Generating p-extremal graphs

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Last Week

1. Discussed generation of combinatorial objects.
2. “Defined” symmetry in terms of automorphism groups.
3. Presented **canonical deletion**, a method to remove isomorphic duplicates.
4. Discussed example for generating connected graphs by vertex augmentations.

Perfect Matchings

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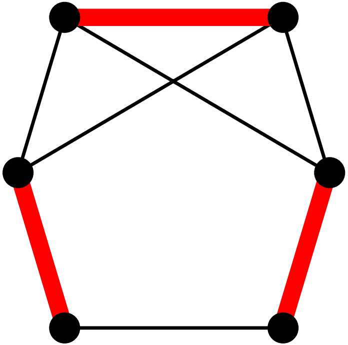

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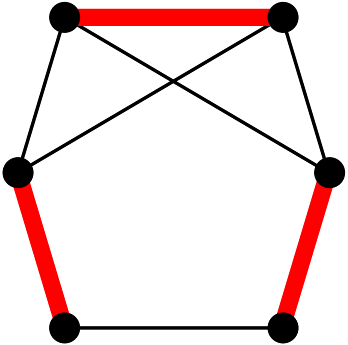

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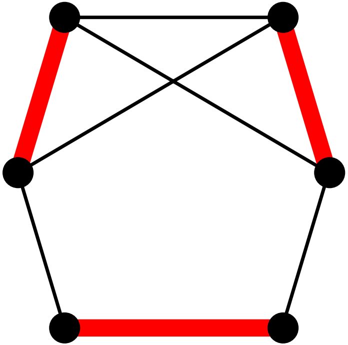

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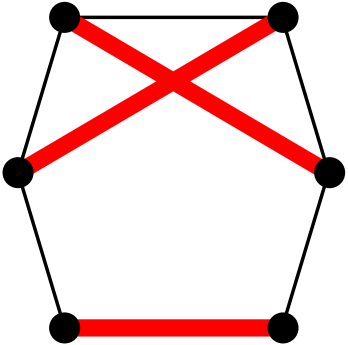

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8 edges

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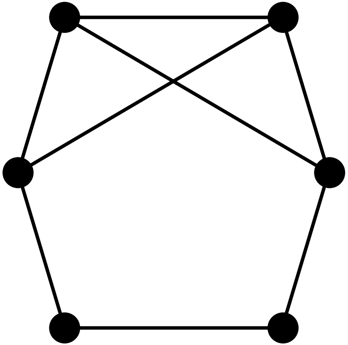

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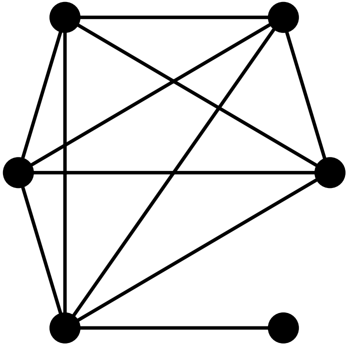

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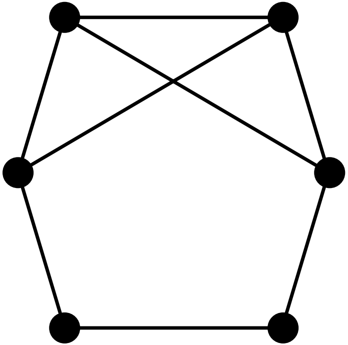

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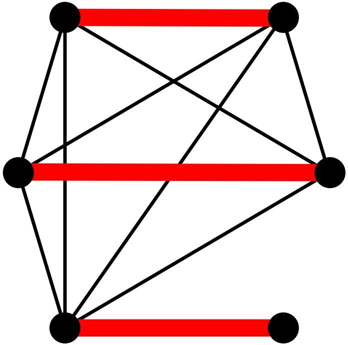

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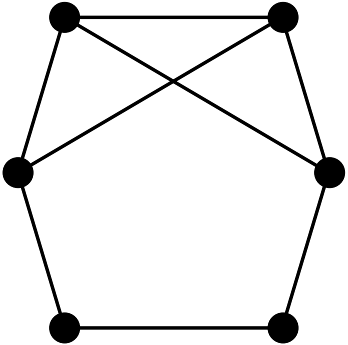

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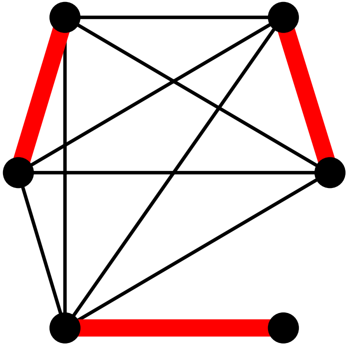

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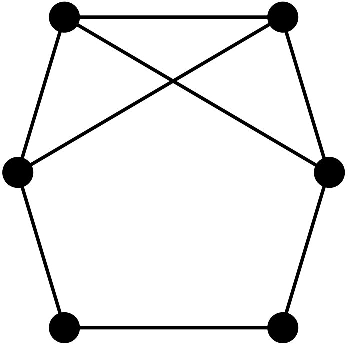

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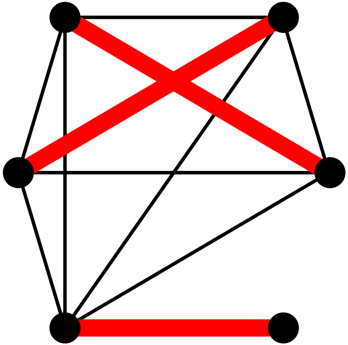

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Perfect Matchings

A **perfect matching** is a set of edges which cover each vertex exactly once.

Question (Dudek, Schmitt, '12) What is the maximum number of edges in a graph with exactly n vertices and p perfect matchings?

