A Linear Programming Approach to the Manickam-Miklós-Singhi Conjecture

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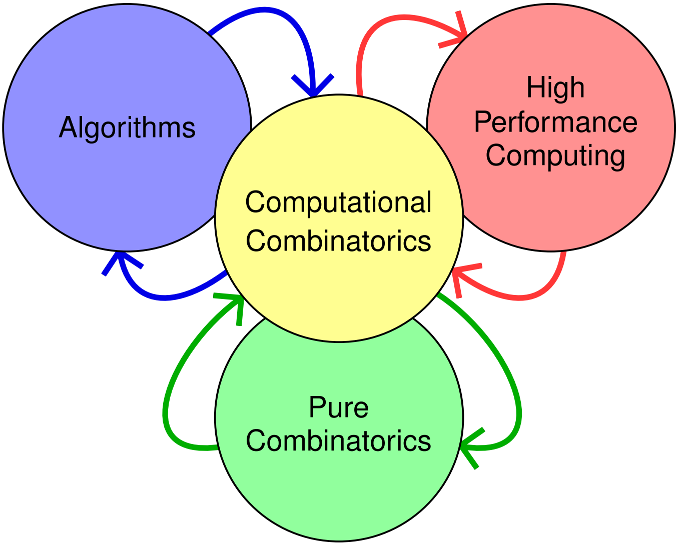
Rochester Institute of Technology

S. G. Hartke, D. Stolee, A linear programming approach to the Manickam-Miklós-Singhi Conjecture, the European Journal of Combinatorics (2014).

Computational Combinatorics



Computational Combinatorics

Computational Combinatorics is using a combination of

- **pure mathematics**,
- **algorithms**, and
- **computational resources**

to solve problems in pure combinatorics by

- **providing evidence** for conjectures,
- finding **examples** and **counterexamples**, and
- **discovering and proving theorems.**

Research Goal

Determine if certain **combinatorial objects** exist with given **structural** or **extremal** properties.

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1. Is there a **projective plane** of order 10?
2. When do **strongly regular graphs** exist?
3. How many **Steiner triple systems** are there of order 19?

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Examples:

1. Is there a **projective plane** of order 10?
   (Lam, Thiel, Swiercz, 1989)
2. When do **strongly regular graphs** exist?
   (Spence 2000, Coolsaet, Degraer, Spence 2006, many others)
3. How many **Steiner triple systems** are there of order 19?
   (Kaski, Östergård, 2004)

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Determine if certain **combinatorial objects** exist with given **structural** or **extremal** properties.

Example: Does there exist a 2-coloring of $E(K_n)$ such that no monochromatic copy of G or H exists?

Stanisław P. Radziszowski

Research Goal

Determine if certain **combinatorial objects** exist with given **structural** or **extremal** properties.

Example: What properties do **uniquely *H*-saturated graphs** exhibit?

Paul S. Wenger

Combinatorial Objects

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Our goal is to use **proof** and **algorithms** to extend the reach of computer check!

Combinatorial Objects

What if the object does not come from a finite space?

Combinatorial Object

Real vectors $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$

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Consider the family of **nonnegative partial k-sums**:
subsets $S \in \binom{[n]}{k}$ such that $\sum_{i \in S} x_i \geq 0$.

A Question

Q: For a nonnegative sum $\sum_{i=1}^{n} x_i \geq 0$, how many partial sums are nonnegative?

The Answer

Theorem (Bier–Manickam, ’87) If $\sum_{i=1}^{n} x_i \geq 0$, then there are at least 2^{n-1} nonnegative partial sums $\sum_{i \in S} x_i \geq 0$.

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S has nonnegative sum if and only if it contains 1.

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Theorem. If $\mathcal{F} \subseteq 2^{[n]}$ is an intersecting family, then $|\mathcal{F}| \leq 2^{n-1}$.

Research Question

Q: For a nonnegative sum $\sum_{i=1}^{n} x_i \geq 0$, how many partial ***k*-sums** are nonnegative?

The Answer?

Conjecture (Manickam–Miklós–Singhi, '88) If $n \geq 4k$ and $\sum_{i=1}^{n} x_i \geq 0$, then there are at least $\binom{n-1}{k-1}$ nonnegative partial **k-sums** $\sum_{i \in S} x_i \geq 0$, where $|S| = k$.

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Example: $x_1 = n - 1, \quad x_2 = \cdots = x_n = -1.$
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Theorem (Erdős–Ko–Rado, '61) If $n \geq 2k$ and $\mathcal{F} \subseteq \binom{[n]}{k}$ is an intersecting family, then $|\mathcal{F}| \leq \binom{n-1}{k-1}$.

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For $k \geq 2$ and $n = 3k+1$, we have $\binom{n-3}{k} < \binom{n-1}{k-1}$, and

$$x_1 = \cdots = x_{n-3} = 3, \qquad x_{n-2} = x_{n-1} = x_n = -(n-3)$$

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A: 4 is the next integer.

It works eventually!

Definition Let $g(n, k)$ be the minimum number of nonnegative k-sums in a nonnegative sum $\sum_{i=1}^{n} x_i \geq 0$.

Theorem (Bier–Manickam, '87) There exists a minimum integer $f(k)$ such that $g(n, k) = \binom{n-1}{k-1}$ for all $n \geq f(k)$.

...eventually...

Bier–Manickam, '87:

$$f(k) \leq k(k-1)^k(k-2)^k + k(k-1)^2(k-2) + k^2$$

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Blinovsky ’13?: $f(k) \overset{?}{\leq} 4k$

Fixed k

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$$f(7) = 23$$

$$f(k) = 3k + 2 \text{ for } 2 \leq k \leq 7.$$

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Lemma (Chowdhury, '12)

If $g(n,k) = \binom{n-1}{k-1}$, then $g(n+k,k) = \binom{n+k-1}{k-1}$.

The Endgame

If we find $g(n, k) = \binom{n-1}{k-1}$ for k consecutive values of n, then we are done!

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Theorem (Bier–Manickam, ’87)

$$\text{If } k \text{ divides } n\text{, then } g(n,k) = \binom{n-1}{k-1}.$$

Multiples of k

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$$\text{For each matching } M_j, \sum_{S \in M_j} \sum_{i \in S} x_i = \sum_{i=1}^{n} x_i \geq 0.$$

Our Method (Again)

$$x_1 \geq x_2 \geq x_3 \geq x_4 \geq x_5 \geq x_6 \geq x_7 \geq x_8$$

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		T		T			T

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$S \quad S \quad T \quad \quad T \quad \quad S \quad T$

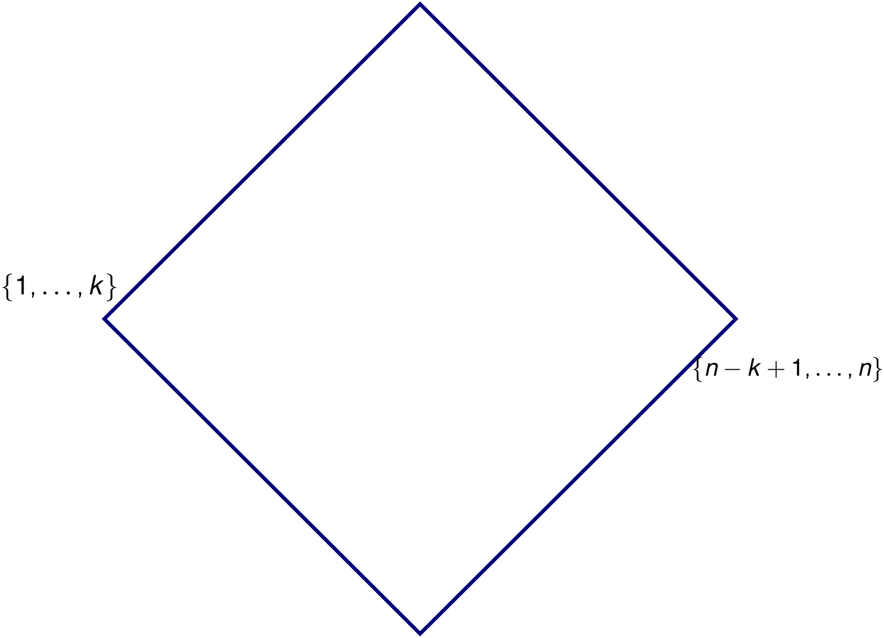
Define $S \succeq T$ (S is to the left of T) if

