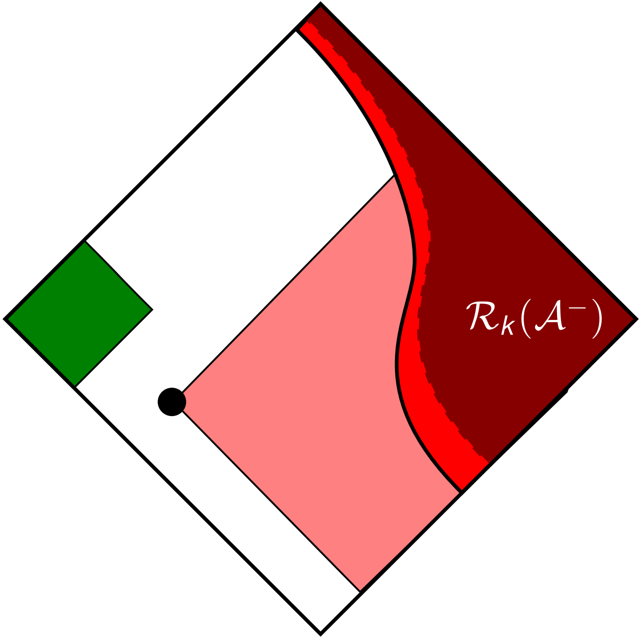
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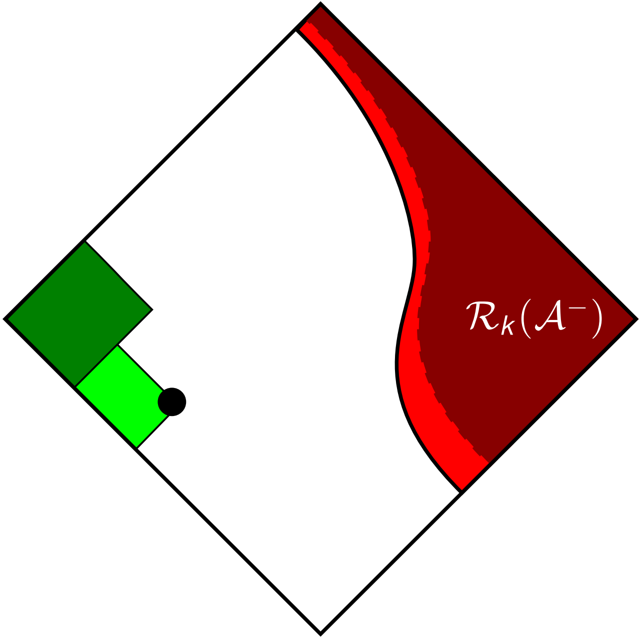
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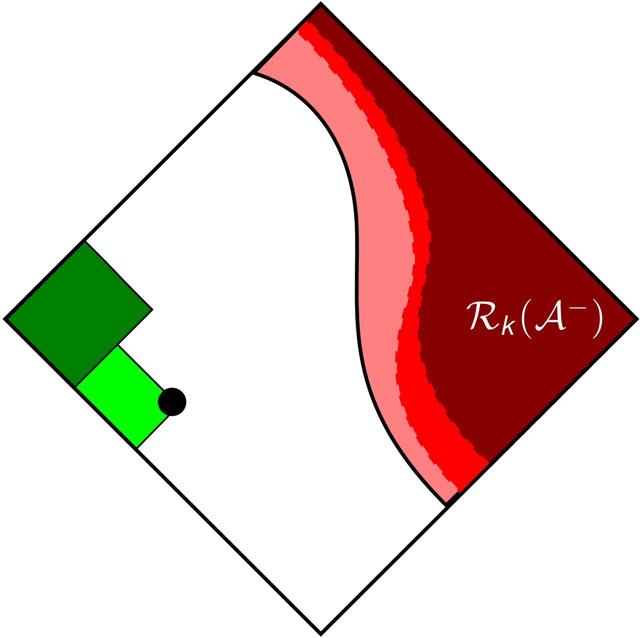
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Our Results

After executing our full algorithm, we discover $g(n, k)$ for all $k \in \{3, 4, 5, 6, 7\}$ and all n.