Definition Let n be an even number and fix $p \geq 1$.

$$f(n,p) = \max\{|E(G)| : |V(G)| = n, \Phi(G) = p\}.$$

Graphs attaining this number of edges are ***p*-extremal**.

Hetyei's Theorem

Theorem (Hetyei's Theorem, 1986) For all even $n \geq 2$,

$$f(n, 1) = \frac{n^2}{4}.$$

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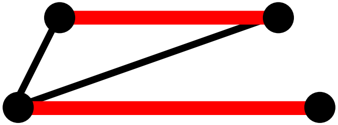

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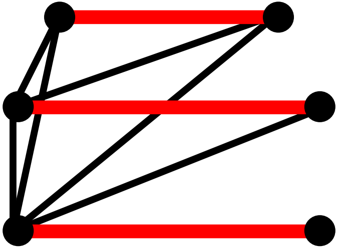

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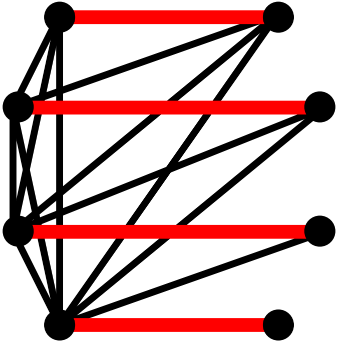
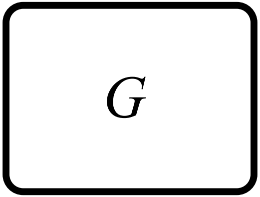


The Form of $f(n, p)$

Theorem (Dudek & Schmitt)

For each p, there exist constants n_p, c_p so that for all $n \geq n_p$,

$$f(n, p) = \frac{n^2}{4} + c_p.$$



Take G with $\frac{n^2}{4} + c$ edges.

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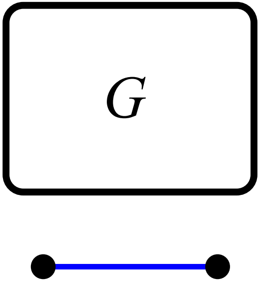

Add two new vertices.

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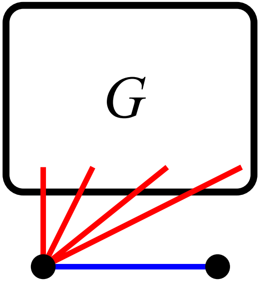

Add edges to get $\frac{(n+2)^2}{4} + c$ edges.

The Excess of a Graph

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$$c(G) = |E(G)| - \frac{|V(G)|^2}{4}.$$

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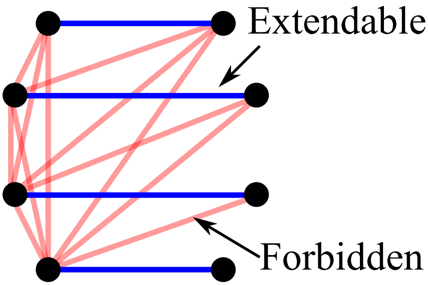
In this sense, lower bounds on c_p are "easy"
(any G with $\Phi(G) = p$, has $c(G) \leq c_p$).

Upper bounds are hard: must prove NO graph achieves a higher constant!

Edge Types

Let $\Phi(G) > 0$ and $e \in E(G)$.

- e is **extendable** if there exists a perfect matching containing e.
- e is **forbidden** otherwise.



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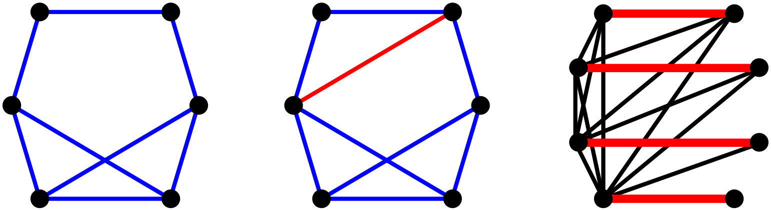
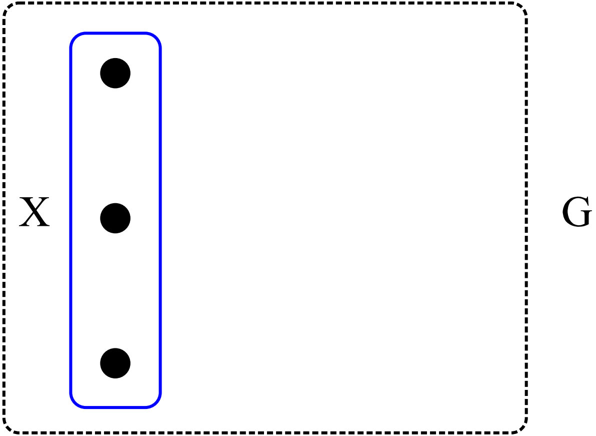


Barriers

Let $\Phi(G) > 0$. A set $X \subset V(G)$ is a **barrier** if

$$|X| = \text{\# of odd connected components in } G - X$$



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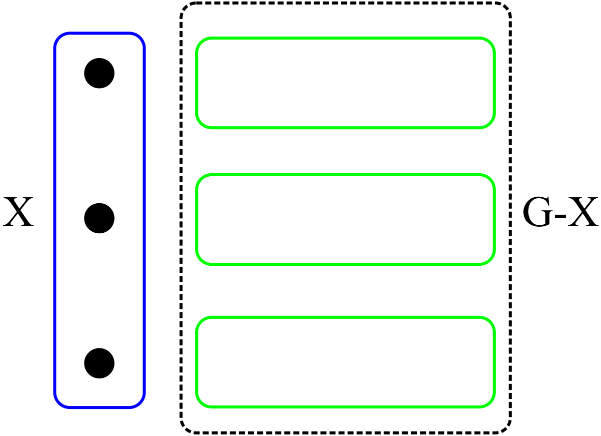

