On the independence ratio of distance graphs

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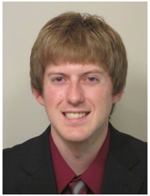

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Distance Graphs

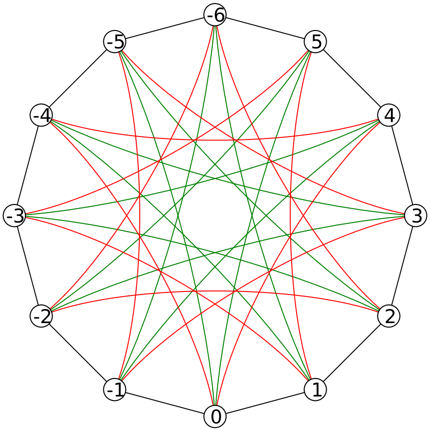
For a set S of positive integers, the **distance graph** $G(S)$ is the infinite graph with vertex set $\mathbb{Z}$ where

two integers i and j are adjacent if and only if $|i - j| \in S$.

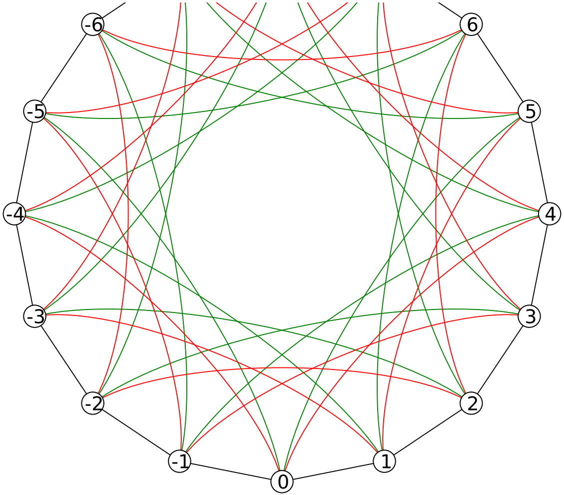
Circulant Graphs

For an integer n, the **circulant graph** $G(n, S)$ is the graph whose vertices are the integers modulo n where

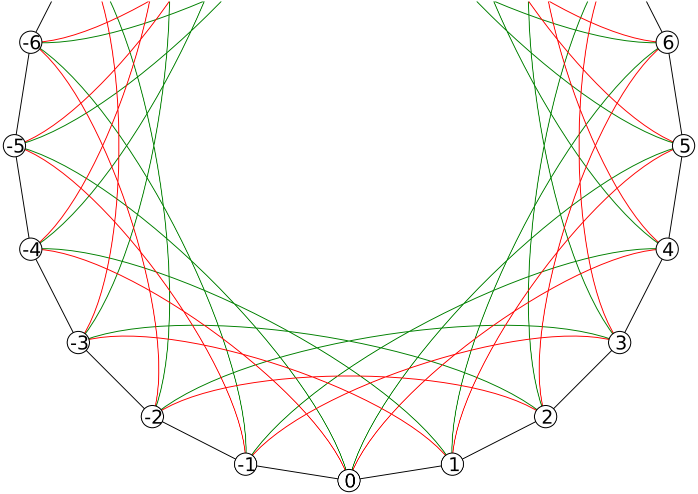
two integers i and j are adjacent if and only if $|i - j| \equiv k \pmod{n}$, for some $k \in S$.