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After executing our full algorithm, we discover $g(n, k)$ for all $k \in \{3, 4, 5, 6, 7\}$ and all n.

We also know of **sharp examples**: $\mathbf{x} \in F_n$ with exactly $g(n, k)$ nonnegative k-sums!

Sharp Examples

All of our examples have the form

$$\mathbf{x} = a^b \, (-b)^a$$

for $a + b = n$, where $\mathbf{x}$ is given as

$$\underbrace{x_1 = \cdots = x_b = a,}_{b \text{ copies of } a} \qquad \underbrace{x_{b+1} = \cdots = x_n = -b}_{a \text{ copies of } -b}$$

Sharp Examples

$$(n-1)^1\ (-1)^{n-1}$$

w has $\binom{n-1}{k-1}$ nonnegative k-sums.

$$3^{n-3}\ (-(n-3))^3$$

has $\binom{n-3}{k}$ nonnegative k-sums when $n > 3k$.

Sharp Examples

k	n	$g(n,k)$	Sharp Example
6	7	1	$1^6\ (-6)^1$
6	8	7	$1^7\ (-7)^1$
6	9	28	$1^8\ (-8)^1$
6	10	70	$8^2\ (-2)^8$
6	11	126	$9^2\ (-2)^9$
6	12	462	
6	13	462	$2^{11}\ (-11)^2$
6	14	924	$2^{12}\ (-12)^2$
6	15	1705	$12^3\ (-3)^{12}$
6	16	2431	$13^3\ (-3)^{13}$
6	17	3367	$14^3\ (-3)^{14}$
6	18	6188	
6	19	8008	$3^{16}\ (-16)^3$

Strong Examples

A vector is **strong** if $x_1 + \sum_{i=n-k+2}^{n} x_i < 0$.

k	n	Strong Example
6	20	$3^{17}\ (-17)^{3}$
6	21	$17^{4}\ (-4)^{17}$
6	22	$18^{4}\ (-4)^{18}$
6	23	$19^{4}\ (-4)^{19}$
6	24	$33^{1}\ 1^{16}\ (-7)^{7}$
6	25	$104^{1}\ 4^{16}\ (-21)^{8}$

Our Conjecture

Conjecture (Hartke, Stolee, '13+) For all $k \geq 2$, and $n < 4k$, the least number of nonnegative k-sums in a strong vector $\mathbf{x} \in F_n$ is achieved by a vector of the form $a^b\ (-b)^a$.

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$$\lim_{k\to\infty} \frac{f(k)}{k} = \lim_{k\to\infty} \frac{N_k}{k} =$$

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$$\lim_{k\to\infty} \frac{f(k)}{k} = \lim_{k\to\infty} \frac{N_k}{k} = 3.$$

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$$\lim_{k\to\infty} \frac{f(k)}{k} = \lim_{k\to\infty} \frac{N_k}{k} = 3.1$$

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$$\lim_{k\to\infty} \frac{f(k)}{k} = \lim_{k\to\infty} \frac{N_k}{k} = 3.14$$