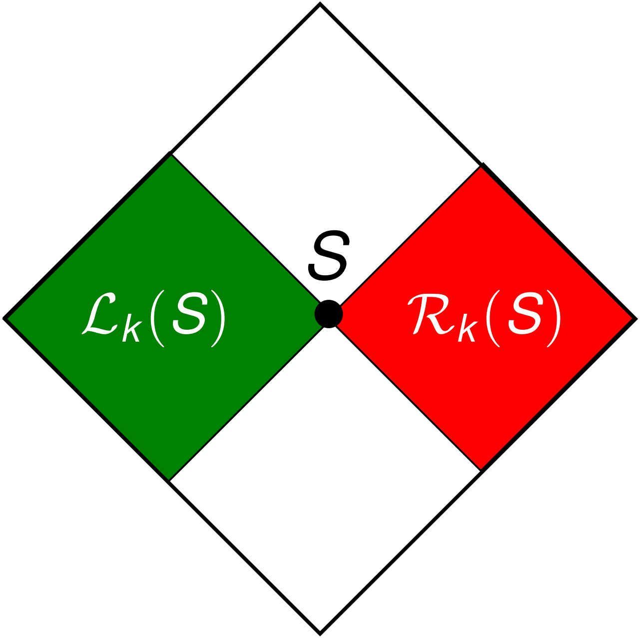
$$S = \{i_1 \leq i_2 \leq \cdots \leq i_k\}, T = \{j_1 \leq j_2 \leq \cdots \leq j_k\},$$

and $i_\ell \leq j_\ell$ for all $\ell \in \{1, \dots, k\}$.

Equivalently:

$$x_{i_\ell} \geq x_{j_\ell} \text{ for all } \ell \in \{1, \dots, k\} \text{ and all } \mathbf{x} \in F_n.$$


$\{1, \ldots, k\}$
$\{n-k+1, \ldots, n\}$


S
$\mathcal{L}_k(S)$
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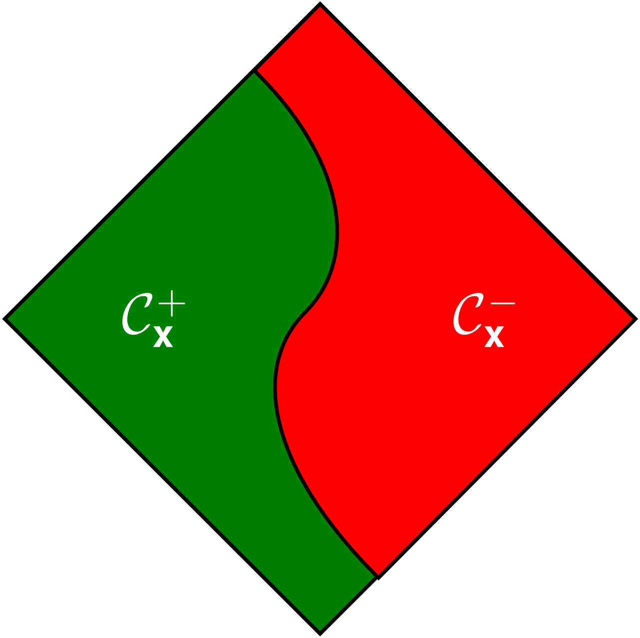

$\mathcal{C}_{\mathbf{x}}^{+}$
$\mathcal{C}_{\mathbf{x}}^{-}$

MMSSearch(n,k,t, $\mathcal{A}^+$, $\mathcal{A}^-$):
Determine if there is an $\mathbf{x} \in F_n$ with $\mathcal{C}_{\mathbf{x}}^+ \supseteq \mathcal{A}^+, \mathcal{C}_{\mathbf{x}}^- \supseteq \mathcal{A}^-$, and fewer than t nonnegative k-sums

if $|\mathcal{L}_k(\mathcal{A}^+)| \geq t$ **then**
  return Null
end if

if $\mathcal{L}_k(\mathcal{A}^+) \cup \mathcal{R}_k(\mathcal{A}^-) = \binom{[n]}{k}$ **then**
  output $(\mathcal{A}^+, \mathcal{A}^-)$
end if

Select $S \notin \mathcal{L}_k(\mathcal{A}^+) \cup \mathcal{R}_k(\mathcal{A}^-)$

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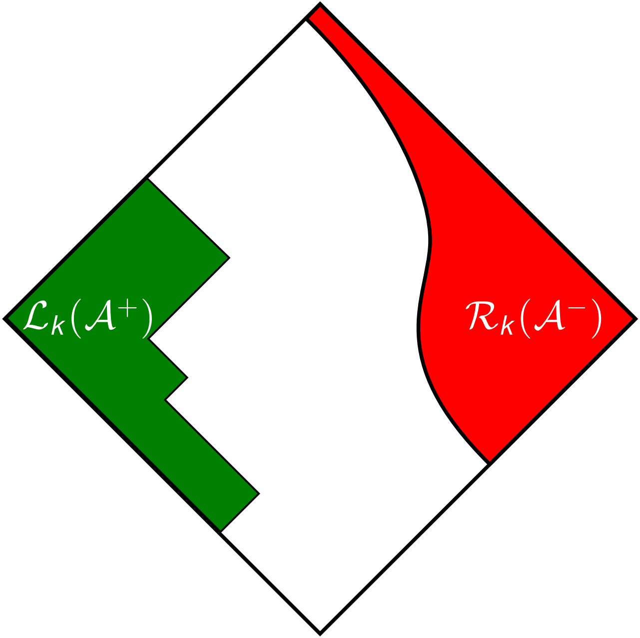
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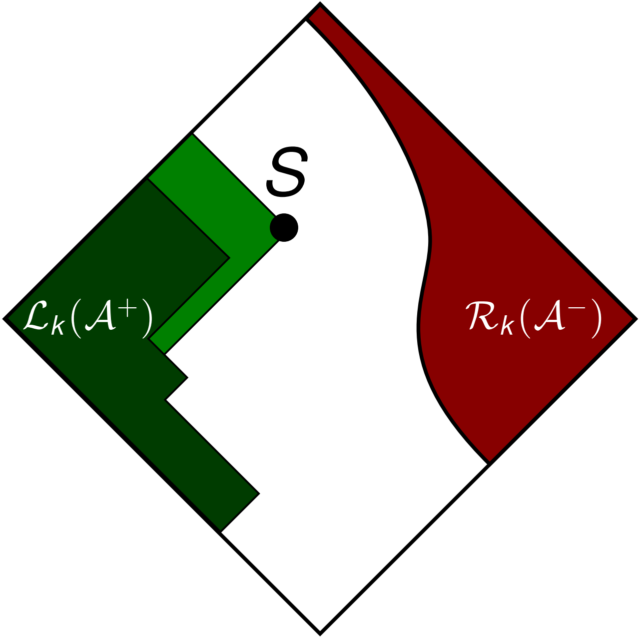

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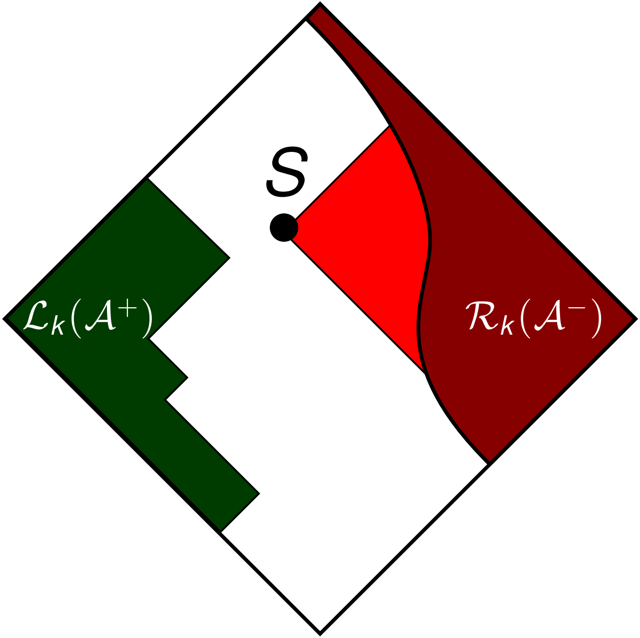
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Refining the Algorithm

Even if $\mathcal{L}_k(\mathcal{A}^+) \cup \mathcal{R}_k(\mathcal{A}^-)$ is a partition of the k-sets, there **may not exist** a vector $\mathbf{x} \in F_n$ with $\mathcal{C}^+_{\mathbf{x}} = \mathcal{L}_k(\mathcal{A}^+)$.