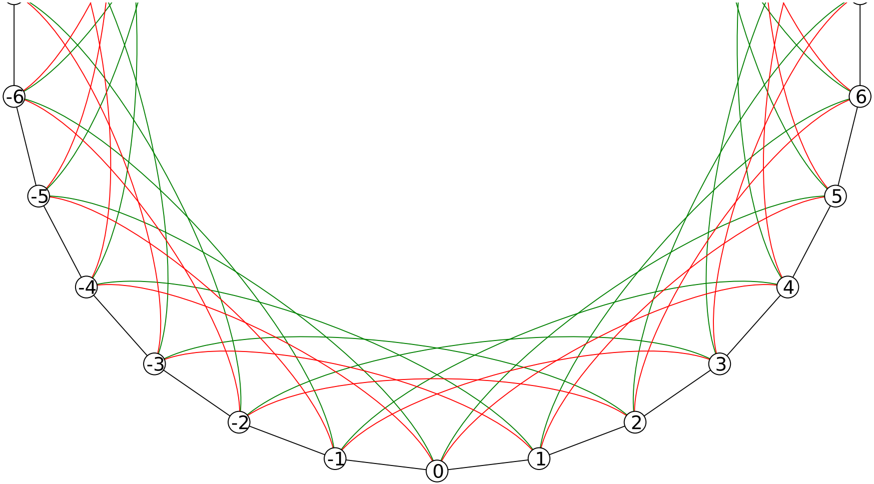
$G(12, \{1, 4, 5\})$



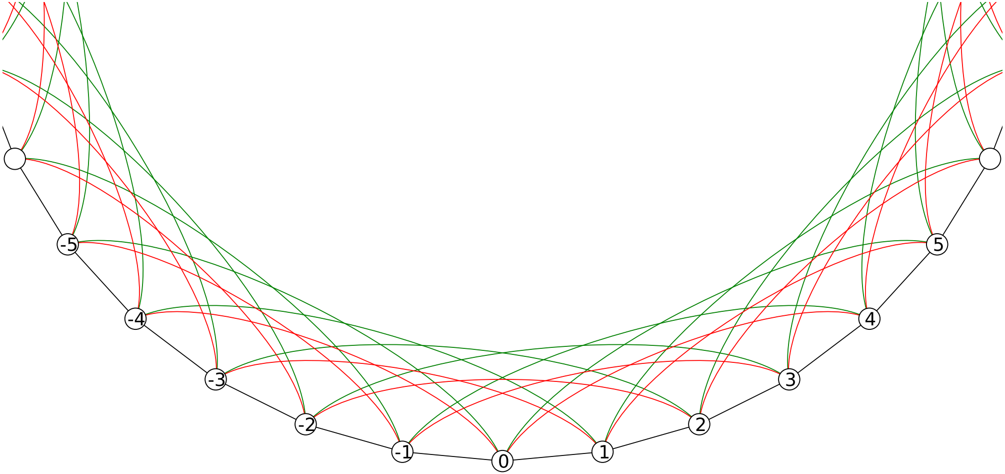
$G(18, \{1, 4, 5\})$



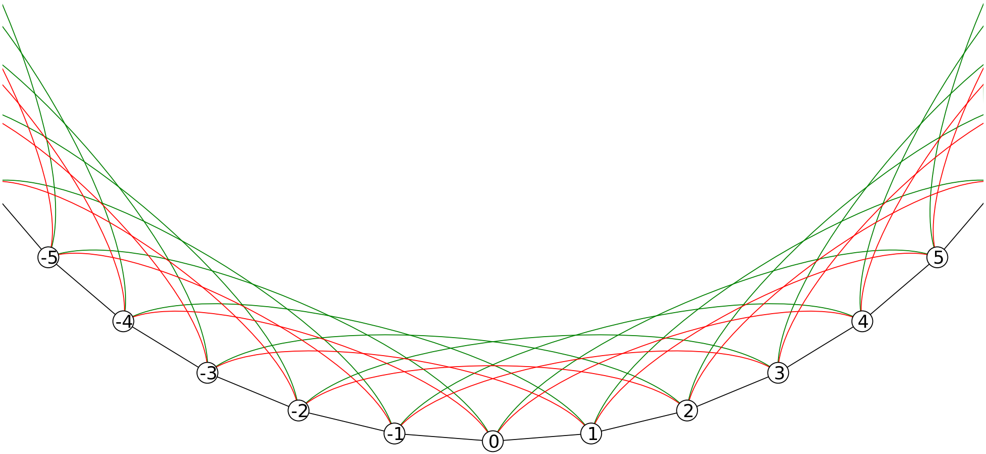
$G(28, \{1, 4, 5\})$



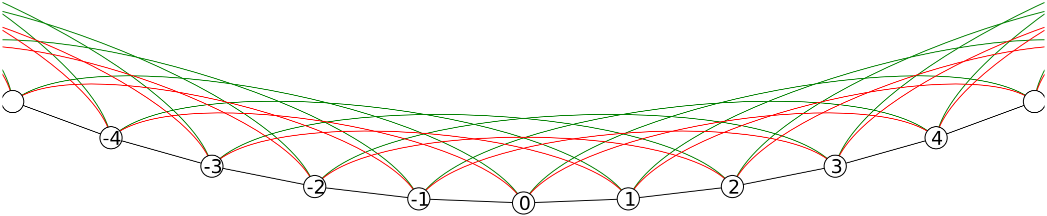
$G(40, \{1, 4, 5\})$



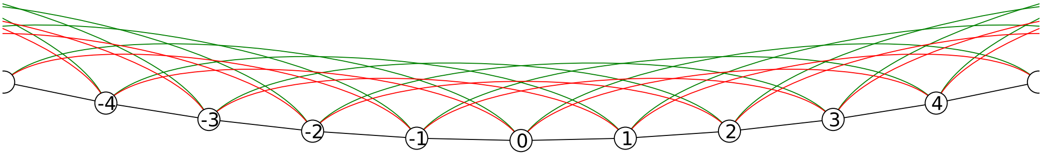
$G(60, \{1, 4, 5\})$



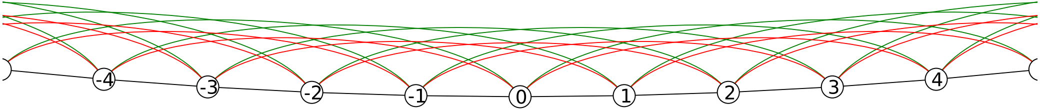
$G(128, \{1, 4, 5\})$



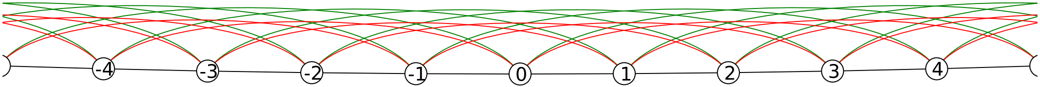
$G(256, \{1, 4, 5\})$



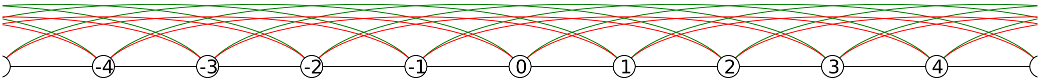
$G(512, \{1, 4, 5\})$



$G(1024, \{1, 4, 5\})$



$G(2048, \{1, 4, 5\})$



$$G(\infty, \{1, 4, 5\}) = G(\{1, 4, 5\})$$

Chromatic Number of Distance Graphs

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Fractional Chromatic Number

A **fractional coloring** of a graph G is a feasible solution to the following linear program:

$$\begin{array}{rlcll} \min & \sum_{I \in \mathcal{I}} c_I & & & \\ & \sum_{I \ni v} c_I & \geq & 1 & \forall v \in V(G) \\ & \quad c_I & \geq & 0 & \forall I \in \mathcal{I} \end{array}$$

where $\mathcal{I}$ is the collection of independent sets in G.

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where $\mathcal{I}$ is the collection of independent sets in G.

The **fractional chromatic number** $\chi_f(G)$ is the minimum value of a fractional coloring, and provides a lower bound on the chromatic number.

Fractional Chromatic Number of Distance Graphs

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Independence Ratio

For an independent set A in $G(S)$ the **density** $\delta(A)$ is equal to

$$\delta(A) = \limsup_{N\to\infty} \frac{|A \cap [-N, N]|}{2N+1}.$$

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Theorem (Lih, Liu, and Zhu, ’99) Let S be a finite set of positive integers.

$$\chi_f(G(S)) = \frac{1}{\overline{\alpha}(S)}.$$

Periodic Independent Sets to Fractional Colorings

Suppose $X \subset \mathbb{Z}$ is a **periodic** independent set in $G(S)$ with period p and density

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Assign $c_{X_i} = \frac{1}{d}$. Every vertex appears in d sets X_i, so $d \cdot \frac{1}{d} \geq 1$.

Then

$$\chi_f(G(S)) \leq \sum_{i=0}^{p-1} c_{X_i} = \frac{p}{d} = \frac{1}{\delta(X)}.$$

Periodic Independent Sets

Theorem (CGHRS, '14+) Let S be a finite set of positive integers and let $s = \max S$.

There exists a periodic independent set A in $G(S)$ with period at most $s2^s$ where $\delta(A) = \overline{\alpha}(S)$.

Cycle Lemma

Let G be a finite digraph with weights on the vertices.

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Lemma (Cycle Lemma) Let G be a finite, vertex-weighted digraph. The **supremum of upper average weights** of infinite walks on G is equal to the upper average weight of some infinite walk given by **repeating a simple cycle**.

Proof of Cycle Lemma

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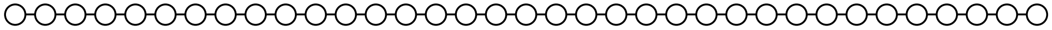

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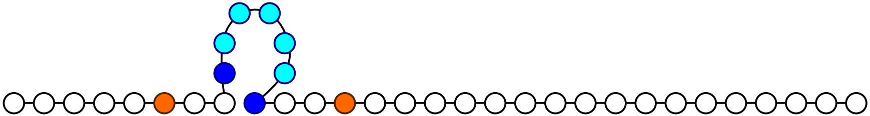

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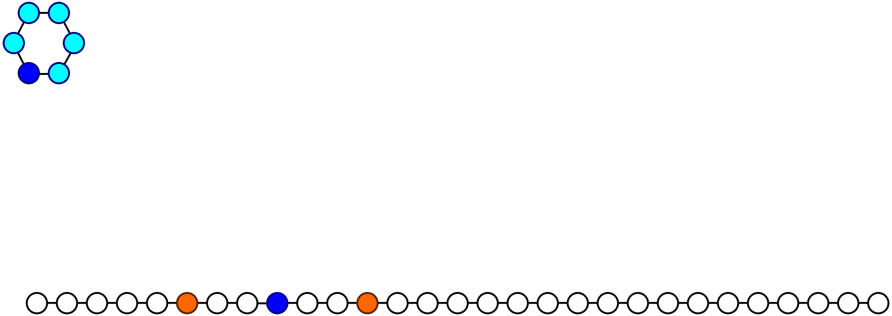

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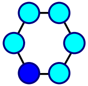

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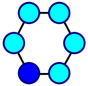

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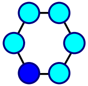

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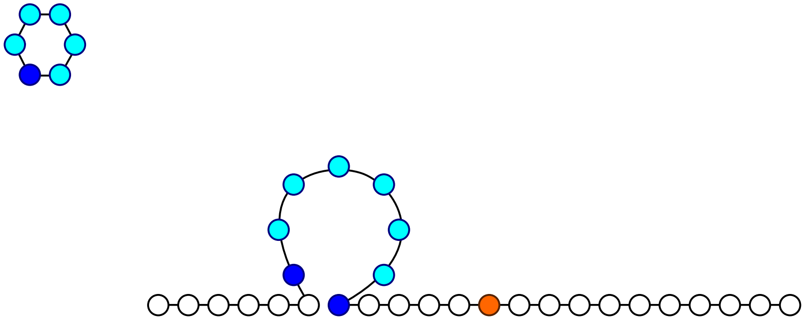

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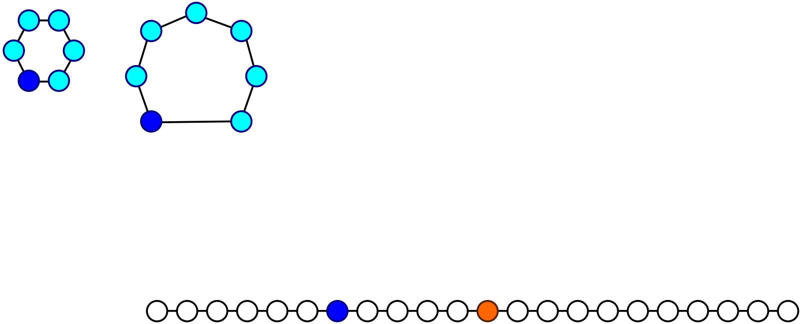

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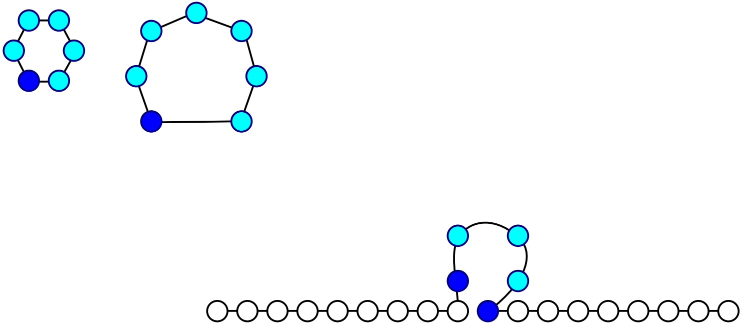

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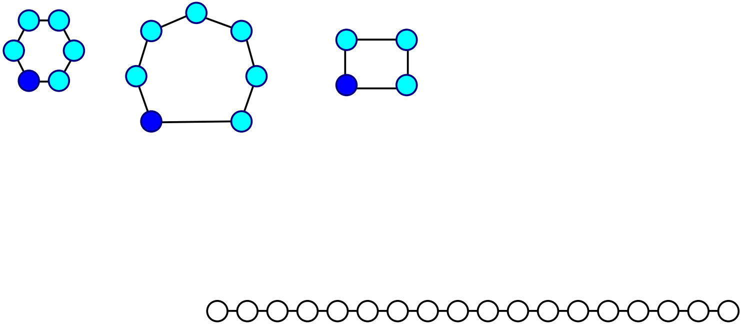

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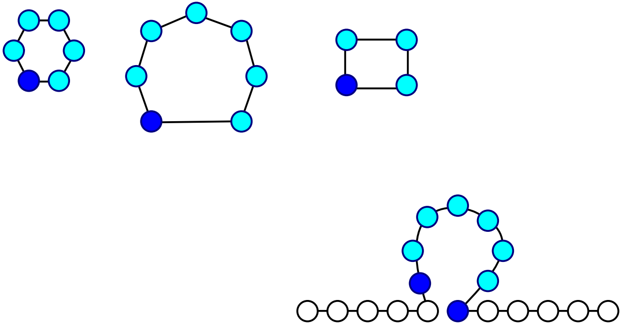