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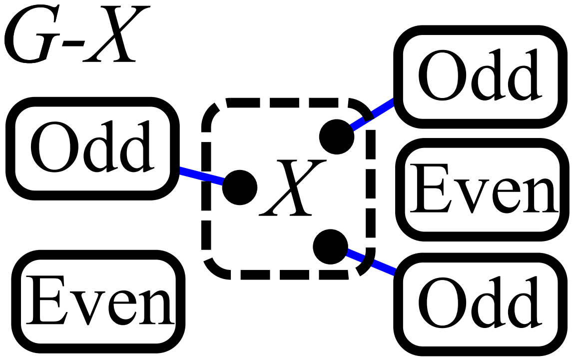

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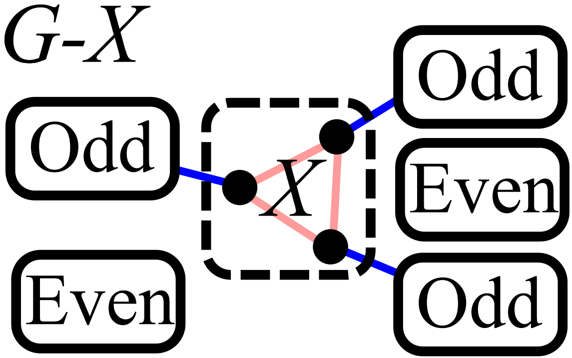
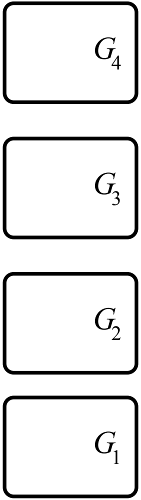


Spires

Let $G_1, \ldots, G_k$ be chambers with barriers $X_1, \ldots, X_k$ where X_i is of maximum size in G_i.



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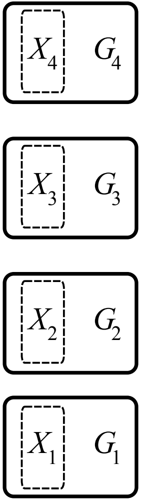

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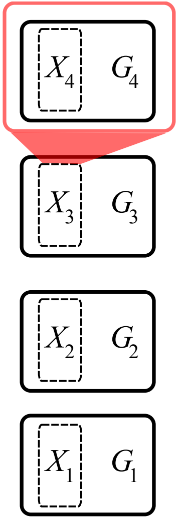

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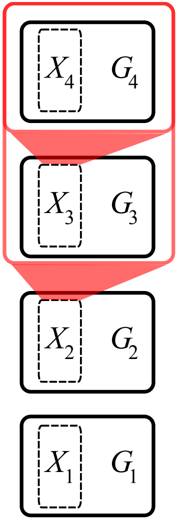

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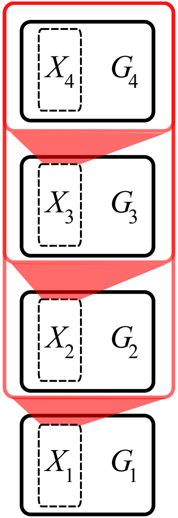

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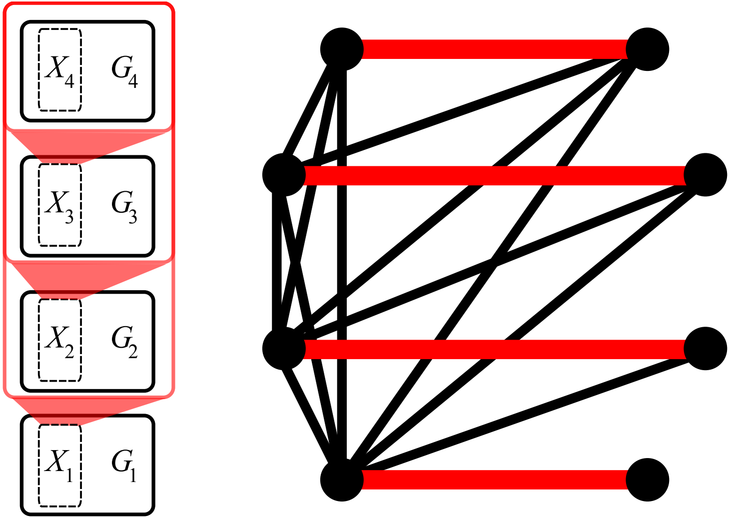

p-Extremal Graphs are Spires

Theorem (Hartke, Stolee, West, Yancey '12) If G is p-extremal, then G is a spire of chambers $G_1, \ldots, G_k$, with barriers $X_i \subseteq V(G_i)$ of maximum size.

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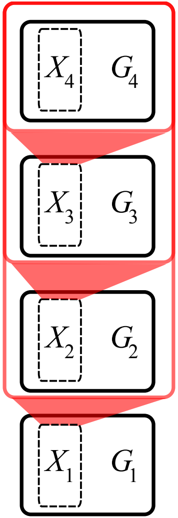
3. There are at most $N_{p_i} = O(\sqrt{p_i})$ vertices in G_i, and $c(G_i) \leq c_{p_i}$.
4. $c_p = c(G) \leq \sum_{i=1}^{k} c(G_i)$ with equality if and only if $\frac{|X_i|}{|V(G_i)|} = \frac{1}{2}$ for all $i < k$.