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Even if $\mathcal{L}_k(\mathcal{A}^+) \cup \mathcal{R}_k(\mathcal{A}^-)$ is a partition of the k-sets, there **may not exist** a vector $\mathbf{x} \in F_n$ with $\mathcal{C}_{\mathbf{x}}^+ = \mathcal{L}_k(\mathcal{A}^+)$.

We need a connection between the **discrete** and **continuous**!

The Linear Program

$$\mathcal{P}(k, n, \mathcal{A}^+, \mathcal{A}^-):$$

$$\begin{aligned}
\text{minimize} \quad & x_1 \\
\text{subject to} \quad & \sum_{i=1}^{n} x_i \geq 0 \\
& x_i - x_{i+1} \geq 0 && \forall i \in \{1, \dots, n-1\} \\
& \sum_{i \in S} x_i \geq 0 && \forall S \in \mathcal{A}^+ \\
& \sum_{i \in T} x_i \leq -1 && \forall T \in \mathcal{A}^- \\
& x_1, \dots, x_n \in \mathbb{R}
\end{aligned}$$

Linear Programming Formulation

Lemma (Hartke, Stolee, '13+) Fix subsets $\mathcal{A}^+, \mathcal{A}^- \subset \binom{[n]}{k}$.

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Lemma (Hartke, Stolee, '13+) Fix subsets $\mathcal{A}^+, \mathcal{A}^- \subset \binom{[n]}{k}$.

There exists a vector $\mathbf{x} \in F_n$ with $\mathcal{C}_{\mathbf{x}}^+ \supseteq \mathcal{A}^+$ and $\mathcal{C}_{\mathbf{x}}^- \supseteq \mathcal{A}^-$ if and only if $\mathcal{P}(n, k, \mathcal{A}^+, \mathcal{A}^-)$ has a nonempty feasible set.

MMSSearch(n,k,t, $\mathcal{A}^+$, $\mathcal{A}^-$):
Determine if there is an $\mathbf{x} \in F_n$ *with* $\mathcal{C}_{\mathbf{x}}^+ \supseteq \mathcal{A}^+, \mathcal{C}_{\mathbf{x}}^- \supseteq \mathcal{A}^-$*,*
and fewer than t *nonnegative* k*-sums*

if $|\mathcal{L}_k(\mathcal{A}^+)| \geq t$ **or** $\mathcal{P}(n, k, \mathcal{A}^+, \mathcal{A}^-)$ is infeasible **then**
return Null
end if

if solution to $\mathcal{P}(n, k\mathcal{A}^+, \mathcal{A}^-)$ has fewer than t nonnegative k-sums **then**
output solution to $\mathcal{P}(n, k, \mathcal{A}^+, \mathcal{A}^-)$
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Select $S \notin \mathcal{L}_k(\mathcal{A}^+) \cup \mathcal{R}_k(\mathcal{A}^-)$

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Assume $\mathbf{x} \in F_n$ has fewer than t nonnegative k-sums.

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Lemma. If $|\mathcal{L}_k(S)| \geq t$, then $\sum_{i \in S} x_i < 0$.

Lemma. If $t \leq \binom{n-1}{k-1}$, $1 \in S$, and $|\mathcal{L}_k(S)| + g(n-k,k) \geq t$, then $\sum_{i \in S} x_i < 0$.

Proof. Let $T = \{1, n-k+2, \dots, n\}$.

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Proof. Let $T = \{1, n-k+2, \ldots, n\}$. $|\mathcal{L}_k(T)| = \binom{n-1}{k-1}$, so $\sum_{i \in T} x_i < 0$.

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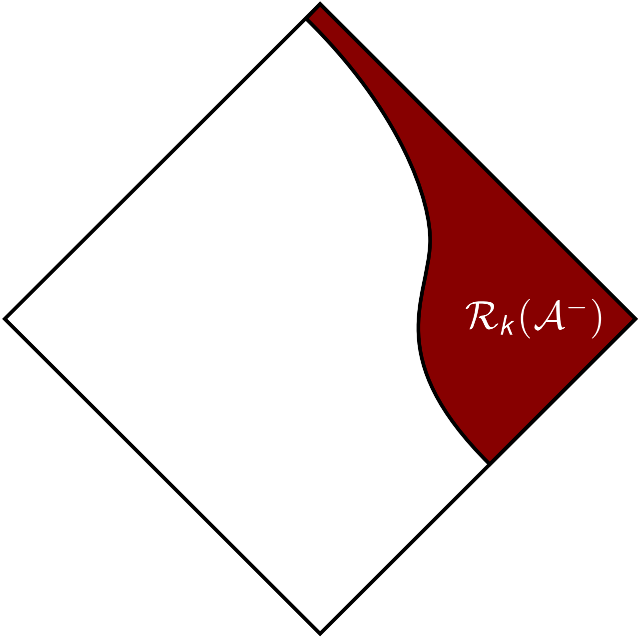
Lemma. If $t \leq \binom{n-1}{k-1}$, $1 \in S$, and $|\mathcal{L}_k(S)| + g(n-k,k) \geq t$, then $\sum_{i\in S} x_i < 0$.

Proof. Let $T = \{1, n-k+2, \ldots, n\}$. $|\mathcal{L}_k(T)| = \binom{n-1}{k-1}$, so $\sum_{i\in T} x_i < 0$.

$$\underset{T}{x_1} \geq \underbrace{x_2 \geq x_3 \geq x_4 \geq x_5 \geq x_6 \geq x_7 \geq x_8 \geq x_9}_{\sum_{i=2}^{n-k+1} x_i \geq 0} \geq \underset{T}{x_{10}} \geq \underset{T}{x_{11}} \geq \underset{T}{x_{12}}$$

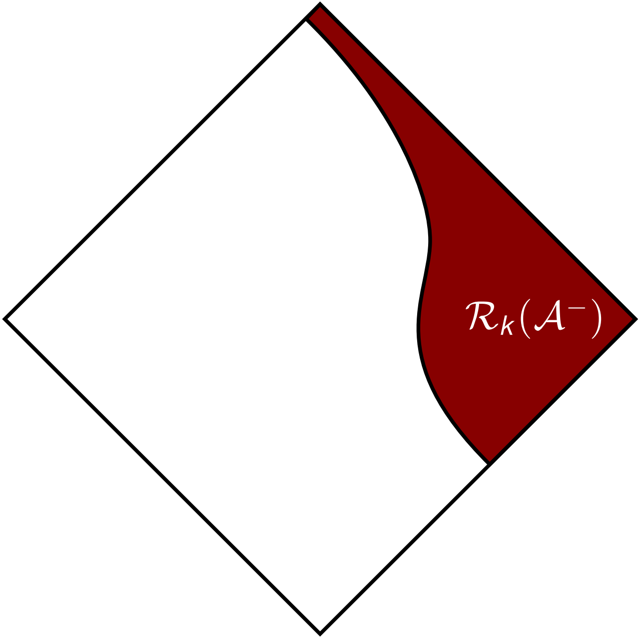
Thus there are at least $g(n-k,k)$ nonnegative k-sums with min coordinate at least 2. □

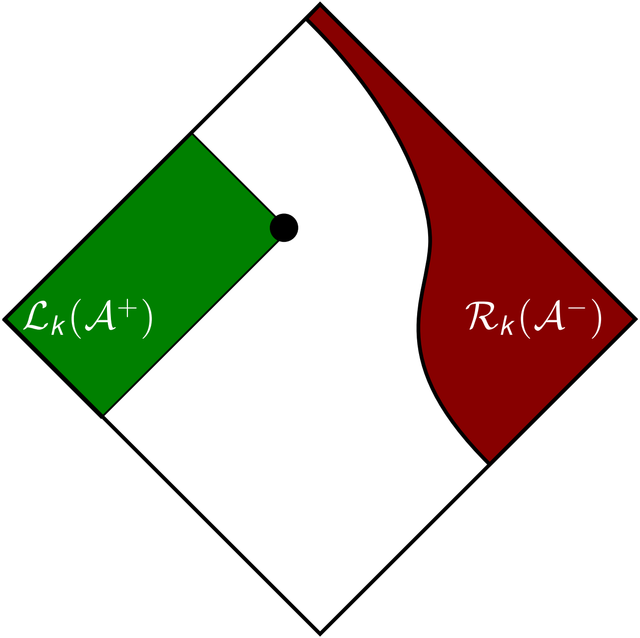

$\mathcal{R}_k(\mathcal{A}^-)$

Learning more about $\mathcal{A}^-$

Define $L^*(S) = |\mathcal{L}_k(S) \setminus \mathcal{L}_k(\mathcal{A}^+)|$.