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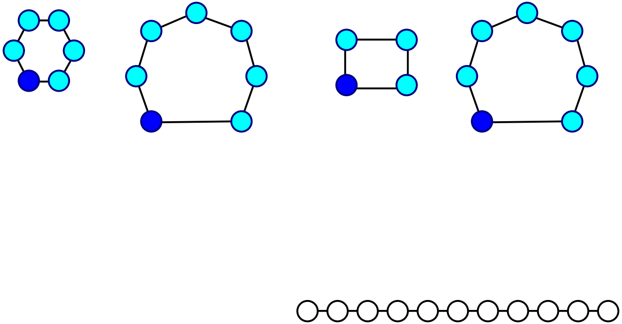

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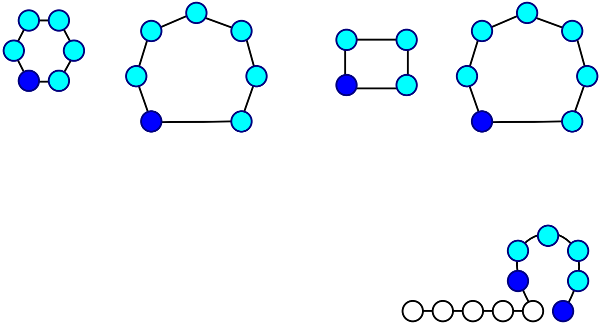

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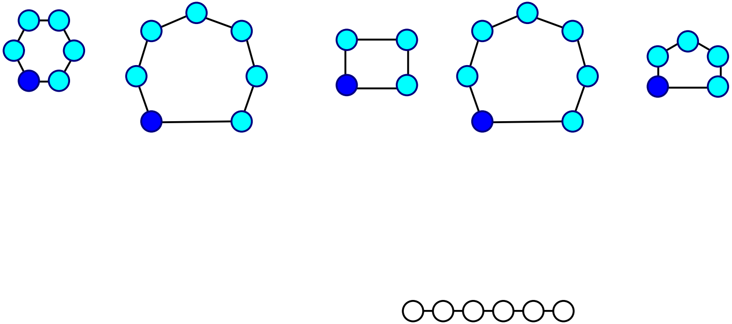

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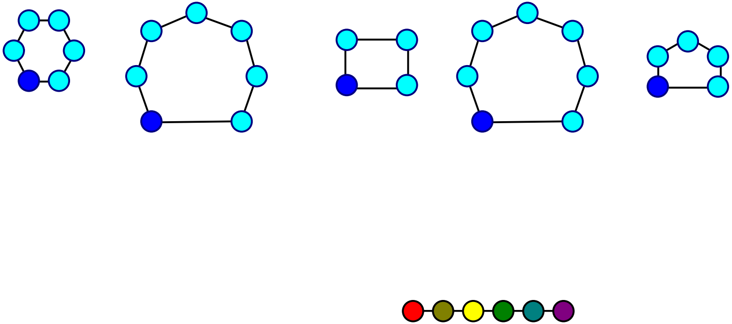

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There exist cycles $C_1, \ldots, C_t$ such that $\dfrac{\sum_{i=-N}^{N} w(x_i)}{2N+1}$ is closely approximated by a convex combination of $\overline{w}(C_1), \ldots, \overline{w}(C_t)$.

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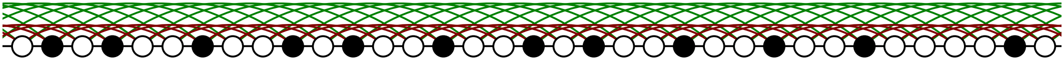
Thus, in the limit, $\overline{w}(W) \leq \max\{ \overline{w}(C) : C$ is a cycle in $G\}$.

Proof of Periodic Sets



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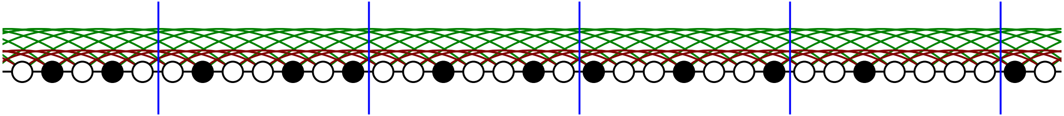

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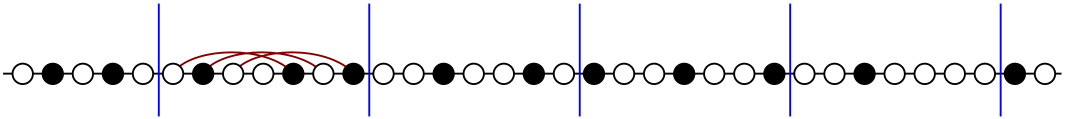

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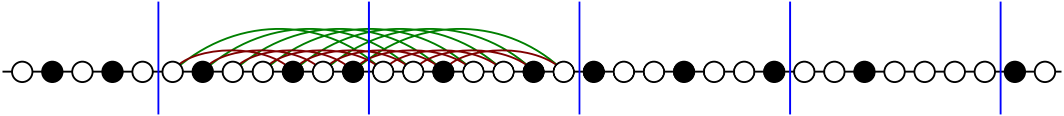

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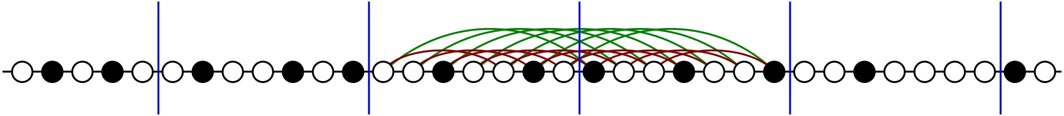

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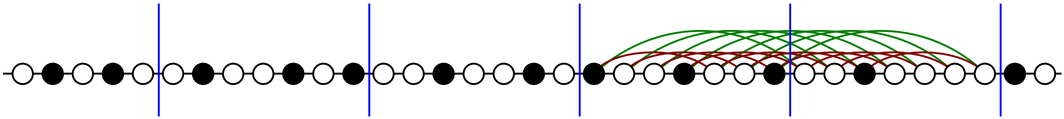

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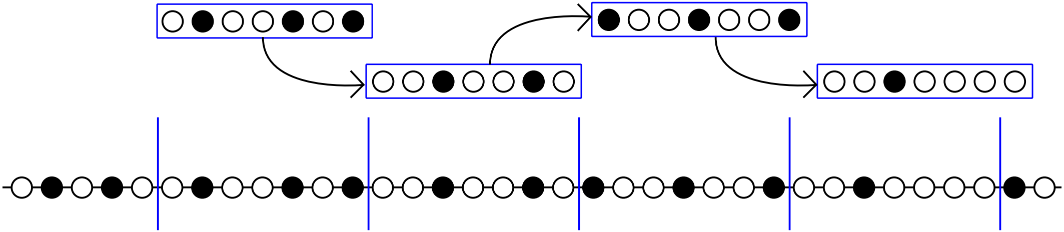

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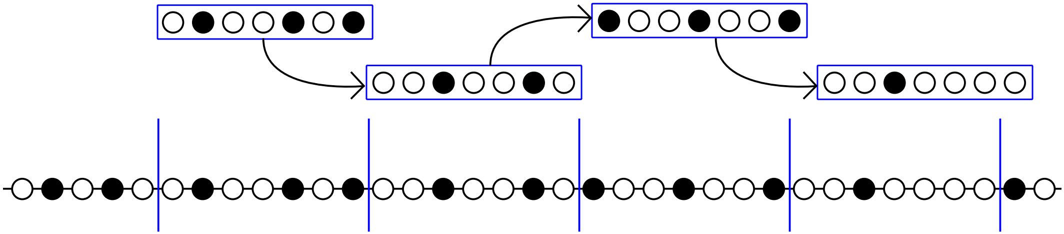

A **state** σ is a subset of $\{0, \ldots, s-1\}$ (there are 2^s such states).

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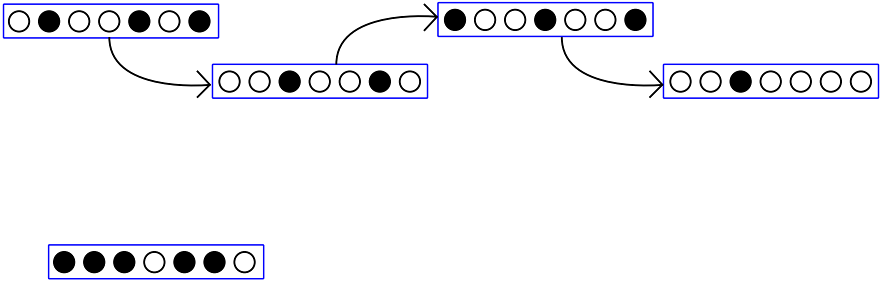

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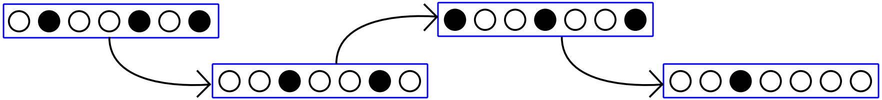

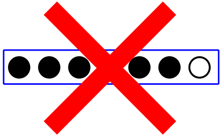

A state is **allowed** if σ is independent in $G(S)$.

