1 Introduction

Definition 1.1. Let G be a graph. A *matching* in G is a set $M \subseteq E(G)$ so that for any pair of edges $e_1, e_2 \in M$, e_1 and e_2 have no common endpoint (i.e. $e_1 \cap e_2 = \emptyset$). A matching M is *perfect* if $2|M| = |V(G)|$.

Definition 1.2. Given a graph G, $\Phi(G)$ is the number of perfect matchings in G. A graph G is *saturated* if for every edge $e \in E(\overline{G})$, $\Phi(G+e) > \Phi(G)$.

There are two main structural theorems in the theory of matchings:

1. **The Edmonds–Gallai Structure Theorem** specializes in the structure of graphs with no perfect matchings. It says nothing about graphs with perfect matchings (two of the three parts of the given partition are empty).
2. **The Lovász Cathedral Theorem** specializes in the structure of saturated graphs with perfect matchings.

Note that while the Edmonds–Gallai structure theorem gives essentially no information on a graph with perfect matchings, there is a graph lurking below the surface with no perfect matchings: $G - v$ for any vertex $v \in V(G)$. The Lovász Cathedral Theorem uses the Edmonds–Gallai structure of these subgraphs extensively in the proofs.

Our goal is to develop a working knowledge of the definitions and statements of the Lovász Cathedral Theorem in order to apply it to extremal problems on perfect matchings. Hence, we shall not state the definitions in their full generality and will not prove the theorem itself. An interested reader is directed to Chapters 3 and 5 of Lovász and Plummer's *Matching Theory*, the standard reference in this topic. These notes are based on that book and we borrow its notation for consistency.

1.1 An Introductory Matching Problem

Problem 1.3. *Let n, p be integers (assume n is even). Define*

$$f(n,p) = \max\{|E(G)| : |V(G)| = n, \Phi(G) = p\}.$$

That is, $f(n,p)$ is the maximum number of edges in a graph on n vertices with exactly p perfect matchings.

Theorem 1.4 (Hetyei)**.** $f(n,1) = \frac{n^2}{4}$.

We shall only prove the lower bound at the moment.

Proof that $f(n,1) \geq \frac{n^2}{4}$. Note that if $n = 2$, K_2 has exactly one perfect matching and $\frac{2^2}{4} = 1$ edge. Proceed by induction on even n.

If G is a graph with one perfect matching on n vertices with $\frac{n^2}{4}$ edges, then join G with a vertex (add an edge from each vertex in $V(G)$ to a new vertex) and add a leaf at the new vertex. This is given as $H = (G \cup K_1) \vee K_1$. H has $n+2$ vertices and $\frac{n^2}{4} + (n+1) + 1$ edges, which equals $\frac{(n+2)^2}{4}$ edges. Moreover, the leaf can only be matched with its only neighbor, so any perfect matching must match vertices in $V(G)$ with $V(G)$, giving the unique perfect matching. ∎

Later, we will show a quick proof that this is also the largest number of edges using the Cathedral Theorem.

Theorem 1.5 (DS)**.** *For all* $p \geq 1$*, there exist constants* n_p, c_p *so that* $-(p-1)(p-2) \leq c_p \leq p-1$ *and for all* $n \geq n_p$*,*

$$f(n,p) = \frac{n^2}{4} + c_p.$$

There are multiple points to this theorem, but the most important is that as n increases, the difference between $f(n,p)$ and $\frac{n^2}{4}$ can only increase.

Claim 1.5.1. $f(n+2,p) \geq \frac{(n+2)^2}{4} + \left(f(n,p) - \frac{n^2}{4}\right)$

Proof. Let G have n vertices and $f(n,p)$ edges. Set $c_{n,p} = f(n,p) - \frac{n^2}{4}$. Consider $H = (G \cup K_1) \vee K_1$. H has $n+2$ vertices and $f(n,p) + n + 1$ edges. This works out to be $\frac{(n+2)^2}{4} - \frac{n^2}{4} + f(n,p) = \frac{(n+2)^2}{4} + c_{n,p}$ edges. ∎

Since $c_{n,p}$ is an increasing sequence in n, but is bounded above by $p-1$, we know it has a limit. This is the value c_p. Dudek and Schmitt conjectured that this constant is positive for $p \geq 2$. We answered this *positively.*

Theorem 1.6 (HSWY)**.** *Let* $p \geq 2$ *be a constant. Then, for all* $n \geq 2\lfloor \log_2 p \rfloor + 2$*,* $c_{n,p} \geq \mathrm{wt}_2(p-1)$ *(where* $\mathrm{wt}_2(m)$ *is the number of* 1*s in the binary expansion of an integer* m*).*

In particular, $c_p \geq \mathrm{wt}_2(p-1) \geq 1$.