Lemma. If $L^*(S) + |\mathcal{L}_k(\mathcal{A}^+)| \geq t$, then $\sum_{i \in S} x_i < 0$ for all $\mathbf{x} \in F_n$ with $\mathcal{C}_{\mathbf{x}}^+ \supseteq \mathcal{A}^+$.


$\mathcal{R}_k(\mathcal{A}^-)$


$\mathcal{L}_k(\mathcal{A}^+)$
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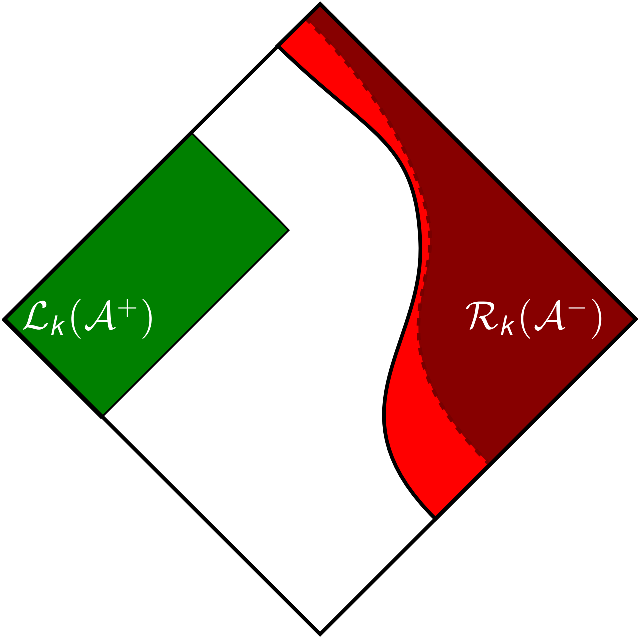

$\mathcal{L}_k(\mathcal{A}^+)$
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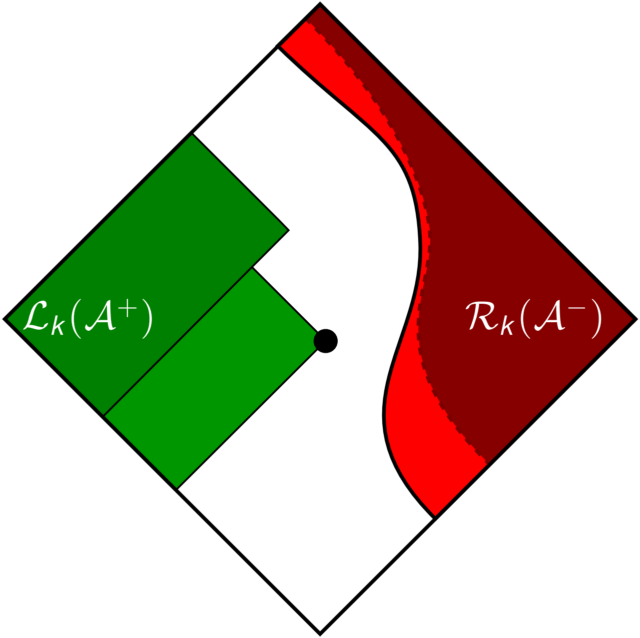
$\mathcal{L}_k(\mathcal{A}^+)$
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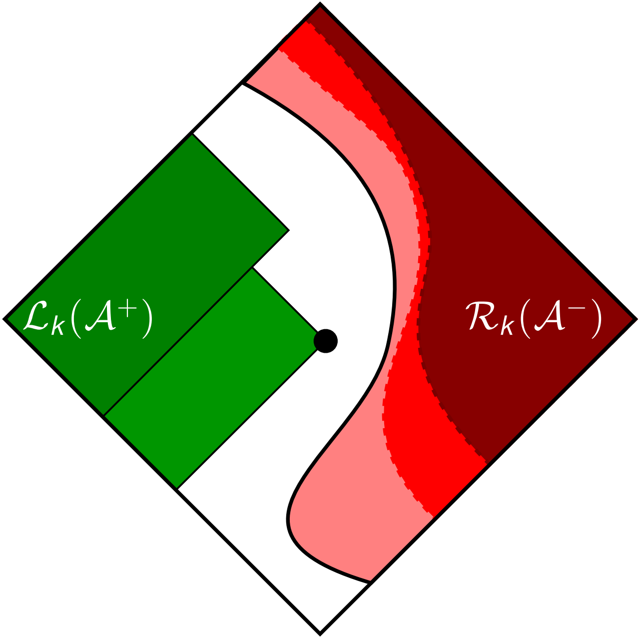

$\mathcal{L}_k(\mathcal{A}^+)$
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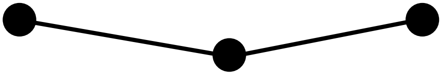
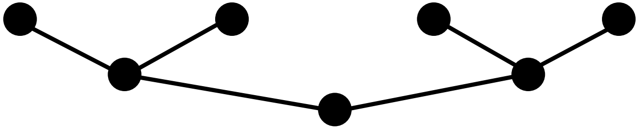
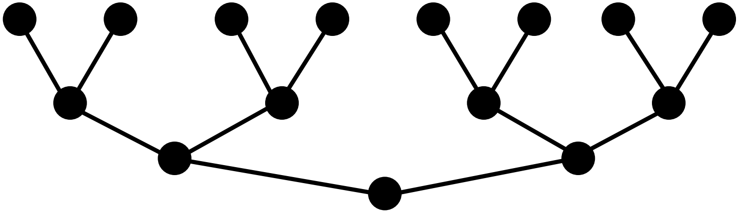
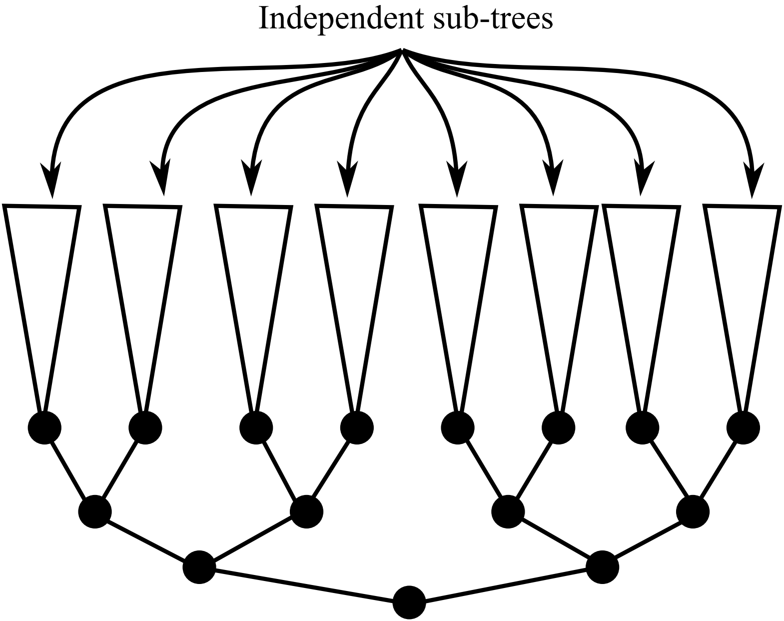
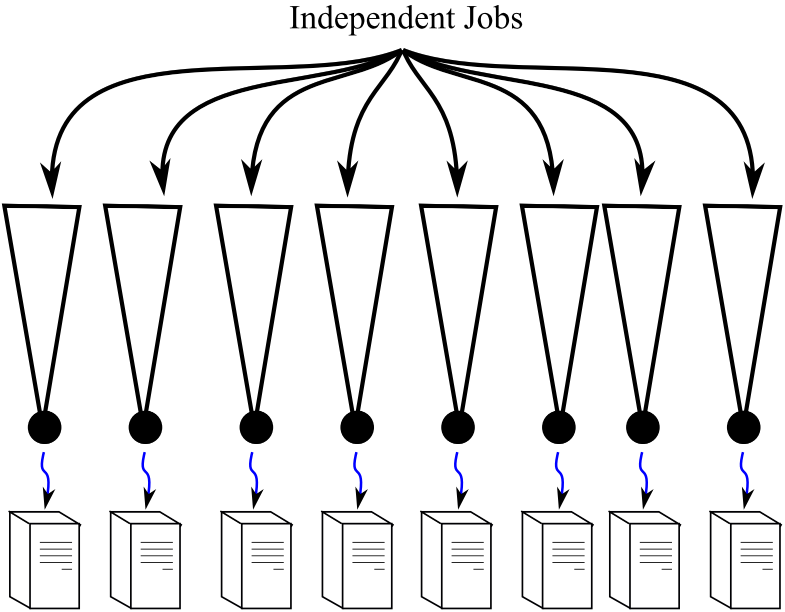