Order of Chambers



$$\frac{|X_4|}{|V(G_4)|} \leq \frac{|X_3|}{|V(G_3)|} \leq \frac{|X_2|}{|V(G_2)|} \leq \frac{|X_1|}{|V(G_1)|}$$

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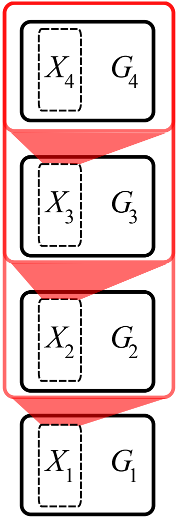

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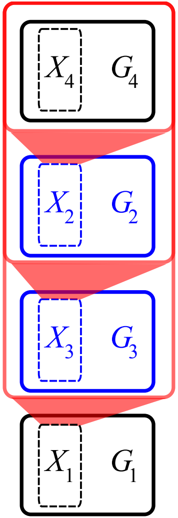

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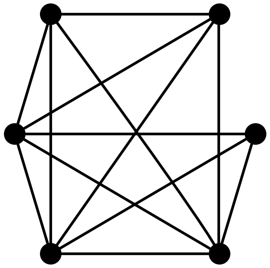
For $p \leq 10$, $N_p \leq 12$ and `geng` can enumerate all possible graphs.

Example: $p = 7$

Theorem (HSWY, '12) For even n with $n \geq 6$, the unique 7-extremal graph has $\frac{n^2}{4} + 3$ edges and is a spire with $k = n/2 - 2$ chambers $G_1, \ldots, G_k$ are given by $G_i = K_2$ for $i < k$ and G_k given below.



Example: Generating Graphs by Vertex Additions

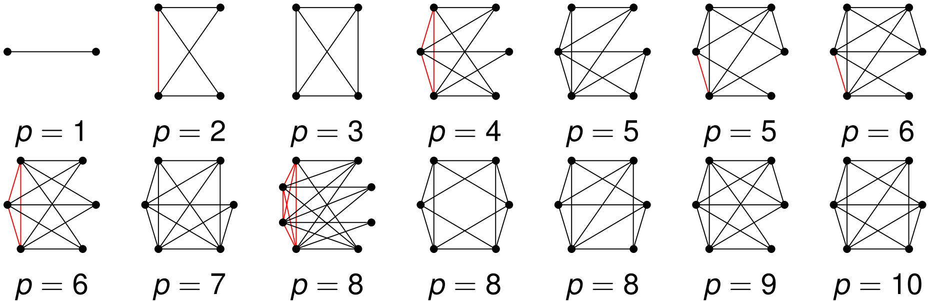
Let's generate all graphs of order n by adding vertices one-by-one.

Augmentation: Add a vertex adjacent to a set $S \subset V(G)$.

IMPORTANT: Only one augmentation per orbit!

Deletion: Select a vertex $v \in V(G)$ to delete, $G' = G - v$.

Extremal Chambers for $p \leq 10$



Values of c_p for $p \leq 10$

p	1	2	3	4	5	6	7	8	9	10
n_p	2	4	4	6	6	6	6	6	6	6
c_p	0	1	2	2	2	3	3	3	4	4
N_p	2	4	6	8	8	10	10	12	12	12
	Dudek & Schmitt						HSWY			

Table: Known values of n_p and c_p.

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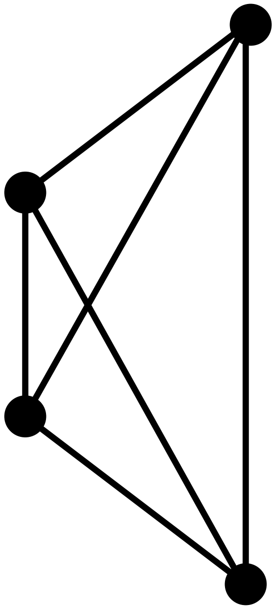

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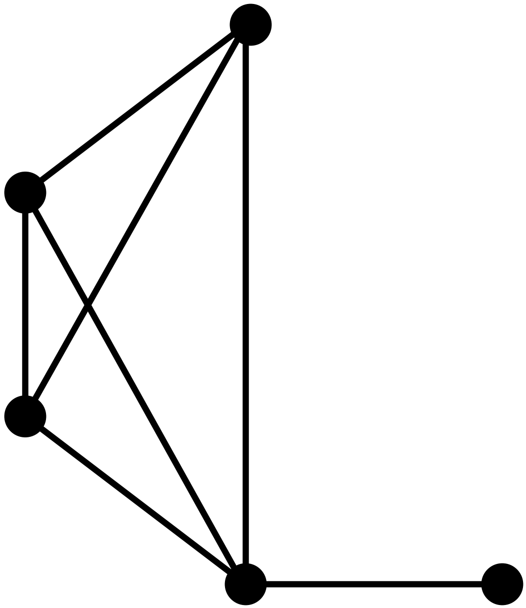

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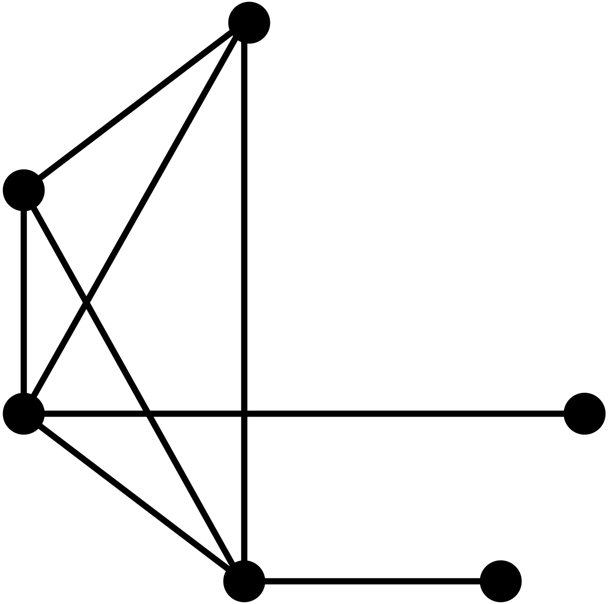