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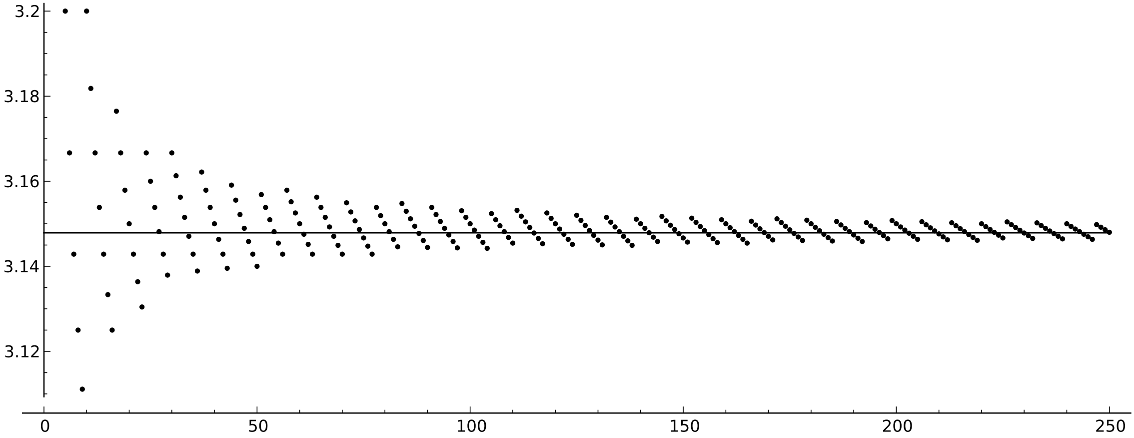
Our Conjecture



Values of N_k / k for $k \in \{5, \ldots, 250\}$.

A linear programming approach to the Manickam-Miklós-Singhi Conjecture

Stephen G. Hartke   Derrick Stolee*

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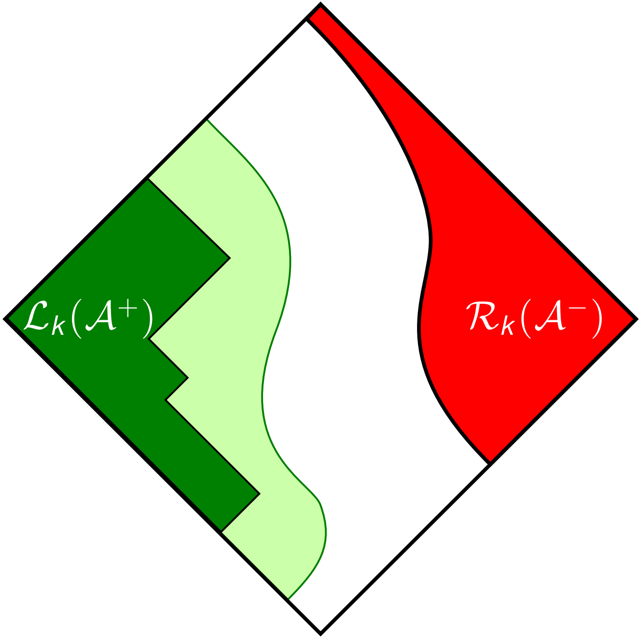
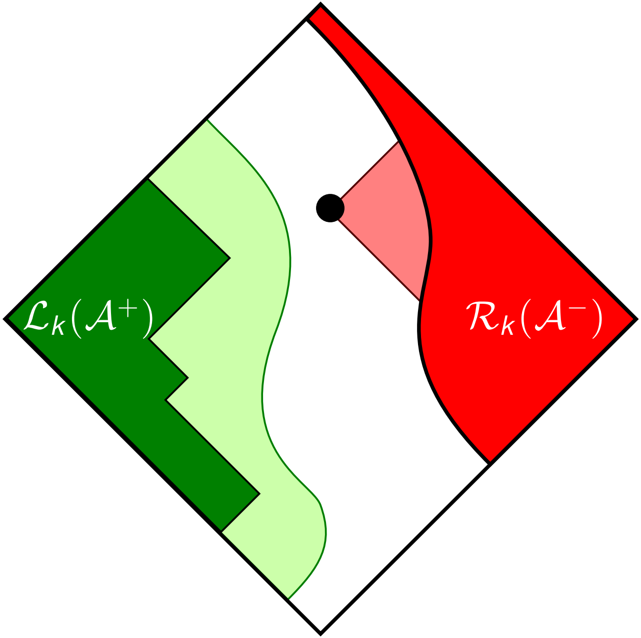
October 5, 2013