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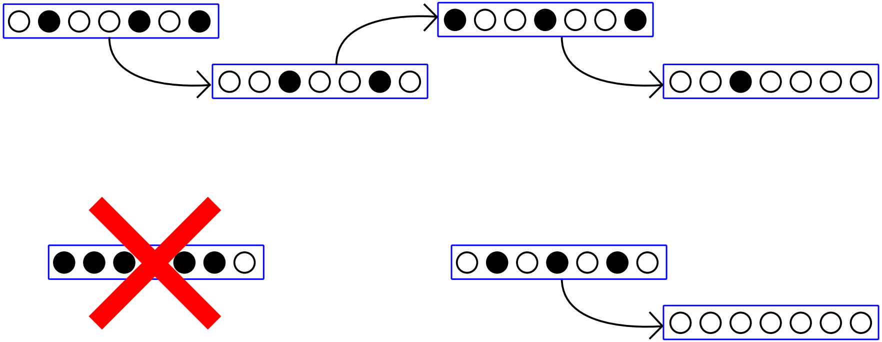

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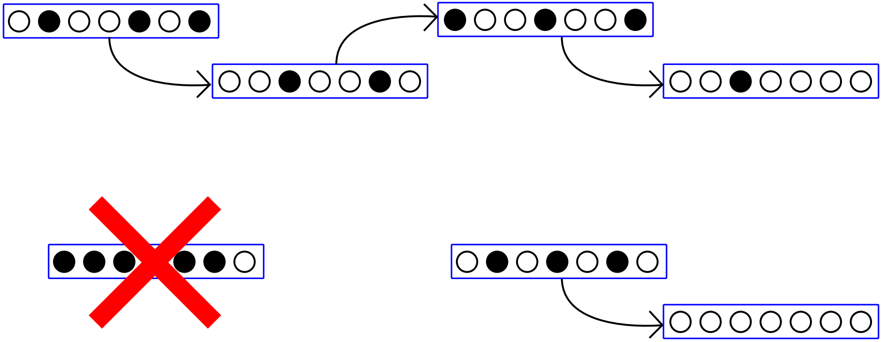

The **state diagram** of allowed states is a digraph where an ordered pair (σ_1, σ_2) of states be an edge if and only if $\sigma_1 \cup (s + \sigma_2)$ is independent in $G(S)$.

Proof of Periodic Sets

The **independent sets** X in $G(S)$ are in **bijection** with the **infinite walks** W in the state diagram, and the **density** of X equals the **average weight** of its corresponding walk, W_X.

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By the cycle lemma, the maximum density of an infinite walk in the state diagram is achieved by repeating a simple cycle C.

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By the cycle lemma, the maximum density of an infinite walk in the state diagram is achieved by repeating a simple cycle C. The independent set X_C given by those states has maximum density $\delta(X_C) = \overline{\alpha}(S)$.

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The length of C is at most 2^s, so the period of X_C is at most $s2^s$.

Other Periodic Sets

Let S be a finite set of positive integers and set $s = \max S$.

Theorem (CGHRS, '14+) The minimum density of a **dominating set** in $G(S)$ is achieved by a periodic set with period at most $(2s)2^{2s}$.

Theorem (CGHRS, '14+) The minimum density of a **1-identifying code** in $G(S)$ is achieved by a periodic set with period at most $(6s)2^{6s}$.

Corollary (CGHRS, '14+) The minimum density of an ***r*-identifying code** in $G(S)$ is achieved by a periodic set with period at most $(6sr)2^{6sr}$.

Theorem (Eggleton, Erdős, and Skilton, '90) For $k = \chi(G(S))$, there exists a periodic proper k-coloring c with minimum period at most sk^s.

Example Theorem ($S = \{1, 2, k\}$)

Theorem (Zhu, '02)

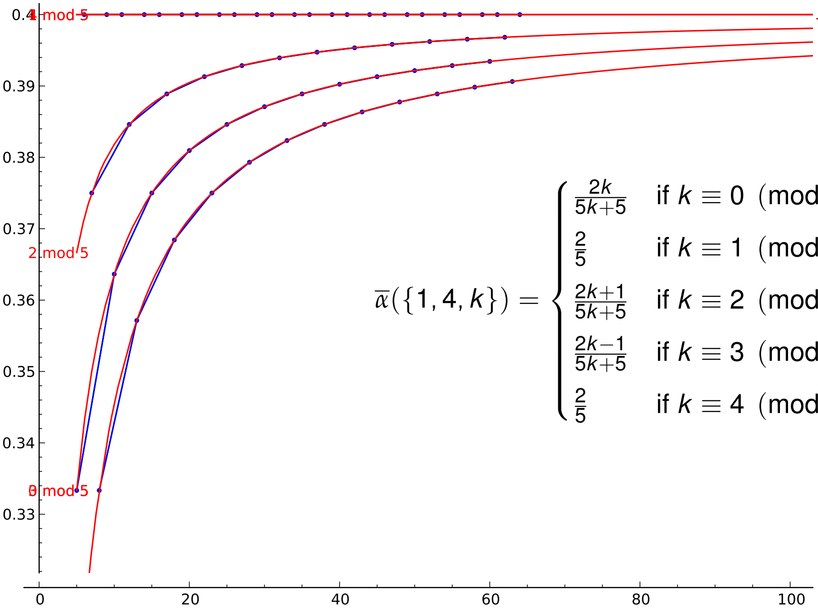
$$\overline{\alpha}(\{1, 2, k\}) = \begin{cases} \frac{k}{3k+3} & \text{if } k \equiv 0 \pmod 3 \\ \frac{1}{3} & \text{if } k \equiv 1 \pmod 3 \\ \frac{1}{3} & \text{if } k \equiv 2 \pmod 3. \end{cases}$$

Example Theorem ($S = \{1, 4, k\}$)

Theorem (CGHRS, '14+) For $k > 4$,

$$\overline{\alpha}(\{1,4,k\}) = \begin{cases} \frac{2k}{5k+5} & \text{if } k \equiv 0 \pmod 5 \\ \frac{2}{5} & \text{if } k \equiv 1 \pmod 5 \\ \frac{2k+1}{5k+5} & \text{if } k \equiv 2 \pmod 5 \\ \frac{2k-1}{5k+5} & \text{if } k \equiv 3 \pmod 5 \\ \frac{2}{5} & \text{if } k \equiv 4 \pmod 5. \end{cases}$$

Example Theorem ($S = \{1, 4, k\}$)



$\longrightarrow \frac{2}{5}$

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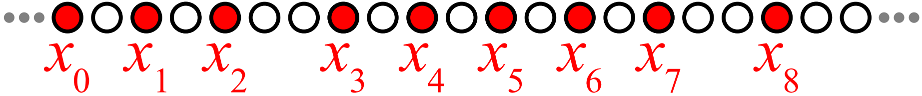
Discharging

Suppose $X = \{x_i : i \in \mathbb{Z}\} \subseteq \mathbb{Z}$ is an infinite independent set in $G(S)$.

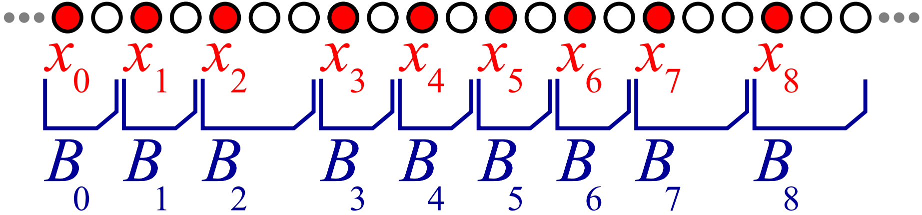
Discharging

Elements are labeled $\ldots, x_{-1}, x_0, x_1, \ldots, x_i, \ldots$.



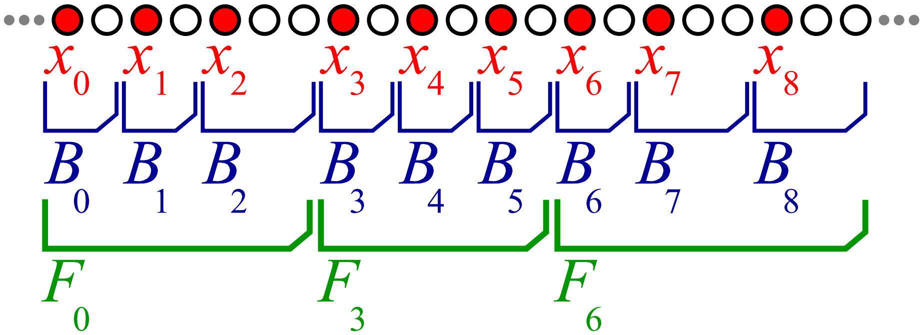
Discharging

Blocks are sets $B_k = \{x_k, x_k + 1, \ldots, x_{k+1} - 1\}$.
("Intervals" closed on element x_k and open on x_{k+1})



Discharging

Frames are collections $F_j = \{B_j, B_{j+1}, \ldots, B_{j+t-1}\}$.
(There are t blocks in each frame.)



Local Discharging Lemma

Let a, b, c, t be nonnegative integers. Let X be a periodic independent set in $G(S)$.