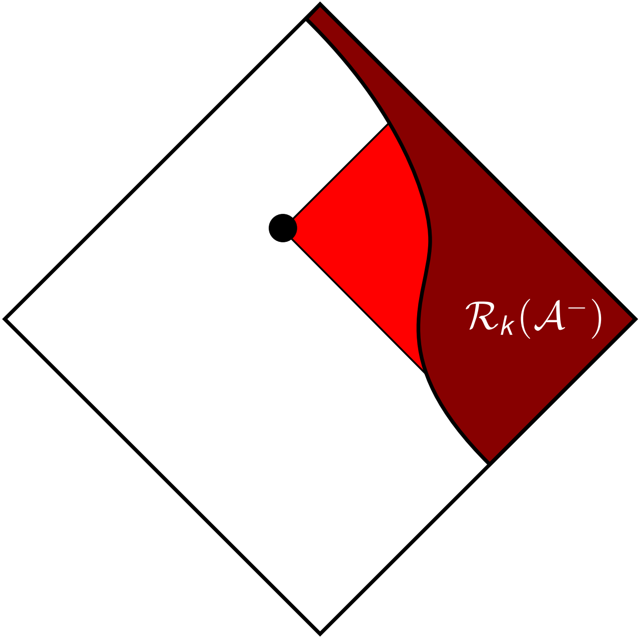

Independent sub-trees

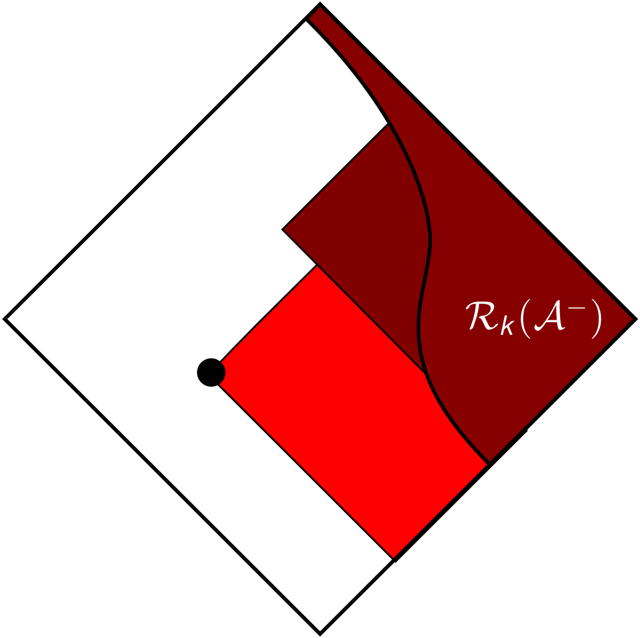

Independent Jobs

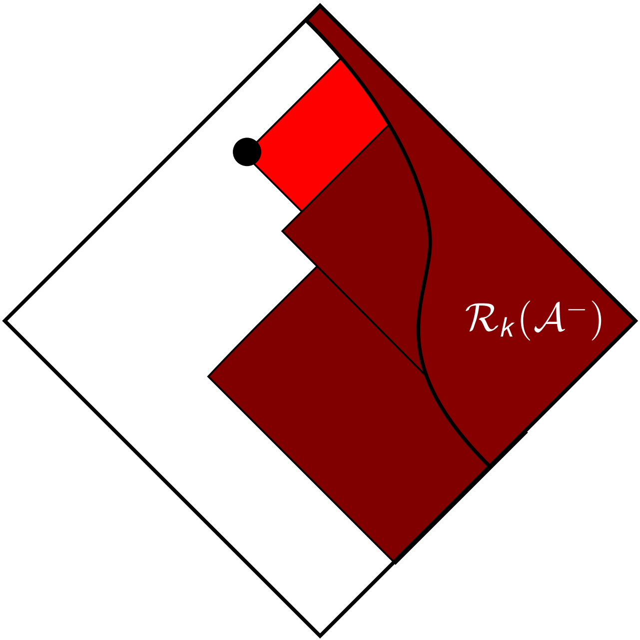
Implementation

My **TreeSearch** library enables parallelization in the Condor scheduler.

Executes on the **Open Science Grid**, a collection of supercomputers around the country.


$\mathcal{R}_K(\mathcal{A}^-)$


$\mathcal{R}_k(\mathcal{A}^-)$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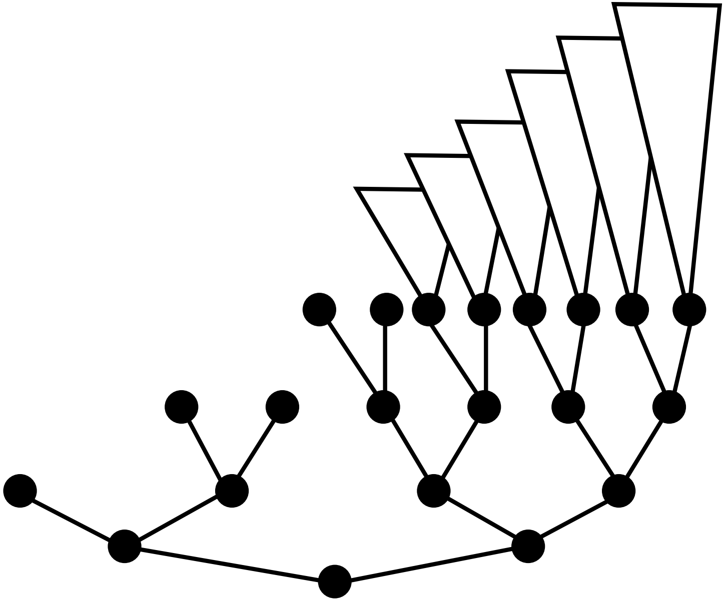

Learning More About $\mathcal{A}^+$

If $\mathcal{P}(n, k, \mathcal{A}^+, \mathcal{A}^- \cup \{S\})$ is infeasible, then all vectors $\mathbf{x} \in F_n$ with $\mathcal{C}_{\mathbf{x}}^+ \supseteq \mathcal{A}^+$ and $\mathcal{C}_{\mathbf{x}}^- \supseteq \mathcal{A}^-$ have $\sum_{i \in S} x_i \geq 0$.