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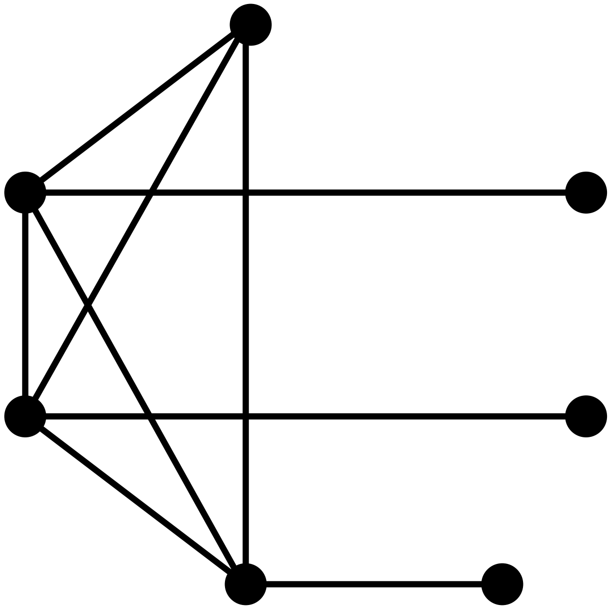

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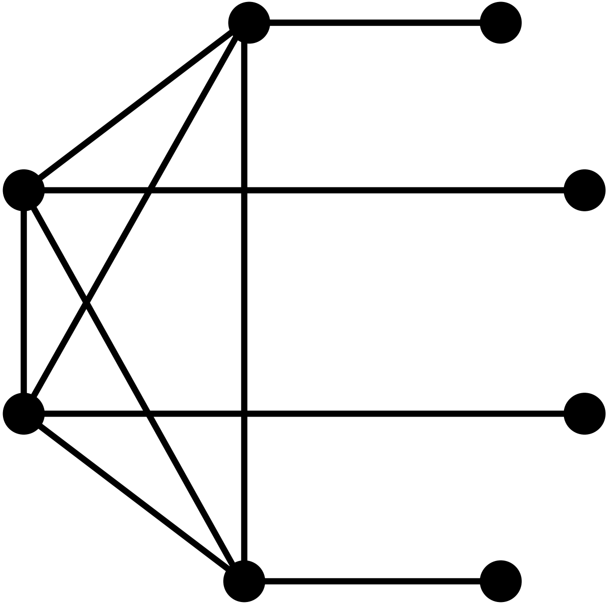

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We break chambers into the **extendable** and **forbidden** edges.

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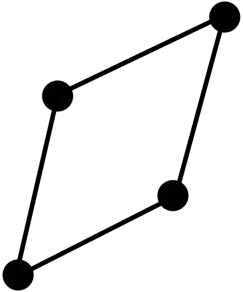

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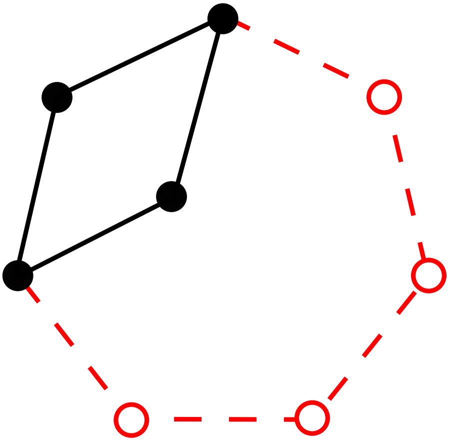

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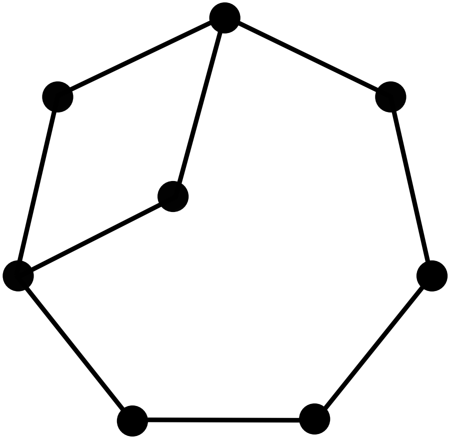

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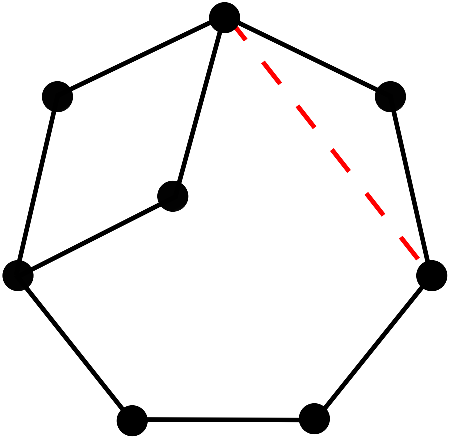

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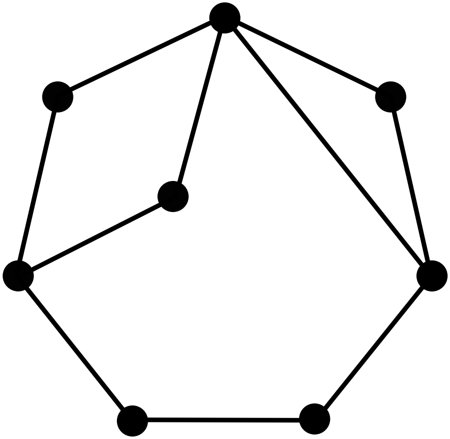

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Graphs which appear "between" two extendable graphs in a two-ear augmentation are *almost extendable* graphs.