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$$
\begin{tikzcd}[column sep=large]
\textbf{Stage 1: } \textit{Blocks} & \mu(B_j) = a|B_j| - b \arrow[r, "\text{discharge}"] & \mu^*(B_j) \arrow[d, "\text{defines}"] & \\
\textbf{Stage 2: } \textit{Frames} & & \nu^*(F_j) \arrow[r, "\text{discharge}"] & \nu'(F_j) \geq c
\end{tikzcd}
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Local Discharging Lemma

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Stage 1: *Blocks*

Stage 2: *Frames*

$$\begin{tikzcd} \mu(B_j) = a|B_j| - b \arrow[r, "\text{discharge}"] & \mu^*(B_j) \arrow[d, "\text{defines}"] & \\ & \nu^*(F_j) \arrow[r, "\text{discharge}"] & \nu'(F_j) \geq c \end{tikzcd}$$

If $\nu'(F_j) \geq c$ for all frames, then

$$\delta(X) \leq \frac{at}{bt + c}.$$

Example Discharging Argument ($S = \{1, 4, k\}$)

Theorem (CGHRS, ’14+) For $k > 4$,

$$\overline{\alpha}(\{1,4,k\}) = \begin{cases} \frac{2k}{5k+5} & \text{if } k \equiv 0 \pmod 5 \\ \frac{2}{5} & \text{if } k \equiv 1 \pmod 5 \\ \frac{2k+1}{5k+5} & \text{if } k \equiv 2 \pmod 5 \\ \frac{2k-1}{5k+5} & \text{if } k \equiv 3 \pmod 5 \\ \frac{2}{5} & \text{if } k \equiv 4 \pmod 5. \end{cases}$$

Always, let $a = 2$ and $b = 5$.

Residue class	t	c	Extremal Set
$k{=}5i$	$t = 2i$	$c = 2$	$(2\ 3)^{i-1}\ 3^2$
$k{=}5i+1$	$t = 1$	$c = 0$	$2\ 3$
$k{=}5i+2$	$t = 2i+1$	$c = 1$	$(2\ 3)^i\ 3$
$k{=}5i+3$	$t = 2i+1$	$c = 3$	$(2\ 3)^{i-1}\ 3^3$
$k{=}5i+4$	$t = 1$	$c = 0$	$2\ 3$

Block Size	μ-charge	μ^*-charge
2	-1	0
3	1	0, 1
5	5	4, 5
6	7	6, 7

Example Discharging Argument ($S = \{1, 4, k\}$)

For all cases, let $a = 2$ and $b = 5$.

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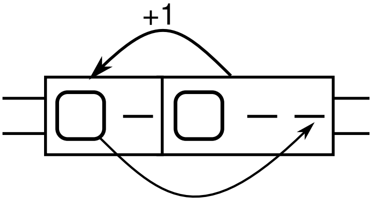
$k{=}5i+4$	$t = 1$	$c = 0$	$2\ 3$

Block Size	μ-charge	μ^*-charge
2	-1	0
3	1	0, 1
5	5	4, 5
6	7	6, 7

(S1) Every 2-block B_j pulls one unit of charge from B_{j+1}.



Example Discharging Argument ($S = \{1, 4, k\}$)

Case $k = 5i + 3$:

Let $t = 2i + 1$ and $c = 3$. Thus $\frac{at}{bt+c} = \frac{4i+2}{10i+8} = \frac{2i+1}{5i+4} = \frac{2k-1}{5k+5}$.

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(S2) If $\sigma(F_j) = \sum_{B_\ell \in F_j} |B_\ell| = 5i + 2$, then F_j pulls 1 unit of charge from each of F_{j+1}, F_{j+2}, and F_{j+3}.

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It remains to show that:

1. if $\nu^*(F_j) < 3 = c$, then $\sigma(F_j) = 5i + 2$ and F_j pulls charge by (S2).

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It remains to show that:

1. if $\nu^*(F_j) < 3 = c$, then $\sigma(F_j) = 5i + 2$ and F_j pulls charge by (S2).

2. if F_j loses charge by (S2), then F_j contains a 5-block and $\nu^*(F_j) \geq 4$.

Densities for $S = \{1, 1+k, 1+k+i\}$

k \ i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
1	1/4	1/3	1/3	2/7	1/3	1/3	3/10	1/3	1/3	4/13	1/3	1/3	5/16	1/3	1/3	6/19	1/3	1/3	7/22	1/3
2	2/7		1/3		4/11		5/13		2/5		7/17		8/19		3/7		10/23		11/25	
3	1/3	2/5	3/8	1/3	2/5	4/11	2/5	5/13	5/14	2/5	3/8	2/5	7/18	7/19	2/5	8/21	2/5	9/23	3/8	2/5
4	2/7		1/3		1/3		6/17		7/19		8/21		9/23		2/5		11/27		12/29	
5	4/13	3/7	2/5	3/8	5/12	1/3	3/7	2/5	3/7	7/17	9/23	8/19	2/5	3/7	9/22	3/7	5/12	2/5	11/26	11/27
6	1/3		4/11		3/8		1/3		6/17		7/19		3/8		11/29		12/31		13/33	
7	3/10	4/9	7/19	2/5	3/7	4/11	7/16	1/3	4/9	5/13	4/9	3/7	12/29	10/23	12/31	11/25	13/33	4/9	3/7	4/9
8	6/19		5/13		2/5		2/5		1/3		8/21		9/23		2/5		2/5		11/29	
9	1/3	5/11	5/14	5/12	2/5	5/13	4/9	4/11	9/20	1/3	5/11	3/8	5/11	7/17	3/7	4/9	15/37	13/29	5/13	14/31
10	4/13		2/5		7/17		5/12		12/31		1/3		2/5		11/27		12/29		5/12	
11	8/25	6/13	3/8	3/7	5/13	2/5	13/31	8/21	5/11	5/13	11/24	1/3	6/13	10/27	6/13	2/5	18/41	3/7	18/43	5/11
12	1/3		11/29		8/19		3/7		3/7		14/37		1/3		16/41		13/31		14/33	
13	5/16	7/15	7/18	7/16	2/5	7/17	2/5	9/23	16/37	2/5	6/13	2/5	13/28	1/3	7/15	12/31	7/15	13/33	21/47	5/12
14	10/31		7/19		3/7		10/23		7/16		17/41		3/8		1/3		18/47		3/7	
15	1/3	8/17	2/5	4/9	9/22	8/19	7/17	2/5	17/41	11/27	19/43	7/17	7/15	9/23	15/32	1/3	8/17	2/5	8/17	15/37
16	6/19		8/21		16/39		11/25		4/9		4/9		19/47		7/18		1/3		14/37	
17	12/37	9/19	5/13	9/20	5/12	3/7	8/19	9/22	2/5	12/29	20/47	13/31	22/49	8/19	8/17	5/13	17/36	1/3	9/19	11/28
18	1/3		9/23		2/5		4/9		13/29		9/20		22/51		2/5		2/5		1/3	
19	7/22	10/21	3/8	5/11	11/26	10/23	3/7	5/12	3/7	2/5	7/17	14/33	23/53	3/7	5/11	3/7	9/19	8/21	19/40	1/3
20	14/43		2/5		11/27		3/7		14/31		5/11		5/11		8/19		9/22		24/61	
21	1/3	11/23	5/13	11/24	3/7	11/25	13/30	11/26	10/23	11/27	2/5	3/7	8/19	16/37	26/59	10/23	28/61	13/31	10/21	9/23
22	8/25		19/49		12/29		5/12		5/11		16/35		11/24		27/61		17/41		5/12	
23	16/49	12/25	11/28	6/13	22/53	4/9	7/16	3/7	11/25	12/29	25/59	2/5	25/61	17/39	3/7	18/41	29/65	11/25	31/67	7/17
24	1/3		11/29		13/31		11/26		26/59		17/37		6/13		6/13		29/67		19/45	
25	9/28	13/27	2/5	13/28	13/32	13/29	15/34	13/30	4/9	13/31	4/9	11/27	2/5	14/33	28/67	19/43	10/23	4/9	32/71	4/9
26	18/55		12/31		14/33		3/7		3/7		6/13		19/41		13/28		32/71		20/47	
27	1/3	14/29	23/59	7/15	7/17	14/31	4/9	7/16	17/38	14/33	13/29	12/29	29/69	2/5	9/22	4/9	31/73	21/47	11/25	13/29
28	10/31		13/33		3/7		16/37		13/30		31/69		20/43		7/15		7/15		34/77	
29	20/61	15/31	13/34	15/32	5/12	5/11	29/67	15/34	9/20	3/7	14/31	18/43	32/73	13/31	2/5	8/19	5/12	22/49	34/79	23/51
30	1/3		2/5		28/67		17/39		7/16		32/73		7/15		22/47		15/32		37/81	
31	11/34	16/33	7/18	8/17	8/19	16/35	14/33	4/9	19/42	16/37	5/11	19/45	5/11	14/33	33/79	2/5	21/50	18/41	11/26	24/53
32	22/67		9/23		16/39		18/41		15/34		3/7		36/79		23/49		8/17		8/17	
33	1/3	17/35	15/38	17/36	17/40	17/37	3/7	17/38	5/11	17/39	21/46	20/47	16/35	3/7	36/83	3/7	2/5	18/43	23/54	5/11
34	12/37		5/13		17/41		19/43		4/9		4/9		37/83		8/17		25/53		17/36	
35	24/73	18/37	2/5	9/19	3/7	6/13	19/44	9/20	4/9	18/41	11/24	3/7	17/37	22/51	13/29	16/37	37/89	2/5	3/7	10/23
36	1/3		16/41		18/43		4/9		21/47		17/38		38/87		41/89		26/55		9/19	
37	13/40	19/39	31/79	19/40	34/81	19/41	10/23	19/42	17/39	19/43	23/50	19/44	6/13	23/53	6/13	17/39	40/93	20/47	2/5	5/12
38	26/79		17/43		19/45		37/85		22/49		9/20		3/7		14/31		9/19		28/59	
39	1/3	20/41	17/44	10/21	19/46	20/43	7/16	5/11	18/41	4/9	6/13	10/23	25/54	24/55	19/41	25/57	43/97	18/41	17/41	2/5
40	14/43		2/5		20/47		3/7		23/51		19/42		42/95		43/97		46/99		29/61	
41	28/85	21/43	9/23	21/44	5/12	7/15	11/25	21/46	19/43	21/47	43/95	7/16	13/28	3/7	20/43	26/59	41/90	19/43	41/96	11/26
42	1/3		35/89		3/7		22/51		24/53		5/11		5/11		27/62		26/57		10/21	
43	15/46	22/45	19/48	11/23	21/50	22/47	23/52	11/24	4/9	22/49	4/9	11/25	27/58	22/51	7/15	27/61	7/15	4/9	29/66	4/9
44	30/91		19/49		8/19		23/53		5/11		26/57		21/46		3/7		35/78		29/62	
45	1/3	23/47	2/5	23/48	11/26	23/49	4/9	23/50	25/56	23/51	21/47	23/52	7/15	23/53	29/62	4/9	22/47	29/65	41/91	21/47
46	16/49		20/51		22/53		24/55		25/56		27/59		11/24		37/84		34/77		29/63	
47	8/25	24/49	13/33	12/25	23/54	8/17	38/87	6/13	13/29	24/53	22/49	4/9	39/85	24/55	15/32	3/7	23/49	30/67	41/89	22/49
48	1/3		21/53		23/55		25/57		11/25		28/61		23/50		37/82		10/23		29/64	
49	17/52	25/51	7/18	25/52	3/7	25/53	25/58	25/54	9/20	5/11	23/51	25/56	37/82	25/57	31/66	22/51	8/17	31/69	8/17	32/71
50	17/53		2/5		8/19		26/59		23/52		29/63		6/13		6/13		30/71		29/65	