There are many open questions on this problem, including:

1. Is c_p monotone in p?
2. What is the maximum growth rate of c_p?
3. What is the growth rate of the minimum n_p?

2 Saturated and Elementary Graphs

Definition 2.1. An edge is

- *forbidden* if it is not contained in any perfect matching.
- *allowable* if it is contained in some perfect matching.
- *vital* if it is contained in every perfect matching.

Definition 2.2. A connected graph is

- *1-extendable* if every edge is allowable;
- *elementary* if the set of allowable edges is connected;
- *saturated* if adding any edge increases the number of perfect matchings.

If $\Phi(G) > 0$, then there are allowable edges in G. The components given by the subgraph of allowable edges induce the maximal elementary subgraphs of G. The Cathedral Theorem will take a saturated graph G, split it into its maximal elementary subgraphs and then control the way they are connected.

Note that if $\Phi(G) > 0$ but G is not saturated, there exists a saturated supergraph $G' \supset G$ so that $\Phi(G') = \Phi(G)$ by adding edges as long as they do not increase the number of perfect matchings. If G has $f(|V(G)|, \Phi(G))$ edges, it is automatically saturated.

2.1 Barriers

Theorem 2.3 (Tutte's Theorem)**.** *A graph G has a perfect matching if and only if for every set $S \subseteq V(G)$, the number of odd components in $G - S$ is at most the number of vertices in S. That is, $c_o(G - S) \leq |S|$.*

The intuition of Tutte's Theorem is that odd components cannot have perfect matchings themselves, so they must be matched with vertices in S. If S is too small, then no perfect matching can exist.

This gives a *certificate* of lacking a perfect matching: a *Tutte set* is a set $S \subseteq V(G)$ so that $c_o(G - S) > |S|$. If $\Phi(G) > 0$, then we have no Tutte sets, but we can consider the sets which are as close as possible to such a set.

Definition 2.4. If G has at least one perfect matching, a *barrier* is a set $X \subseteq V(G)$ so that the number of odd components in $G - X$ is equal to the number of vertices in X.

Lemma 2.5. *Given a graph G with $\Phi(G) > 0$, the set of maximal barriers partitions the vertex set. This partition is denoted $\mathcal{P}(G)$.*

Proposition 2.6. *If G is a saturated graph, each barrier $S \in \mathcal{P}(G)$ is a clique in G.*

Proof. All of the perfect matchings in G must match S to a vertex in an odd component of $G - S$. Adding missing edges between vertices of S does not change the connectivity of $G - S$, and hence does not increase the number of perfect matchings. ∎

3 The Cathedral Construction

Definition 3.1 (The Cathedral Construction)**.** Let G_0 be a saturated elementary graph. Let $\mathcal{P}(G_0) = \{S_1, \ldots, S_k\}$ and denote G_{S_i} as a (possibly null) saturated graph associated with the barrier $S_i \in \mathcal{P}(G_0)$. The *cathedral* given by $G_0, G_{S_1}, \ldots, G_{S_k}$ is the graph given by the disjoint union of $G_0 \bigcup \cup_{i=1}^{k} G_{S_i}$ and edges added between each vertex of S_i to each vertex of G_{S_i} for each $i \in \{1, \ldots, k\}$.

The graph G_0 is called the *foundation*. The graphs G_{S_i} are the *towers*.

Theorem 3.2 (Lovász's Cathedral Theorem)**.** *A graph G is saturated if and only if there is a saturated elementary graph G_0 and a saturated graph G_S for each $S \in \mathcal{P}(G_0)$ so that G is isomorphic to the Cathedral Construction given by these graphs. Moreover, the graph G_0 is uniquely determined by the Edmonds–Gallai structure theorem of graphs $G - v$.*

Note that this theorem implies that the graphs G_S are cathedrals themselves, giving a recursive structure.

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We will not prove that every saturated graph is of this type, but we shall prove that every cathedral is itself saturated.

Proof. Let G_0 be saturated elementary graph with a tower G_S for each $S \in \mathcal{P}(G_0)$ and G is the resulting cathedral. Use induction to guarantee that G_S is itself a cathedral.

Claim 3.2.1. *All allowed edges in the cathedral are in the elementary subgraphs.*

Note that for each $S \in \mathcal{P}(G_0)$, $c_o(G_0 - S) = |S|$. This holds also for $c_o(G - S)$, since S cuts G_0 into components which are connected to some number of towers (which are even graphs themselves) and cuts G_S into its own connected component.

Now, all vertices in S must be matched to vertices of odd components in $G - S$, but these edges must be in G_0. So, all vertices in G_0 are matched to vertices in G_0 and also all matchings on G_S are matched within G_S (by induction gives that the edges are within the elementary parts).

Claim 3.2.2. $\Phi(G) = \Phi(G_0) \cdot \prod_{S \in \mathcal{P}} \Phi(G_S)$

Since the matchings are independently chosen from the foundation and the towers, they multiply.

