Coloring the Integers with Rainbow Arithmetic Progressions

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A *k-term arithmetic progression* is a finite sequence of k terms of the form $\{a, a+d, a+2d, ..., a+(k-1)d\}$, where k, a, and d are nonnegative integers.

$$3, 10, 17, 24, 31$$

anti-van der Waerden numbers

An exact r-coloring of a set S is a function $c : S \to C$, such that $|C| = r$.

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A set S is rainbow under an r-coloring c, if $c(s_1) \neq c(s_2)$, for each distinct $s_1, s_2 \in S$.

anti-van der Waerden numbers

Given positive integers n and k with $k \leq n$, the anti-van der Waerden number, denoted by $aw(n, k)$, is the least positive integer r such that every exact r-coloring of $[n]$ contains a rainbow k-term AP.

$aw(8,3) \leq 5$

$$1\ 2\ 3\ 4\ 5\ 6\ 7\ 8$$

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1 2 3 4 5 6 7 8

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Properties of $aw(n, k)$

- $k \leq aw(n, k) \leq n$.
- $aw(n, k) = n$ if and only if $k \geq \frac{n}{2} + 1$.
- $aw(n, k) \leq aw(n-1, k) + 1$.

Small anti-van der Waerden numbers

$n \backslash k$	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22
3	3																			
4	4																			
5	4	5																		
6	4	6	6																	
7	4	6	7	7																
8	5	6	8	8	8															
9	4	7	8	9	9	9														
10	5	8	9	10	10	10	10													
11	5	8	9	10	11	11	11	11												
12	5	8	10	11	12	12	12	12	12											
13	5	8	11	11	12	13	13	13	13	13										
14	5	8	11	12	13	14	14	14	14	14	14									
15	5	9	11	13	14	14	15	15	15	15	15	15								
16	5	9	12	13	15	15	16	16	16	16	16	16	16							
17	5	9	13	13	15	16	16	17	17	17	17	17	17	17						
18	5	10	14	14	16	17	17	18	18	18	18	18	18	18	18					
19	5	10	14	15	17	17	18	18	19	19	19	19	19	19	19	19				
20	5	10	14	16	17	18	19	19	20	20	20	20	20	20	20	20	20			
21	5	11	14	16	17	19	20	20	20	21	21	21	21	21	21	21	21	21		
22	6	12	14	17	18	20	21	21	21	22	22	22	22	22	22	22	22	22	22	
23	6	12	14	17	19	20	21	22	22	22	23	23	23	23	23	23	23	23	23	23
24	6	12	15	18	20	20	22	23	23	23	24	24	24	24	24	24	24	24	24	24
25	6	12	15	19	21	21	23	23	24	24	24	24	25	25	25	25	25	25	25	
26	6	12																		
27	5																			
28	6																			
29	6																			
30	6																			
31	6																			
32	6																			

Lowerbound for $aw(n,3)$

Theorem

$aw(\frac{n}{3},3)+1 \leq aw(n,3)$.

1 2 3 4 5 6 7 8 9....

Lowerbound for $aw(n, 3)$

Theorem

$aw(\frac{n}{3}, 3) + 1 \leq aw(n, 3)$.

1 2 3 4 5 6 7 8 9....

Corollary

$\log_3(n) + 2 \leq aw(n, 3)$.

Upperbound for $aw(n, 3)$

Theorem

$aw(n, 3) \leq aw(\frac{n}{2}, 3) + 1.$

Upperbound for $aw(n, 3)$

Theorem

$aw(n, 3) \leq aw(\frac{n}{2}, 3) + 1.$

Corollary

$aw(n, 3) \leq \log_2(n) + 1.$

Upperbound for $aw(n, 3)$

Theorem

$aw(n, 3) \leq aw(\frac{n}{2}, 3) + 1.$

Corollary

$aw(n, 3) \leq \log_2(n) + 1.$

Conjecture

$aw(n, 3) \leq \log_3(n) + 4.$

Conjecture

$aw(3^m, 3) \leq m + 2.$

$aw(\mathbb{Z}_n, k)$

A k-term arithmetic progression in $\mathbb{Z}_n$ is a finite sequence of k terms of the form $\{a, a+d, a+2d, ..., a+(k-1)d\}(\text{mod } n)$, where k, a, and d are integers.

Small anti-van der Waerden numbers of $\mathbb{Z}_n$

$n \backslash k$	4	5	6	7	8	9	10	11	12	13	14	15	16	17
4	4													
5	4	5												
6	5	5	6											
7	4	5	6	7										
8	6	6	7	7	8									
9	5	6	8	8	8	9								
10	6	8	8	8	9	9	10							
11	5	7	8	9	9	10	10							
12	8	9	10	10	11	11	11	12						
13	5	7	9	10	10	11	11	12	12					
14	6	8	10	12	12	12	12	13	13	14				
15	8	11	12	12	12	13	14	14	14	14	15			
16	8	10	10	11	14	14	14	14	15	15	15	15		
17	6	8	10	11	12	13	14	14	15	15	16	16	16	
18	8	10	13	14	14	16	16	16	17	17	17	17	17	18
19	6	9												

Small anti-van der Waerden numbers of $\mathbb{Z}_n$ with $k = 3$

	0	1	2	3	4	5	6	7	8	9
0				3	3	3	4	3	3	4
10	4	3	4	3	4	4	3	4	5	3
20	4	4	4	3	4	4	4	5	4	3
30	5	4	3	4	5	4	5	3	4	4
40	4	4	5	4	4	5	4	3	4	4
50	5	5	4	3	6	4	4	4	4	3
60	5	3	5	5	3	4	5	3	5	4
70	5	3	5	4	4	5	4	4	5	3
80	4	6	5	3	5	5	5	4	4	4
90	6	4	4	5	4	4	4	4	5	

Theorem

$aw(\mathbb{Z}_{2^m}, 3) = 3.$

Theorem

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Theorem

If t is odd and s is either odd or 2^m, then

$$aw(\mathbb{Z}_{st}, 3) \leq aw(\mathbb{Z}_s, 3) + aw(\mathbb{Z}_t, k) - 2.$$

Question: Is $aw(\mathbb{Z}_s, 3) + aw(\mathbb{Z}_t, 3) - 2 \leq aw(\mathbb{Z}_{st}, k)$ when t is odd?

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Question: Given a prime p, does there exist a coloring of $\mathbb{Z}_p$ with $aw(\mathbb{Z}_p, 3) - 1$ colors, no rainbow 3-term APs, and a color class of size 1?

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Question: Given a prime p, does there exist a coloring of $\mathbb{Z}_p$ with $aw(\mathbb{Z}_p, 3) - 1$ colors, no rainbow 3-term APs, and a color class of size 1?

Question: Is $aw(\mathbb{Z}_p, 3) \leq 4$, for all prime p?

Thank You!