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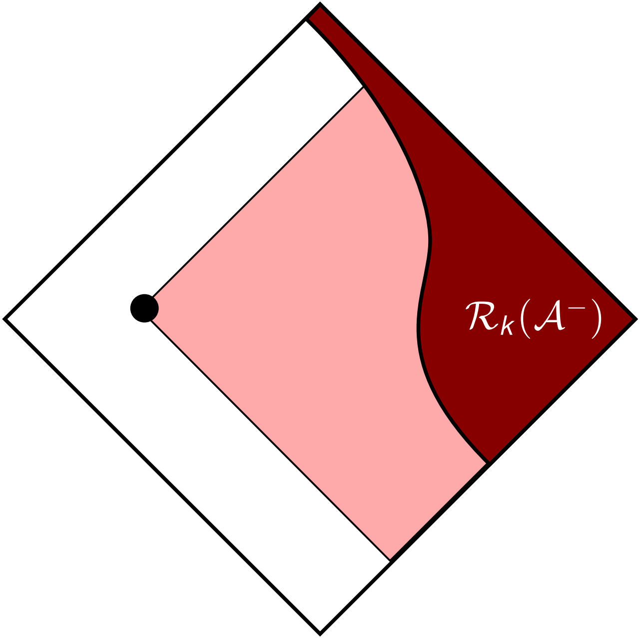
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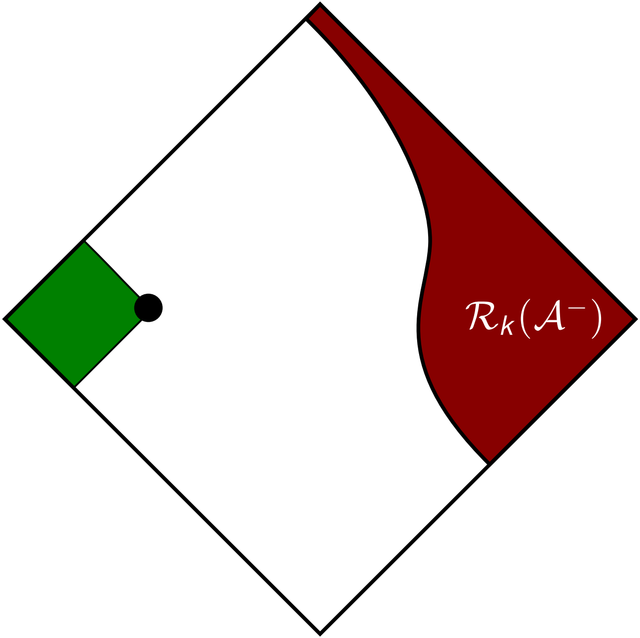
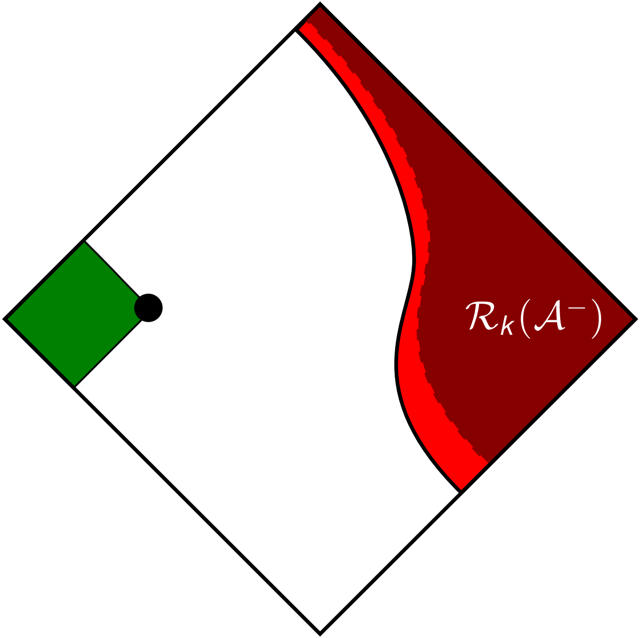


$\mathcal{R}_k(\mathcal{A}^-)$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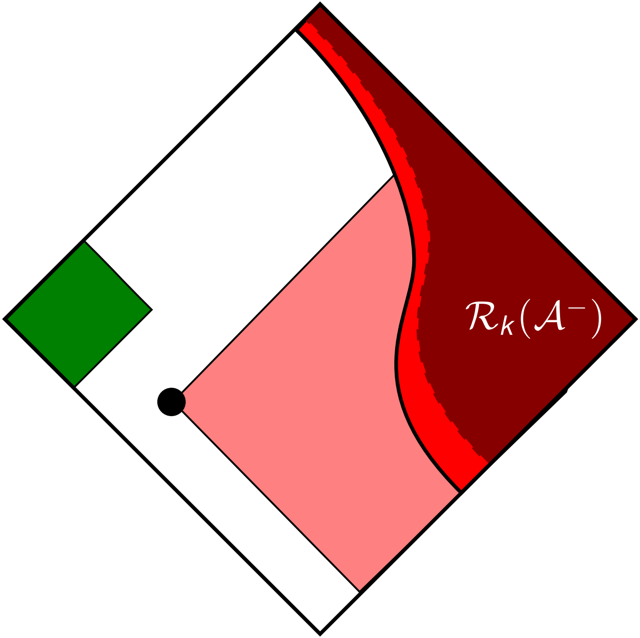
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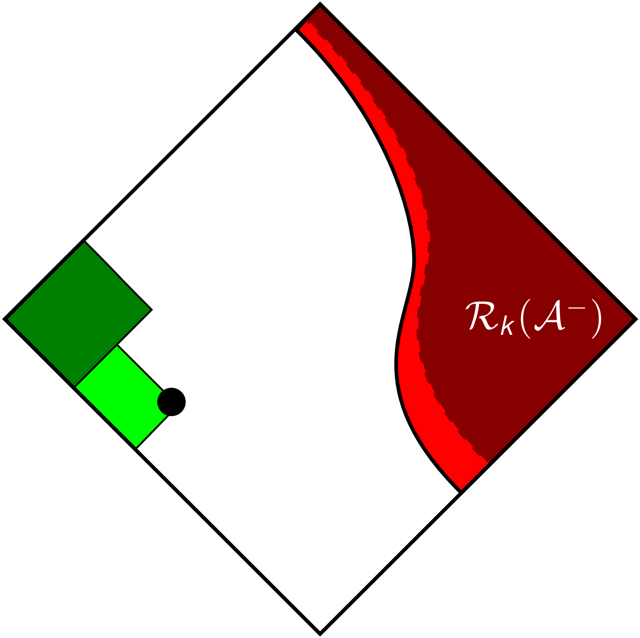
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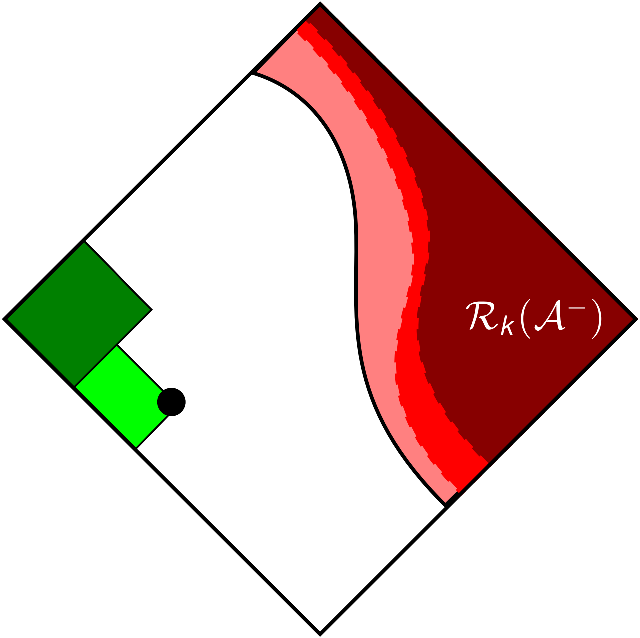
$\mathcal{R}_k(\mathcal{A}^-)$

Computer-Generated Proof that $g(11,3) = \binom{10}{2} = 45$

Proof.