Example: Generating Graphs by Ear Augmentations

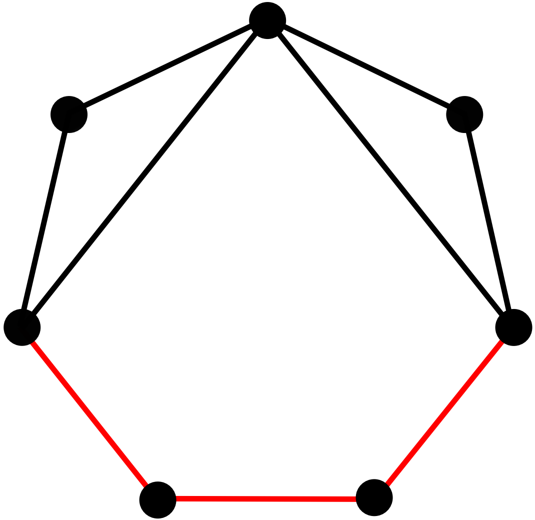
Let's generate all graphs of order n by adding vertices one-by-one.

Initialization: Let G be a cycle.

Augmentation: Let $x, y \in V(G)$ be distinct vertices and ℓ a length.
Add an ear of length ℓ between x and y.

Deletion: Select an ear to delete, such that G remains 2-connected.

Generating with Ear Augmentations



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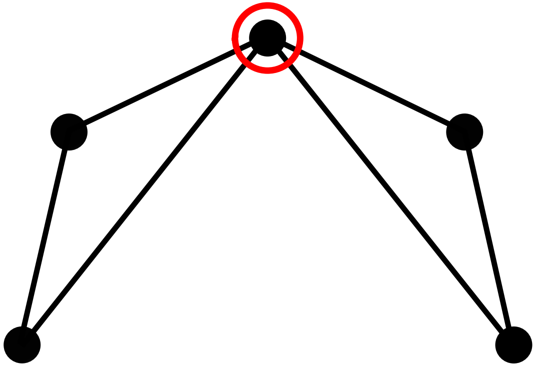

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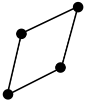