$\mathcal{L}_k(\mathcal{A}^+)$
$\mathcal{R}_k(\mathcal{A}^-)$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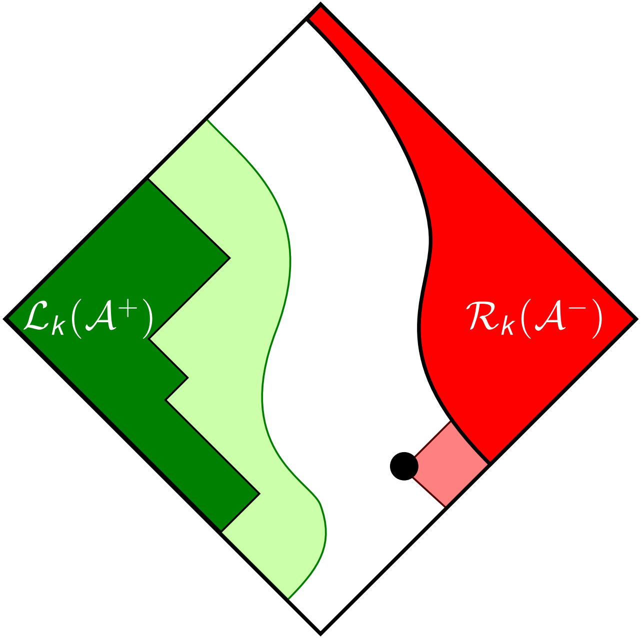

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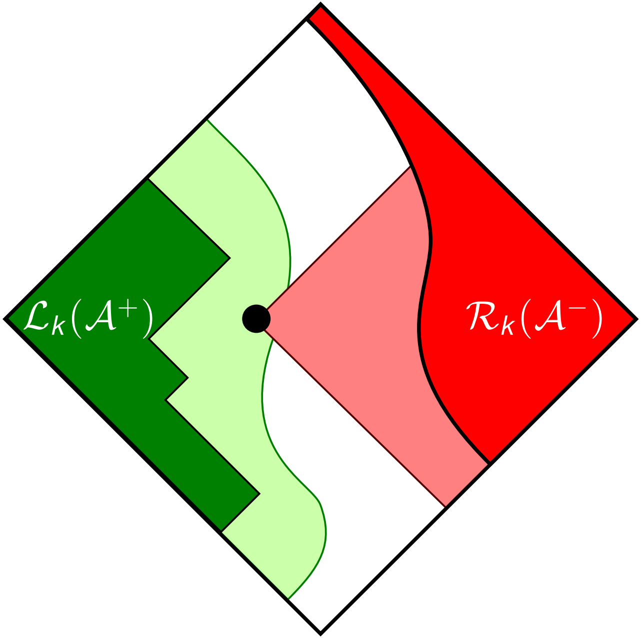

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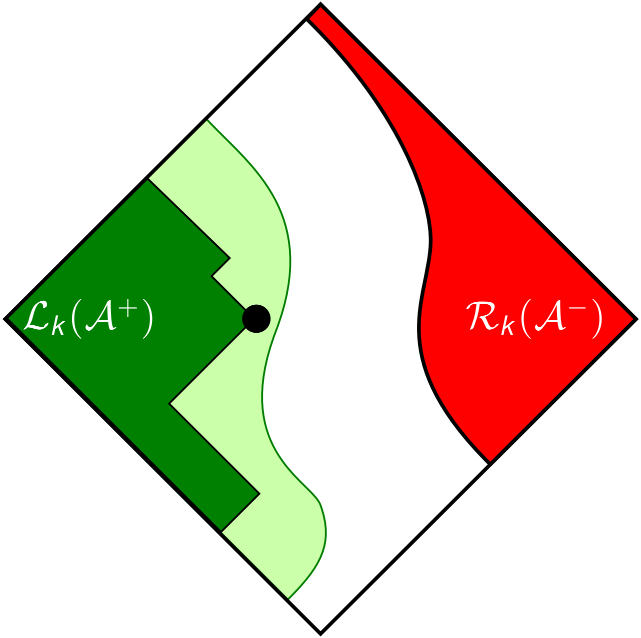