Computer-Generated Proof that $g(11,3) = \binom{10}{2} = 45$

Proof. The following sums must be strictly negative or we have at least 45 of nonnegative sets:

$$
\begin{array}{ccc}
x_1 + x_6 + x_{11} & x_1 + x_8 + x_{10} & x_2 + x_5 + x_{11} \\
x_2 + x_6 + x_{10} & x_2 + x_7 + x_9 & x_3 + x_4 + x_{11} \\
x_3 + x_5 + x_9 & x_3 + x_7 + x_8 & x_4 + x_6 + x_8
\end{array}
$$

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x_3+x_5+x_9 & x_3+x_7+x_8 & x_4+x_6+x_8
\end{array}
$$

The following sums must be nonnegative or else the associated linear program becomes infeasible:

$$
\begin{array}{lll}
x_4+x_6+x_7 & x_4+x_5+x_8 & x_3+x_4+x_{10}
\end{array}
$$

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$$\begin{array}{lll} x_1+x_6+x_{11} & x_1+x_8+x_{10} & x_2+x_5+x_{11} \\ x_2+x_6+x_{10} & x_2+x_7+x_9 & x_3+x_4+x_{11} \\ x_3+x_5+x_9 & x_3+x_7+x_8 & x_4+x_6+x_8 \end{array}$$

The following sums must be nonnegative or else the associated linear program becomes infeasible:

$$\begin{array}{lll} x_4+x_6+x_7 & x_4+x_5+x_8 & x_3+x_4+x_{10} \end{array}$$

These positive sets now force at least 56 nonnegative 3-sums, and our target was 45 nonnegative 3-sums. □

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So, we can add such sets S to $\mathcal{A}^+$.

How do we find such sets?

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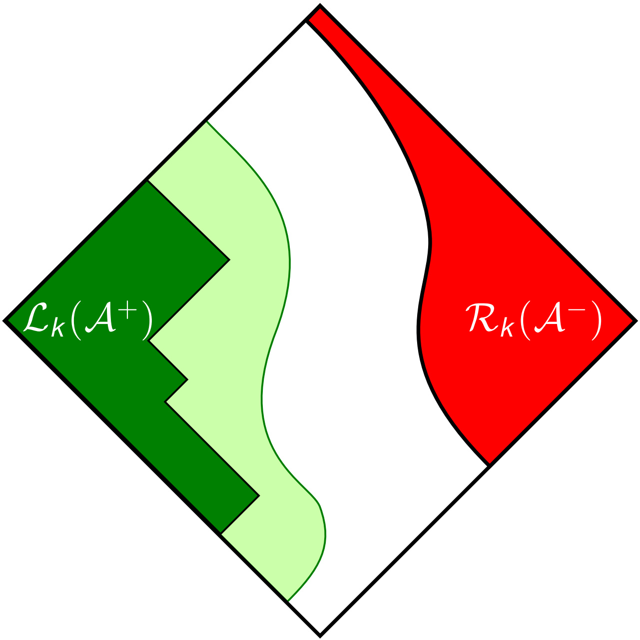
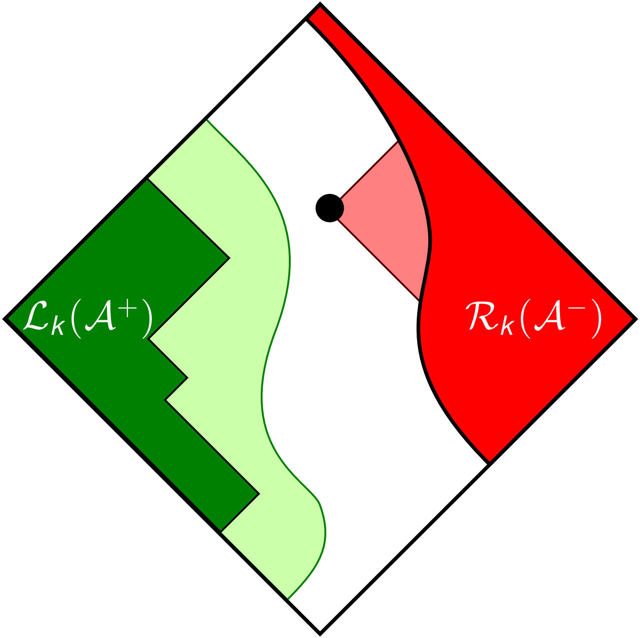
So, we can add such sets S to $\mathcal{A}^+$.

How do we find such sets?

We **randomly sample** a set S to test.



$\mathcal{L}_k(\mathcal{A}^+)$
$\mathcal{R}_k(\mathcal{A}^-)$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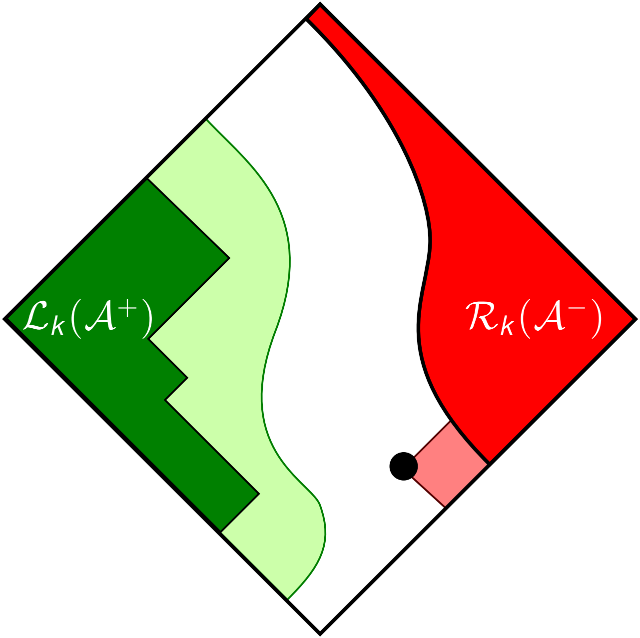
$\mathcal{L}_k(\mathcal{A}^+)$
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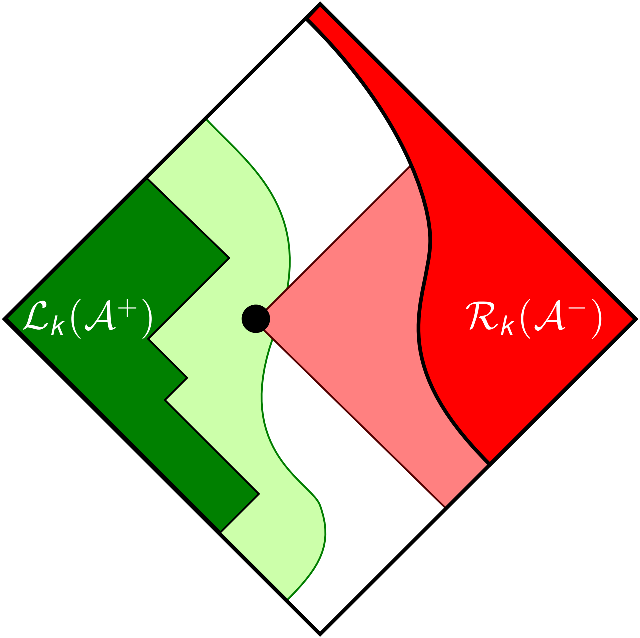
$\mathcal{L}_k(\mathcal{A}^+)$
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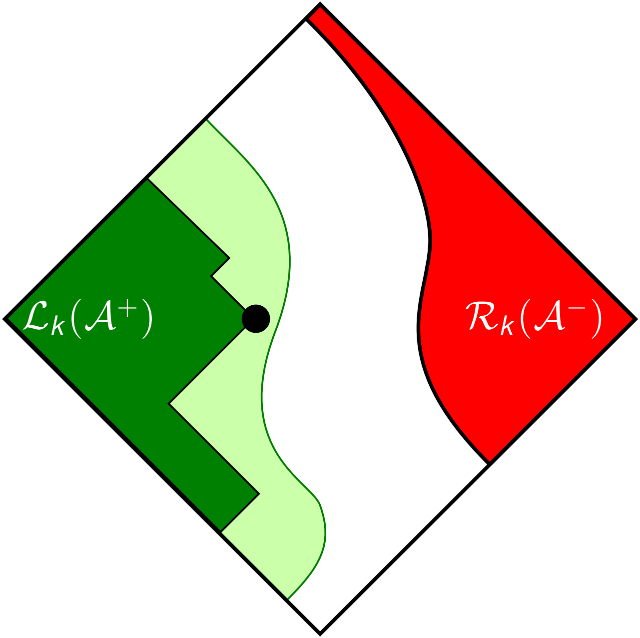
$\mathcal{L}_k(\mathcal{A}^+)$
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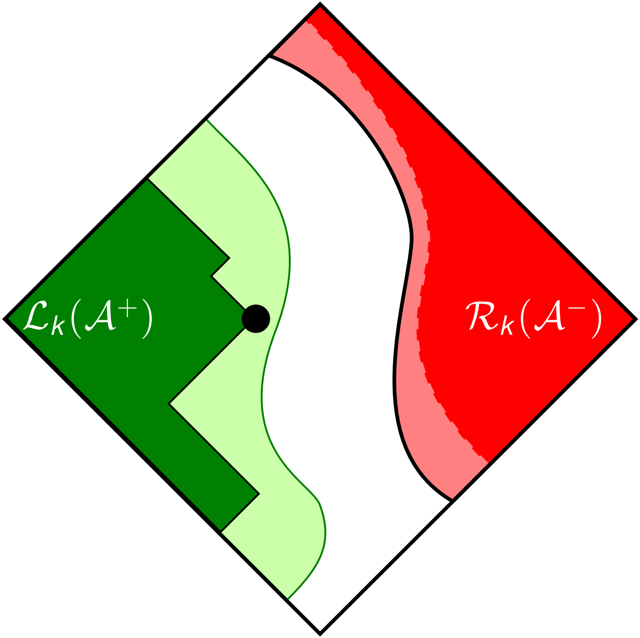
$\mathcal{L}_k(\mathcal{A}^+)$
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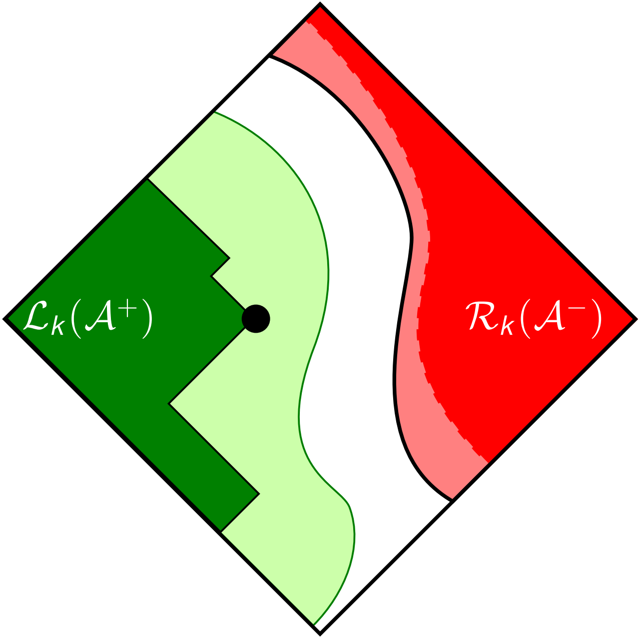
$\mathcal{L}_k(\mathcal{A}^+)$
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Our Results