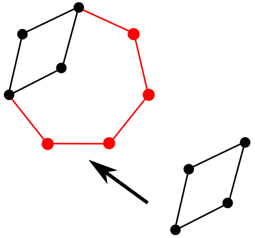

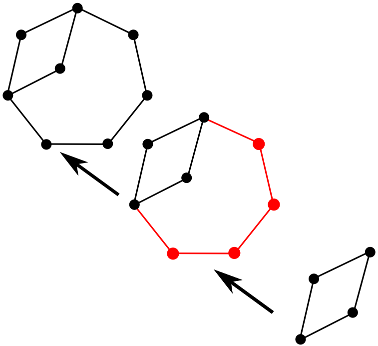

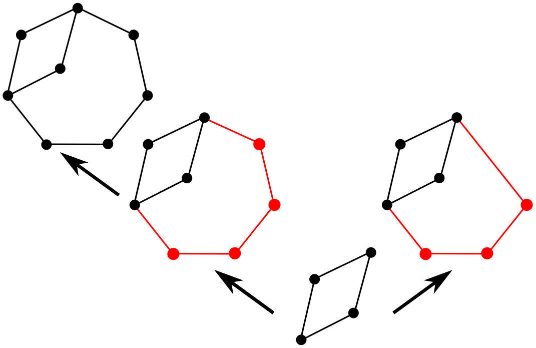

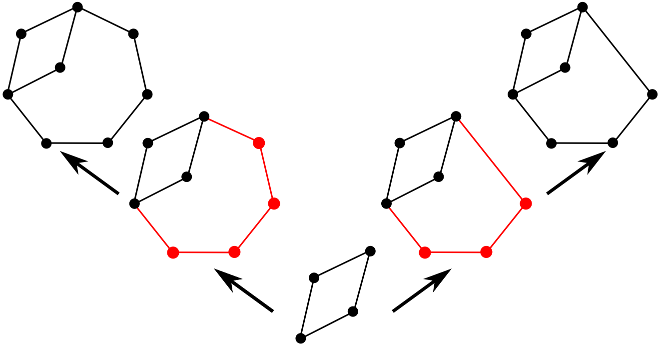

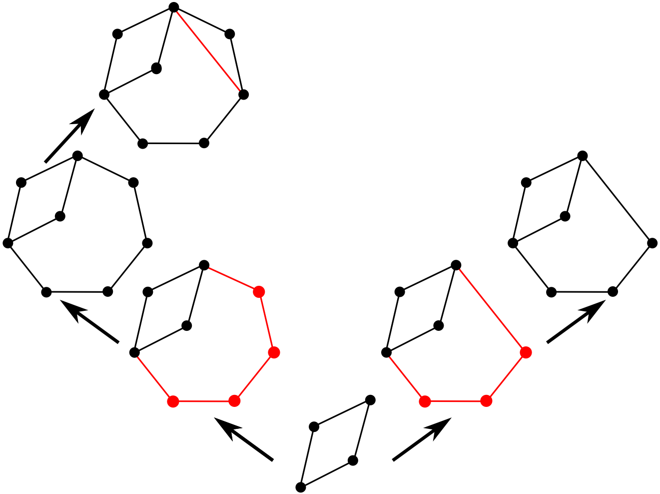

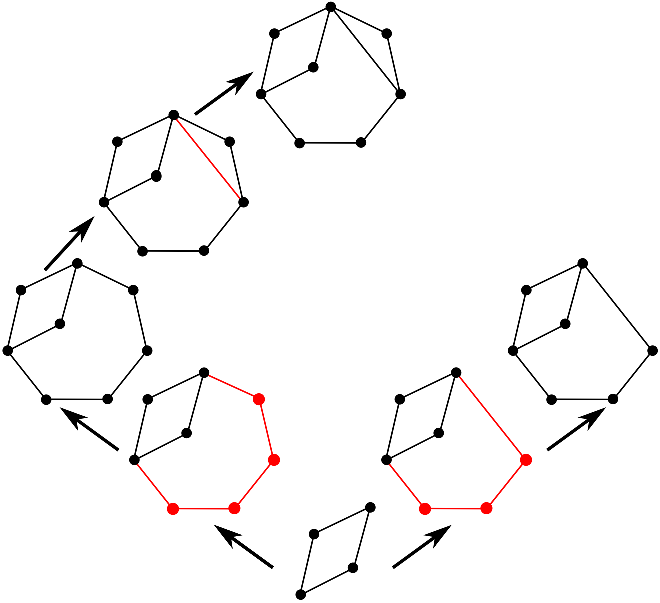

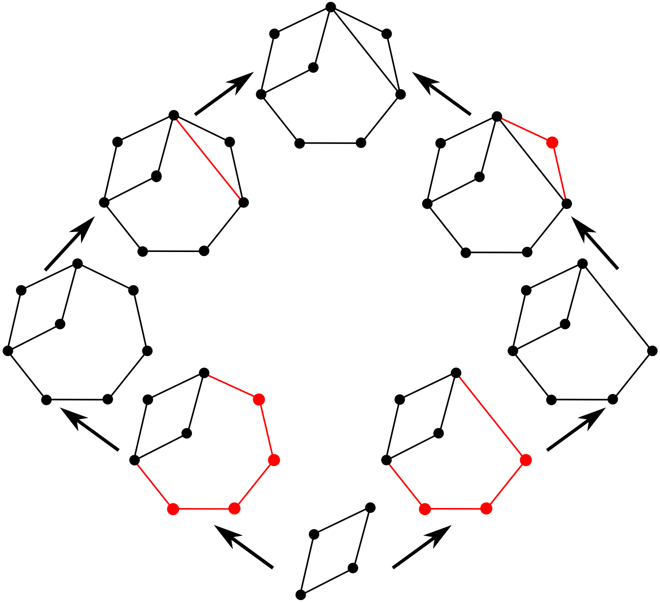

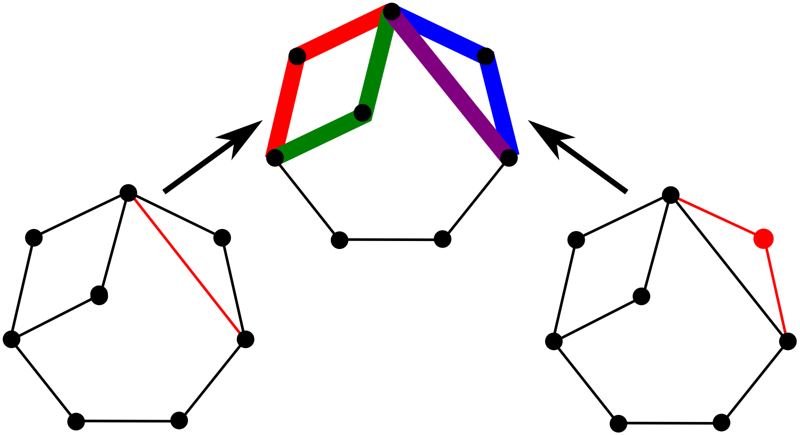

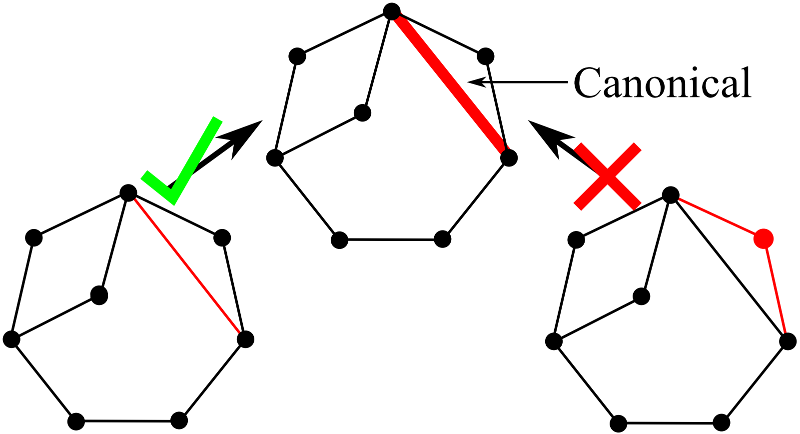
Canonical

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The ear ε is the canonical deletion.

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If H is extendable, then let $\mathcal{E}(H)$ be the collection of supergraphs G where all edges in $E(G) \setminus E(H)$ are forbidden.

The Extremal Two-Ears Theorem

Lemma. Let H be a 1-extendable graph on n vertices with $\Phi(H) = q$. Let H' be a 1-extendable supergraph of H built from H by a graded ear decomposition. Let $\Phi(H') = p > q$ and $N = n(H')$. Choose $G \in \mathcal{E}(H)$ and $G' \in \mathcal{E}(H')$ with the maximum number of edges in each set. Then,

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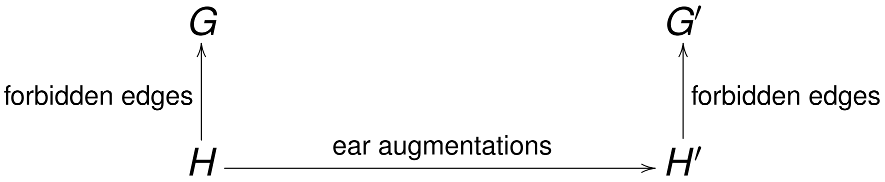

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Begin with p, c, N. Generate all chambers G with p perfect matchings, $c(G) \geq c$, and $n(G) \leq N$.

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7. If $\Phi(H) = p$, then output all maximum G (with $c(G) \geq c$).

