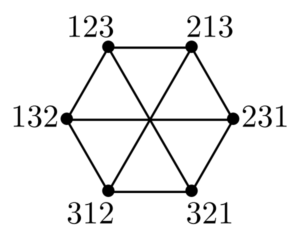
MATH 412, FALL 2012 - HOMEWORK 2

WARMUP PROBLEMS: Section 1.2: #3, 6, 8, 10, 12. Section 1.3: #1, 9. Do not write these up! Think about how to solve them to make sure you understand the material before doing the homework.

EXTRA PROBLEMS: Section 1.2: #15, 23, 25, 28, 32, 37. Section 1.3: #10, 14. Do not write these up! These are interesting problems (related to what we have discussed) to provide extra practice.

WRITTEN PROBLEMS: Do five of the six problems below (students registered for four hours or honors must do all six problems). Due Wednesday, September 12.

1. Let G_n be the graph whose vertices are the permutations of $\{1, \ldots, n\}$, with two permutations $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ adjacent if they differ by switching two entries. Prove that G_n is bipartite (G_3 shown below). (Hint: For each permutation a, count the pairs i, j such that $i < j$ and $a_i > a_j$; these are called *inversions*.)



2. Let G be a simple graph with vertices $v_1, \ldots, v_n$. Let A^k denote the kth power of the adjacency matrix of G under matrix multiplication. Prove that entry i, j of A^k is the number of v_i, v_j-walks of length k in G. Prove that G is bipartite if and only if, for the odd integer r in $\{n-1, n\}$, the diagonal entries of A^r are all 0. (Reminder: A walk is an *ordered* list of vertices and edges.)

3. Use ordinary induction on k or on the number of edges to give another proof that a connected graph with $2k$ odd vertices decomposes into k trails if $k > 0$. Does this remain true without the connectedness hypothesis?

4. Let G be a connected graph with at least three vertices. Prove that G has two vertices x, y such that 1) $G - \{x, y\}$ is connected and 2) x, y are adjacent or have a common neighbor. (Hint: Consider a longest path.)

5. Use induction on k to prove that every connected simple graph with $2k$ edges decomposes into paths of length 2. Does the conclusion remain true without the hypothesis of connectedness?

6. Let W be a closed walk in a graph G. Let H be the subgraph of G consisting of edges used an odd number of times in W. Prove that $d_H(v)$ is even for every $v \in V(G)$.