k \ i	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
1	1/3	8/25	1/3	1/3	9/28	1/3	1/3	10/31	1/3	1/3	11/34	1/3	1/3	12/37	1/3	1/3	13/40	1/3	1/3	14/43
2	4/9		13/29		14/31		5/11		16/35		17/37		6/13		19/41		20/43		7/15	
3	5/13	2/5	11/28	11/29	2/5	12/31	2/5	13/33	13/34	2/5	7/18	2/5	15/38	5/13	2/5	16/41	2/5	17/43	17/44	2/5
4	13/31		14/33		3/7		16/37		17/39		18/41		19/43		4/9		21/47		22/49	
5	3/7	12/29	3/7	13/31	13/32	14/33	7/17	3/7	5/12	3/7	8/19	16/39	17/40	17/41	3/7	18/43	3/7	19/45	19/46	20/47
6	2/5		15/37		16/39		17/41		18/43		19/45		20/47		3/7		22/51		23/53	
7	13/30	8/19	7/16	2/5	15/34	3/7	4/9	16/37	4/9	17/39	20/47	18/41	3/7	19/43	19/44	4/9	10/23	4/9	7/16	3/7
8	12/31		13/33		2/5		2/5		19/47		20/49		7/17		22/53		23/55		8/19	
9	16/41	5/11	18/43	5/11	4/9	10/23	17/38	5/12	9/20	2/5	19/42	11/26	5/11	4/9	5/11	21/47	25/57	22/49	25/59	23/51
10	17/43		2/5		15/37		16/39		17/41		5/12		2/5		19/47		20/49		7/17	
11	2/5	16/35	5/13	17/37	19/49	6/13	7/17	6/13	23/53	4/9	5/11	3/7	21/46	12/29	11/24	2/5	23/50	13/31	6/13	7/16
12	3/7		3/7		20/51		21/53		18/43		19/45		20/47		3/7		3/7		2/5	
13	3/7	11/25	7/17	6/13	2/5	19/41	2/5	20/43	17/44	7/15	24/59	7/15	26/61	14/31	4/9	7/16	6/13	14/33	25/54	7/17
14	16/37		17/39		7/16		8/19		23/59		24/61		3/7		22/51		23/53		24/55	
15	24/53	16/39	24/55	3/7	8/19	13/29	9/22	7/15	7/17	22/47	25/63	23/49	2/5	8/17	21/52	8/17	29/69	16/35	31/71	4/9
16	23/55		18/41		19/43		4/9		4/9		27/65		7/18		9/23		30/71		25/57	
17	9/19	17/41	27/59	18/43	27/61	8/19	3/7	7/16	5/12	5/11	21/50	8/17	8/19	25/53	28/71	26/55	29/73	9/19	12/29	9/19
18	16/41		25/61		4/9		21/47		22/49		9/20		31/71		30/73		2/5		23/59	
19	10/21	12/31	10/21	19/45	6/13	20/47	30/67	3/7	10/23	3/7	11/26	4/9	23/54	17/37	3/7	9/19	3/7	28/59	31/79	29/61
20	1/3		2/5		19/47		10/23		23/51		24/53		5/11		5/11		34/79		9/22	
21	21/44	1/3	11/23	18/47	11/23	3/7	33/71	22/51	33/73	23/53	11/25	10/23	3/7	17/39	25/58	9/20	13/30	19/41	10/23	10/21
22	26/67		1/3		28/71		7/17		32/75		5/11		26/57		27/59		11/24		38/85	
23	11/23	2/5	23/48	1/3	12/25	20/51	12/25	8/19	36/77	24/55	36/79	25/57	4/9	11/25	36/83	3/7	27/62	19/43	7/16	5/11
24	11/26		5/13		1/3		30/77		23/55		8/19		37/83		28/61		29/63		6/13	
25	34/73	11/27	12/25	15/38	25/52	1/3	13/27	2/5	13/27	17/41	39/83	26/59	39/85	27/61	13/29	4/9	39/89	4/9	3/7	10/23
26	3/7		3/7		11/28		22/67		22/57		25/59		26/61		39/89		6/13		31/67	
27	5/11	17/39	37/79	12/29	13/27	16/41	27/56	1/3	14/29	17/43	14/29	25/61	42/89	4/9	6/13	29/65	14/31	30/67	42/95	13/29
28	22/51		13/30		35/83		2/5		1/3		24/61		3/7		28/65		29/67		44/97	
29	4/9	14/31	38/83	3/7	8/17	13/31	14/29	12/31	29/60	1/99	15/31	9/23	15/31	27/65	9/19	7/16	45/97	31/69	5/11	32/71
30	23/53		24/55		7/16		37/89		36/91		1/99		2/5		41/97		10/23		31/71	
31	37/85	5/11	13/29	5/11	41/89	25/59	43/91	14/33	15/31	13/33	31/64	1/99	16/33	26/67	16/33	29/69	29/61	22/51	41/88	33/73
32	13/29		25/57		26/59		15/34		7/17		38/97		1/99		9/23		38/91		32/73	
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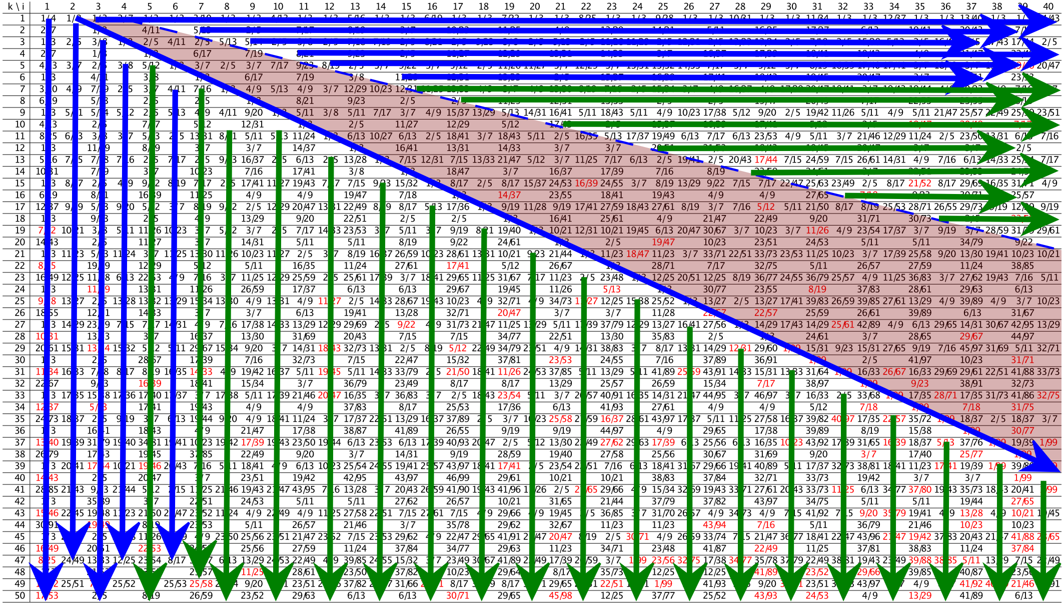
34	6/13		41/93		27/61		4/9		4/9		5/12		7/18		1/99		7/18		31/75	
35	25/58	27/59	16/37	28/61	43/97	17/37	5/11	11/25	27/58	16/37	39/82	40/97	17/35	22/57	35/72	1/99	18/37	2/5	18/37	3/7
36	9/19		44/97		4/9		29/65		17/38		39/89		8/19		15/38		1/99		30/77	
37	13/30	22/49	27/62	29/63	17/39	6/13	25/56	6/13	16/35	10/23	43/92	17/39	31/65	16/39	18/37	5/13	37/76	1/99	19/39	1/99
38	19/40		27/58		25/56		30/67		31/69		9/20		3/7		17/40		25/77		1/99	
39	23/54	22/51	7/16	6/13	18/41	31/67	29/66	19/41	40/89	5/11	17/37	32/73	38/81	18/41	11/23	17/41	19/39	1/99	39/80	1/99
40	10/21		10/21		38/83		37/84		32/71		33/73		19/42		3/7		3/7		1/99	
41	2/5	27/65	29/66	4/9	15/34	32/69	19/43	33/71	27/61	20/43	33/73	11/25	6/13	34/77	37/80	19/43	35/73	18/43	20/41	1/99
42	31/65		21/44		37/79		37/82		43/97		34/75		5/11		5/11		19/44		27/65	
43	19/45	2/5	36/85	3/7	31/70	26/57	4/9	34/73	4/9	7/15	41/92	7/15	9/20	35/79	19/41	4/9	13/28	4/9	10/21	19/45
44	32/67		11/23		11/23		43/94		7/16		5/11		36/79		21/46		10/23		10/23	
45	20/47	8/19	2/5	30/71	4/9	26/59	33/74	7/15	21/47	36/77	18/41	22/47	43/96	21/47	19/42	37/83	20/43	21/47	41/88	28/65
46	11/23		34/71		23/48		41/87		22/49		11/25		37/81		38/83		11/24		37/84	
47	17/39	26/59	3/7	1/99	23/56	32/75	17/38	34/77	35/78	37/79	22/49	38/81	19/43	23/49	39/88	38/85	5/11	13/29	7/15	22/49
48	8/17		35/73		12/25		12/25		41/89		23/52		29/66		39/85		40/87		23/50	
49	29/65	23/51	22/51	13/31	1/99	3/7	41/93	7/16	9/20	37/81	23/51	39/83	43/97	8/17	4/9	8/17	41/92	40/89	21/46	41/91
50	45/98		12/25		37/77		25/52		43/93		24/53		4/9		13/29		41/89		6/13	