Timing

p	N_p	c_p	CPU Time
5	8	2	0.02s
6	10	3	0.04s
7	10	3	0.18s
8	12	3	0.72s
9	12	4	1.46s
10	12	4	5.95s
11	14	3	43.29s
12	14	5	44.01s
13	14	3	6.66m
14	16	4	12.17m
15	16	6	12.71m

p	N_p	c_p	CPU Time
16	16	4	2.02h
17	16	4	6.77h
18	18	5	11.75h
19	18	4	2.71d
20	18	5	4.21d
21	18	5	13.71d
22	20	5	42.84d
23	20	5	118.32d
24	20	6	209.42d
25	20	5	2.52y
26	20	5	7.21y
27	22	6	10.68y

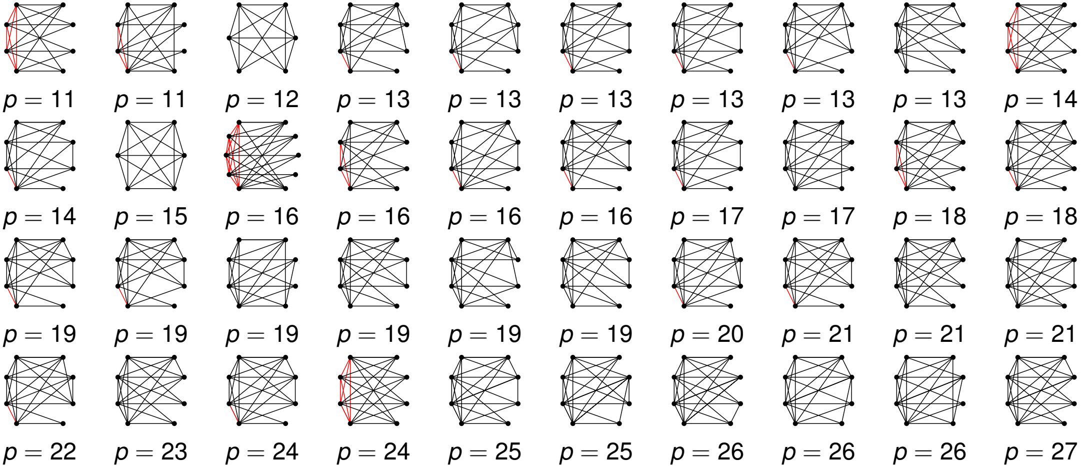
Results

p	1	2	3	4	5	6	7	8	9	10
c_p	**0**	**1**	**2**	**2**	**2**	**3**	**3**	**3**	**4**	**4**
n_p	2	4	4	6	6	6	6	6	6	6

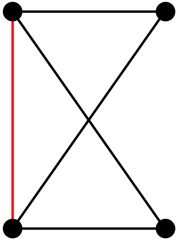
p	11	12	13	14	15	16	17	18	19	20
c_p	3	**5**	3	4	**6**	4	4	5	4	5
n_p	8	6	8	8	6	8	8	8	8	8

p	21	22	23	24	25	26	27			
c_p	5	5	5	**6**	5	5	**6**			
n_p	8	8	8	8	8	8	8			

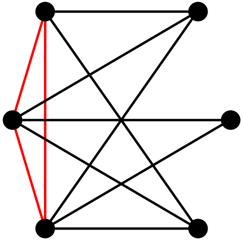
Extremal Chambers for $11 \leq p \leq 27$



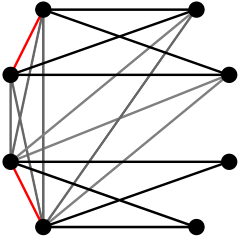
p-Extremal Configurations for $p \in \{2, 4\}$



$p = 2$
$c_2 = 1$



$p = 4$
$c_4 = 2$



$p = 4$
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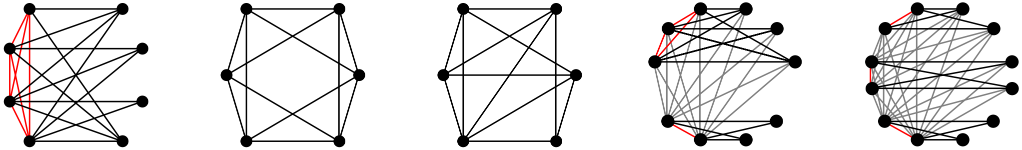
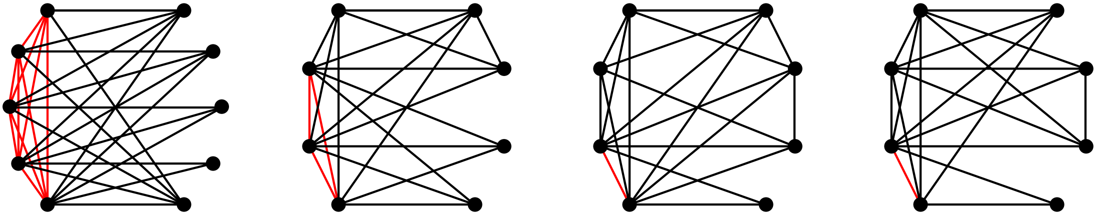
p-Extremal Configurations for $p = 8$



$p = 8$

$c_8 = 3$

p-Extremal Configurations for $p = 16$



$p = 16$

$c_{16} = 4$

Open Problems!

1. Compute more values of c_p.

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Conjecture (Hartke, Stolee, West, Yancey, '12) Let p, k, t be integers so that $k \in \{1, \ldots, 2t\}$ and

$$k(2t-1)!! \leq p < (k+1)(2t-1)!!$$

and set $C_p = t^2 - t + k - 1$. **Always** $c_p \leq C_p$.

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If the conjecture holds, then $c_p \leq O\left(\left(\frac{\log p}{\log \log p}\right)^2\right)$.

If you learned ANYTHING...

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...then it should be that

pairing structural theorems with specialized algorithms can be

very effective!

Generating p-extremal graphs

Derrick Stolee

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dstolee@iastate.edu
http://www.math.iastate.edu/dstolee/

December 9, 2013
ISU MECS Seminar