After executing our full algorithm, we discover $g(n, k)$ for all $k \in \{3, 4, 5, 6, 7\}$ and all n.

Our Results

After executing our full algorithm, we discover $g(n, k)$ for all $k \in \{3, 4, 5, 6, 7\}$ and all n.

We also know of **sharp examples**: $\mathbf{x} \in F_n$ with exactly $g(n, k)$ nonnegative k-sums!

Sharp Examples

All of our examples have the form

$$\mathbf{x} = a^b \left(-b\right)^a$$

for $a + b = n$, where $\mathbf{x}$ is given as

$$\underbrace{x_1 = \cdots = x_b = a,}_{b \text{ copies of } a} \qquad \underbrace{x_{b+1} = \cdots = x_n = -b}_{a \text{ copies of } -b}$$

Sharp Examples

$$(n-1)^1\ (-1)^{n-1}$$

has $\binom{n-1}{k-1}$ nonnegative k-sums.

$$3^{n-3}\ (-(n-3))^3$$

has $\binom{n-3}{k}$ nonnegative k-sums when $n > 3k$.

k	n	$g(n,k)$	Sharp Example
6	7	1	$1^6\ (-6)^1$
6	8	7	$1^7\ (-7)^1$
6	9	28	$1^8\ (-8)^1$
6	10	70	$8^2\ (-2)^8$
6	11	126	$9^2\ (-2)^9$
6	12	462	$11^1\ (-1)^{11}$
6	13	462	$2^{11}\ (-11)^2$
6	14	924	$2^{12}\ (-12)^2$
6	15	1705	$12^3\ (-3)^{12}$
6	16	2431	$13^3\ (-3)^{13}$
6	17	3367	$14^3\ (-3)^{14}$
6	18	6188	$17^1\ (-1)^{17}$
6	19	8008	$3^{16}\ (-16)^3$

A vector is **strong** if $x_1 + \sum_{i=n-k+2}^{n} x_i < 0$.

k	n	Strong Example
6	20	$3^{17}\ (-17)^{3}$
6	21	$17^{4}\ (-4)^{17}$
6	22	$18^{4}\ (-4)^{18}$
6	23	$19^{4}\ (-4)^{19}$
6	24	$33^{1}\ 1^{16}\ (-7)^{7}$
6	25	$104^{1}\ 4^{16}\ (-21)^{8}$

Our Conjecture

Conjecture (Hartke, Stolee, '13+) For all $k \geq 2$, and $n < 4k$, the least number of nonnegative k-sums in a **strong** vector $\mathbf{x} \in F_n$ is achieved by a vector of the form $a^b\ (-b)^a$.

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For $k \leq 250$, the largest examples with fewer than $\binom{n-1}{k-1}$ nonnegative k-sums are of the form $3^{n-3}\ (-(n-3))^3$.

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$$\lim_{k\to\infty} \frac{f(k)}{k} = \lim_{k\to\infty} \frac{N_k}{k} =$$

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$$\lim_{k\to\infty} \frac{f(k)}{k} = \lim_{k\to\infty} \frac{N_k}{k} = 3.$$

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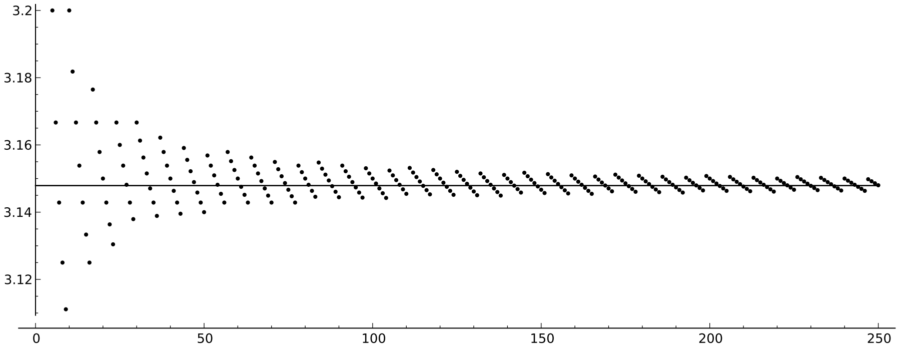
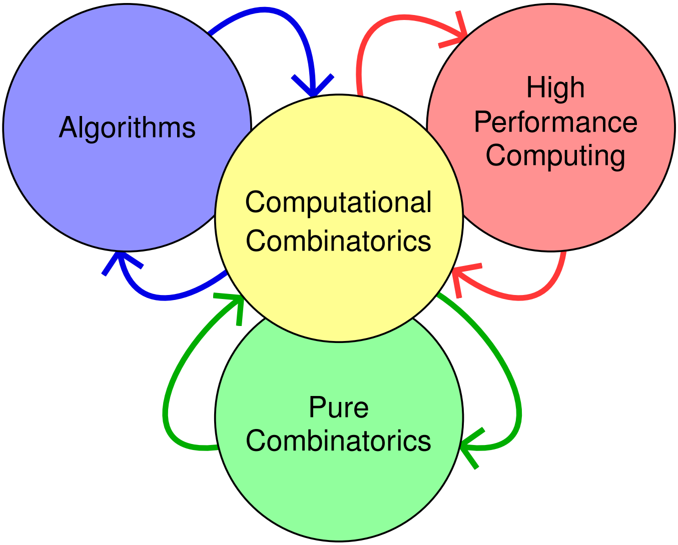
$$\lim_{k\to\infty} \frac{f(k)}{k} = \lim_{k\to\infty} \frac{N_k}{k} = 3.147899...$$

Our Conjecture



Values of N_k / k for $k \in \{5, \ldots, 250\}$.

Computational Combinatorics



A Linear Programming Approach to the Manickam-Miklós-Singhi Conjecture

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