Densities for $S = \{1, 1+k, 1+k+i\}$

k \ i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
1	1/4	1/3	1/3	2/7	1/3	1/3	3/10	1/3	1/3	4/13	1/3	1/3	5/16	1/3	1/3	6/19	1/3	1/3	7/22	1/3	1/3	8/25	1/3	1/3	9/28	1/3	1/3	10/31	1/3	1/3	11/34	1/3	1/3	12/37	1/3	1/3	13/40	1/3	1/3	14/43
2	2/7		1/3		4/11		5/13		2/5		7/17		8/19		3/7		10/23		11/25		4/9		13/29		14/31		5/11		16/35		17/37		6/13		19/41		20/43		7/15	
3	1/3	2/5	3/8	1/3	2/5	4/11	2/5	5/13	5/14	2/5	3/8	2/5	7/18	7/19	2/5	8/21	2/5	9/23	3/8	2/5	5/13	2/5	11/28	11/29	2/5	12/31	2/5	13/33	13/34	2/5	7/18	2/5	15/38	5/13	2/5	16/41	2/5	17/43	17/44	2/5
4	2/7		1/3		1/3		6/17		7/19		8/21		9/23		2/5		11/27		12/29		13/31		14/33		3/7		16/37		17/39		18/41		19/43		4/9		21/47		22/49	
5	4/13	3/7	2/5	3/8	5/12	1/3	3/7	2/5	3/7	7/17	9/23	8/19	2/5	3/7	9/22	3/7	5/12	2/5	11/26	11/27	3/7	12/29	3/7	13/31	13/32	14/33	7/17	3/7	5/12	3/7	8/19	16/39	17/40	17/41	3/7	18/43	3/7	[illegible]	[illegible]	20/47
6	1/3		4/11		3/8		1/3		6/17		7/19		3/8		11/28		12/31		13/33		2/5		15/37		16/39		17/41		18/43		19/45		20/47		3/7		22/51		23/53	
7	3/10	4/9	7/19	2/5	3/7	4/11	7/16	1/3	4/9	5/13	4/9	3/7	12/29	10/23	12/31	11/25	13/33	4/9	3/7	4/9	13/30	8/19	7/16	2/5	15/34	3/7	4/9	16/37	4/9	17/39	20/47	18/41	3/7	19/43	19/44	4/9	10/23	4/9	7/16	3/7
8	6/19		5/13		2/5		2/5		1/3		8/21		9/23		2/5		2/5		11/29		12/31		13/33		2/5		2/5		19/47		20/49		7/17		22/53		23/55		8/19	
9	1/3	5/11	5/14	5/12	2/5	5/13	4/9	4/11	9/20	1/3	5/11	3/8	5/11	7/17	3/7	4/9	15/37	13/29	5/13	14/31	16/41	5/11	18/43	5/11	4/9	10/23	17/38	5/12	9/20	2/5	19/42	11/26	5/11	4/9	5/11	21/47	25/57	22/49	25/59	23/51
10	4/13		2/5		7/17		5/12		12/31		1/3		2/5		11/27		12/29		5/12		17/43		2/5		15/37		16/39		17/41		5/12		2/5		19/47		20/49		7/17	
11	8/25	6/13	3/8	3/7	5/13	2/5	13/31	8/21	5/11	5/13	11/24	1/3	6/13	10/27	6/13	2/5	18/41	3/7	18/43	5/11	2/5	16/35	5/13	17/37	19/49	6/13	7/17	6/13	23/53	4/9	5/11	3/7	21/46	12/29	11/24	2/5	23/50	13/31	6/13	7/16
12	1/3		11/29		8/19		3/7		3/7		14/37		1/3		16/41		13/31		14/33		3/7		3/7		20/51		21/53		18/43		19/45		20/47		3/7		3/7		2/5	
13	5/16	7/15	7/18	7/16	2/5	7/17	2/5	9/23	16/37	2/5	6/13	2/5	13/28	1/3	7/15	12/31	7/15	13/33	21/47	5/12	3/7	11/25	7/17	6/13	2/5	19/41	2/5	20/43	17/44	7/15	24/59	7/15	26/61	14/31	4/9	7/16	6/13	14/33	25/54	7/17
14	10/31		7/19		3/7		10/23		7/16		17/41		3/8		1/3		18/47		3/7		16/37		17/39		7/16		8/19		23/59		24/61		3/7		22/51		23/53		24/55	
15	1/3	8/17	2/5	4/9	9/22	8/19	7/17	2/5	17/41	11/27	19/43	7/17	7/15	9/23	15/32	1/3	8/17	2/5	8/17	15/37	24/53	16/39	24/55	3/7	8/19	13/29	9/22	7/15	7/17	22/47	25/63	23/49	2/5	8/17	21/52	8/17	29/69	16/35	31/71	4/9
16	6/19		8/21		16/39		11/25		4/9		4/9		19/47		7/18		1/3		14/37		23/55		18/41		19/43		4/9		4/9		27/65		7/18		9/23		30/71		25/57	
17	12/37	9/19	5/13	9/20	5/12	3/7	8/19	9/22	2/5	12/29	20/47	13/31	22/49	8/19	8/17	5/13	17/36	1/3	9/19	11/28	9/19	17/41	27/59	18/43	27/61	8/19	3/7	7/16	5/12	5/11	21/50	8/17	8/19	25/53	28/71	26/55	29/73	9/19	12/29	9/19
18	1/3		9/23		2/5		4/9		13/29		9/20		22/51		2/5		2/5		1/3		16/41		25/61		4/9		21/47		22/49		9/20		31/71		30/73		2/5		23/59	
19	7/22	10/21	3/8	5/11	11/26	10/23	3/7	5/12	3/7	2/5	7/17	14/33	23/53	3/7	5/11	3/7	9/19	8/21	19/40	1/3	10/21	12/31	10/21	19/45	6/13	20/47	30/67	3/7	10/23	3/7	11/26	4/9	23/54	17/37	3/7	9/19	3/7	28/59	31/79	29/61
20	14/43		2/5		11/27		3/7		14/31		5/11		5/11		8/19		9/22		24/61		1/3		2/5		19/47		10/23		23/51		24/53		5/11		5/11		34/79		9/22	
21	1/3	11/23	5/13	11/24	3/7	11/25	13/30	11/26	10/23	11/27	2/5	3/7	8/19	16/37	26/59	10/23	28/61	13/31	10/21	9/23	21/44	1/3	11/23	18/47	11/23	3/7	33/71	22/51	33/73	23/53	11/25	10/23	3/7	17/39	25/58	9/20	13/30	19/41	10/23	10/21
22	8/25		19/49		12/29		5/12		5/11		16/35		11/24		27/61		17/41		5/12		26/67		1/3		28/71		7/17		32/75		5/11		26/57		27/59		11/24		38/85	
23	16/49	12/25	11/28	6/13	22/53	4/9	7/16	3/7	11/25	12/29	25/59	2/5	25/61	17/39	3/7	18/41	29/65	11/25	31/67	7/17	11/23	2/5	23/48	1/3	12/25	20/51	12/25	8/19	36/77	24/55	36/79	25/57	4/9	11/25	36/83	3/7	27/62	19/43	7/16	5/11
24	1/3		11/29		13/31		11/26		26/59		17/37		6/13		6/13		29/67		19/45		11/26		5/13		1/3		30/77		23/55		8/19		37/83		28/61		29/63		6/13	
25	9/28	13/27	2/5	13/28	13/32	13/29	15/34	13/30	4/9	13/31	4/9	11/27	2/5	14/33	28/67	19/43	10/23	4/9	32/71	4/9	34/73	11/27	12/25	15/38	25/52	1/3	13/27	2/5	13/27	17/41	39/83	26/59	39/85	27/61	13/29	4/9	39/89	4/9	3/7	10/23
26	18/55		12/31		14/33		3/7		3/7		6/13		19/41		13/28		32/71		20/47		3/7		3/7		11/28		22/57		22/57		25/59		26/61		39/89		6/13		31/67	
27	1/3	14/29	23/59	7/15	7/17	14/31	4/9	7/16	17/38	14/33	13/29	12/29	29/69	2/5	9/22	4/9	31/73	21/47	11/25	13/29	5/11	17/39	37/79	12/29	13/27	16/41	27/56	1/3	14/29	17/43	14/29	25/61	42/89	4/9	6/13	29/65	14/31	30/67	42/95	13/29