Claim 3.2.3. *If $e \in E(\overline{G})$, $\Phi(G + e) > \Phi(G)$.*

If e is added within a tower G_S (which is saturated), we have $\Phi(G_S + e) > \Phi(G_S)$ so $\Phi(G + e) > \Phi(G)$. Similarly if e is added to G_0.

We can assume by induction that e is added between towers or between a vertex x of G_0 and a tower G_S so that $x \notin S$. At minimum, we know that one endpoint of e is in a tower G_S and the other endpoint is not in S. Let M be a matching in G.

There is an M-alternating path from the endpoints of e that takes an edge in each level of the cathedral then goes down a level until it reaches G_0, then travels between barriers using an M-alternating path. By adding e to this path, we have an M-alternating cycle. Take the perfect matching given by M but swapping the M-edges in the cycle with the others, making a new perfect matching.

Therefore, G is saturated. ∎

Corollary 3.3. *If G is saturated with exactly one perfect matching, then the foundation of the cathedral is a single edge.*

Proof. $1 = \Phi(G) = \Phi(G_0) \cdot \prod_{S \in \mathcal{P}(G_0)} \Phi(G_S)$. Since $\Phi(G_0) = 1$, the allowable edges of G_0 are given by the single perfect matching. But these edges should yield a connected graph! Thus, $G_0 = K_2$. ∎

Lemma 3.4 (HSWY)**.** *Fix n, p and let G be a graph with n vertices, p perfect matchings, and $f(n,p)$ edges. Then, the Cathedral structure of G has a single tower of elementary graphs $G_0, G_1, \ldots, G_\ell$ where the barrier chosen in G_i has maximum size in $\mathcal{P}(G_i)$.*

Proof. Let I be an index set on the elementary subgraphs of G, where G_i denotes the elementary subgraph for $i \in I$. Let S_i be a maximum-order barrier in G_i. There is a partial-order $\leq$ on I given by $i \leq j$ if G_j is in a tower above G_i. Note that the poset $(I, \leq)$ is a tree with a root.

Let $\leq'$ be a total order on I that extends $\leq$. Replace G by the graph G', defined as

$$G' = \cup_{i \in I} G_i + \cup_{\substack{i \leq' j \\ i \neq j}} E(S_i \vee G_j).$$

Note that G' has at least as many edges as G, since the edges in each G_i have not changed, and for every $i \leq j$, the number of edges joining a barrier of G_i to G_j in G is at most the number of edges joining S_i to G_j in G'. Moreover, if G has more than one tower, there exists an $\leq$-incomparable pair i, j which are now comparable in $\leq'$ which increases the number of edges. Since $\Phi(G') = \prod_{i \in I} \Phi(G_i) = \Phi(G)$, we conclude that G was not extremal. ∎

We are now well-equipped to prove the other inequality in Hetyei's theorem.

Proof that $f(n,p) \leq \frac{n^2}{4}$. If G has $f(n,p)$ edges, then it has a single tower of $n/2$ copies of K_2 foundations. The barriers of K_2 are the endpoints. Counting the edges gives $\frac{n^2}{4}$. ∎

While the previous lemma gave us some nice control on the structure of extremal graphs with $\Phi(G) = p$, we in fact have even more control, based on the relative sizes of the barriers in each level.

Lemma 3.5 (HSWY)**.** *Let G have n vertices, p perfect matchings, and $f(n,p)$ edges. Let $G_0, G_1, \ldots, G_k$ be the elementary subgraphs of G in order from the foundation to the highest tower. Denote $n_i = |V(G_i)|$ and s_i as the order of the maximum-size barrier in G_i.*

1. *For each pair $i \leq j$, $\frac{1}{2} \geq \frac{s_i}{n_i} \geq \frac{s_j}{n_j}$.*
2. *If the set $\{0, \ldots, k\}$ is partitioned by the equivalence $i \sim j$ if and only if $\frac{s_i}{n_i} = \frac{s_j}{n_j}$, then any reordering of the G_i within each equivalence class gives a graph with the same number of edges.*

Proof. Note that the fraction $\frac{s_i}{n_i}$ is always at most half, since any barrier X_i in G_i has $|X_i| = c_o(G_i - X_i) \leq n(G_i) - |X_i|$.

Let i be the smallest i so that $\frac{s_i}{n_i} < \frac{s_{i+1}}{n_{i+1}}$ (if this does not occur, then (1) holds). Let G' be the graph given by the tower $G_0, \ldots, G_{i-1}, G_{i+1}, G_i, G_{i+2}, \ldots, G_k$ where the barriers are chosen the same. The only edges that have changed are those between G_{i+1} and G_i. There are now $s_j \cdot n_i > s_i \cdot n_j$ edges, which increases the number of edges from G. Hence, G was not extremal.

Note that if $\frac{s_i}{n_i} = \frac{s_j}{n_j}$, then $s_j \cdot n_i = s_i \cdot n_j$. Hence, the number of edges between G_i and G_j does not change if they are reordered in the tower of G. (2) follows. ∎