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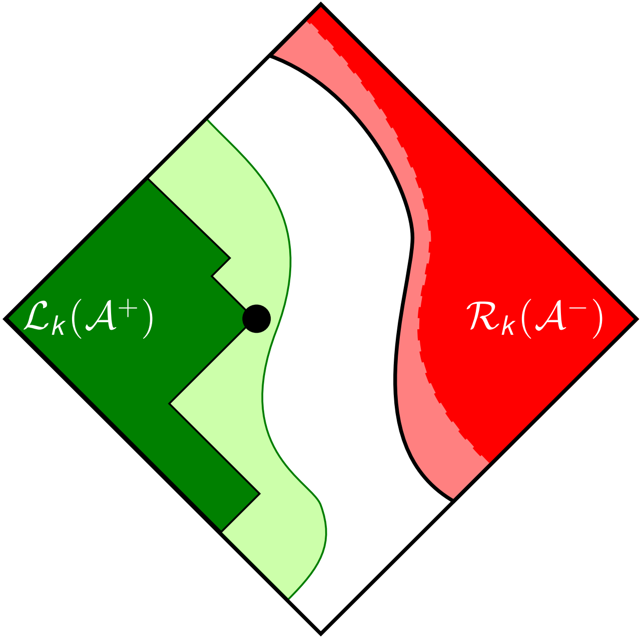

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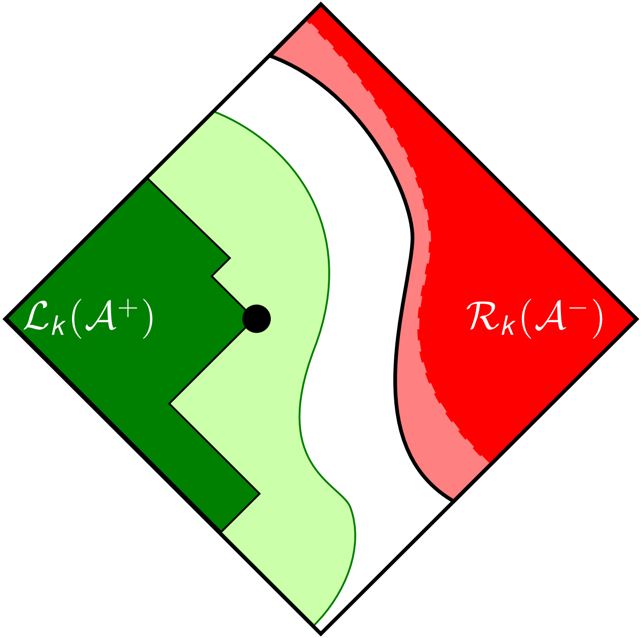
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MMSSearch(n,k,t, $\mathcal{A}^+$, $\mathcal{A}^-$):
Determine if there is an $\mathbf{x} \in F_n$ *with* $\mathcal{C}_\mathbf{x}^+ \supseteq \mathcal{A}^+, \mathcal{C}_\mathbf{x}^- \supseteq \mathcal{A}^-$,
and fewer than t *nonnegative* k*-sums*

if $|\mathcal{L}_k(\mathcal{A}^+)| \geq t$ **or** $\mathcal{P}(n, k, \mathcal{A}^+, \mathcal{A}^-)$ is infeasible **then**
return Null
end if

if solution to $\mathcal{P}(n, k\mathcal{A}^+, \mathcal{A}^-)$ has fewer than t nonnegative k-sums **then**
output solution to $\mathcal{P}(n, k, \mathcal{A}^+, \mathcal{A}^-)$
end if
Propagate to build $\mathcal{A}^-$.
Randomly sample to build $\mathcal{A}^+$.

Select $S \notin \mathcal{L}_k(\mathcal{A}^+) \cup \mathcal{R}_k(\mathcal{A}^-)$

call MMSSearch$(n, k, t, \mathcal{A}^+ \cup \{S\}, \mathcal{A}^-)$

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