28	10/31		13/33		3/7		16/37		13/30		31/69		20/43		7/15		7/15		34/77		22/51		13/30		35/83		2/5		1/3		24/61		3/7		28/65		29/67		44/97	
29	20/61	15/31	13/34	15/32	5/12	5/11	29/67	15/34	9/20	3/7	14/31	18/43	32/73	13/31	2/5	8/19	5/12	22/49	34/79	23/51	4/9	14/31	38/83	3/7	8/17	13/31	14/29	12/31	29/60	1/99	15/31	9/23	15/31	27/65	9/19	7/16	45/97	31/69	5/11	32/71
30	1/3		2/5		28/67		17/39		7/16		32/73		7/15		22/47		15/32		37/81		23/53		24/55		7/16		37/89		36/91		1/99		2/5		41/97		10/23		31/71	
31	11/34	16/33	7/18	8/17	8/19	16/35	14/33	4/9	19/42	16/37	5/11	19/45	5/11	14/33	33/79	2/5	21/50	18/41	11/26	24/53	37/85	5/11	13/29	5/11	41/89	25/59	43/91	14/33	15/31	13/33	31/64	1/99	16/33	26/67	16/33	29/69	29/61	22/51	41/88	33/73
32	22/67		9/23		16/39		18/41		15/34		3/7		36/79		23/49		8/17		8/17		13/29		25/57		26/59		15/34		7/17		38/97		1/99		9/23		38/91		32/73	
33	1/3	17/35	15/38	17/36	17/40	17/37	3/7	17/38	5/11	17/39	21/46	20/47	16/35	3/7	36/83	3/7	2/5	18/43	23/54	5/11	3/7	26/57	40/91	16/35	14/31	21/47	44/95	3/7	46/97	3/7	16/33	2/5	33/68	1/99	17/35	28/71	17/35	31/73	41/86	32/75
34	12/37		5/13		17/41		19/43		4/9		4/9		37/83		8/17		25/53		17/36		6/13		41/93		27/61		4/9		4/9		5/12		7/18		1/99		7/18		31/75	
35	24/73	18/37	2/5	9/19	3/7	6/13	19/44	9/20	4/9	18/41	11/24	3/7	17/37	22/51	13/29	16/37	37/89	2/5	3/7	10/23	25/58	27/59	16/37	28/61	43/97	17/37	5/11	11/25	27/58	16/37	39/82	40/97	17/35	22/57	35/72	1/99	18/37	2/5	18/37	3/7
36	1/3		16/41		18/43		4/9		21/47		17/38		38/87		41/89		26/55		9/19		9/19		44/97		4/9		29/65		17/38		39/89		8/19		15/38		1/99		30/77	
37	13/40	19/39	31/79	19/40	34/81	19/41	10/23	19/42	17/39	19/43	23/50	19/44	6/13	23/53	6/13	17/39	40/93	20/47	2/5	5/12	13/30	22/49	27/62	29/63	17/39	6/13	25/56	6/13	16/35	10/23	43/92	17/39	31/65	16/39	18/37	5/13	37/76	1/99	19/39	1/99
38	26/79		17/43		19/45		37/85		22/49		9/20		3/7		14/31		9/19		28/59		19/40		27/58		25/56		30/67		31/69		9/20		3/7		17/40		25/77		1/99	
39	1/3	20/41	17/44	10/21	19/46	20/43	7/16	5/11	18/41	4/9	6/13	10/23	25/54	24/55	19/41	25/57	43/97	18/41	17/41	2/5	23/54	22/51	7/16	6/13	18/41	31/67	29/66	19/41	40/89	5/11	17/37	32/73	38/81	18/41	11/23	17/41	19/39	1/99	39/8[illegible]	[illegible]
40	14/43		2/5		20/47		3/7		23/51		19/42		42/95		43/97		46/99		29/61		10/21		10/21		38/83		37/84		32/71		33/73		19/42		3/7		3/7		1/99	
41	28/85	21/43	9/23	21/44	5/12	7/15	11/25	21/46	19/43	21/47	43/95	7/16	13/28	3/7	20/43	26/59	41/90	19/43	41/96	11/26	2/5	27/65	29/66	4/9	15/34	32/69	19/43	33/71	27/61	20/43	33/73	11/25	6/13	34/77	37/80	19/43	35/73	18/43	20/41	1/99
42	1/3		35/89		3/7		22/51		24/53		5/11		5/11		27/62		26/57		10/21		31/65		21/44		37/79		37/82		43/97		34/75		5/11		5/11		19/44		27/65	
43	15/46	22/45	19/48	11/23	21/50	21/47	23/52	11/24	4/9	22/49	4/9	11/25	27/58	22/51	7/15	27/61	7/15	4/9	29/66	4/9	19/45	2/5	36/85	3/7	31/70	26/57	4/9	34/73	4/9	7/15	41/92	7/15	9/20	35/79	19/41	4/9	13/28	4/9	10/21	19/45
44	30/91		19/49		8/19		23/53		5/11		26/57		21/46		3/7		35/78		29/62		32/67		11/23		11/23		43/94		7/16		5/11		36/79		21/46		10/23		10/23	
45	1/3	[illegible]	2/5	[illegible]	11/26	[illegible]	4/9	23/50	25/56	23/51	21/47	23/52	7/15	23/53	29/62	4/9	22/47	29/65	41/91	21/47	20/47	8/19	2/5	30/71	4/9	26/59	33/74	7/15	21/47	36/77	18/41	22/47	43/96	21/47	19/42	37/83	20/43	21/47	41/88	28/65
46	16/49		20/51		22/53		24/55		25/56		27/59		11/24		37/84		34/77		29/63		11/23		34/71		23/48		41/87		22/49		11/25		37/81		38/83		11/24		37/84	
47	8/25	24/49	13/33	12/25	23/54	8/17	38/87	6/13	13/29	24/53	22/49	4/9	39/85	24/55	15/32	3/7	23/49	30/67	41/89	22/49	17/39	26/59	3/7	1/99	23/56	32/75	17/38	34/77	35/78	37/79	22/49	38/81	19/43	23/49	39/88	38/85	5/11	13/29	7/15	22/49
48	[illegible]		[illegible]		23/55		25/57		11/25		28/61		23/50		37/82		10/23		29/64		8/17		35/73		12/25		12/25		41/89		23/52		29/66		39/85		40/87		23/50	
49	[illegible]	25/51	[illegible]	25/52	3/7	25/53	25/58	25/54	9/20	5/11	23/51	25/56	37/82	25/57	31/66	22/51	8/17	31/69	8/17	32/71	29/65	23/51	22/51	13/31	1/99	3/7	41/93	7/16	9/20	37/81	23/51	39/83	43/97	8/17	4/9	8/17	41/92	40/89	21/46	41/91
50	17/53		2/5		8/19		26/59		23/52		29/63		6/13		6/13		30/71		29/65		45/98		12/25		37/77		25/52		43/93		24/53		4/9		13/29		41/89		6/13	

Densities for $S = \{1, 1+k, 1+k+i\}$



Future Work

Goal: Characterize $\overline{\alpha}(\{i, j, k\})$ for all $1 \leq i < j < k$. **(or just $i = 1$?)**

Discharging arguments for $|S| > 3$?

Stronger bounds on minimum period?

On the independence ratio of distance graphs

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