MATH 412, FALL 2012 - HOMEWORK 5

WARMUP PROBLEMS: Section 2.3 #2, 3, 4, 5, 22, 24. Section 3.1 #1, 2, 4, 7. Do not write these up!

OTHER INTERESTING PROBLEMS: Section 2.3 #6, 8, 9, 11, 12, 17, 22. Section 3.1 #8, 9, 11, 16, 19, 21. Do not write these up!

WRITTEN PROBLEMS: Five of the following six. Due Wednesday, October 3.

1. *Prim's Algorithm* grows a spanning tree from a given vertex of a connected weighted graph G, iteratively adding the cheapest edge from a vertex already reached to a vertex not yet reached, finishing when all the vertices of G have been reached. (Ties are broken arbitrarily.) Prove that Prim's Algorithm produces a minimum-weight spanning tree of G.

2. Let T be a minimum-weight spanning tree in G, and let T' be another spanning tree in G. Prove that T' can be transformed into T by a list of steps that exchange one edge of T' for one edge of T, such that the edge set is always a spanning tree and the total weight never increases.

3. Prove that the following algorithm correctly finds the diameter of a tree. First, run BFS from an arbitrary vertex w to find a vertex u at maximum distance from w. Next, run BFS from u to reach a vertex v at maximum distance from u. Report $\operatorname{diam} T = d(u, v)$.

4. Two people play a game on a graph G, alternately picking vertices. Player 1 starts at any vertex. Each subsequent choice must be adjacent to the preceding choice (of the other player) and not used before. Thus together they follow a path. The last player who moves wins. Prove that the second player has a winning strategy if G has a perfect matching, and otherwise the first player has a winning strategy. (Hint: Be careful about the second part!)

5. Use the König–Egerváry Theorem to prove that every bipartite graph G has a matching of size at least $e(G)/\Delta(G)$. Use this to conclude that every subgraph of $K_{n,n}$ with more than $(k-1)n$ edges has a matching of size at least k.

6. In an X, Y-bigraph G, the *deficiency* of a set S is $\operatorname{def}(S) = |S| - |N(S)|$; note that $\operatorname{def}(\varnothing) = 0$. Prove that $\alpha'(G) = |X| - \max_{S \subseteq X} \operatorname{def}(S)$. (Hint: Form a bipartite graph G' such that G' has a matching that saturates X if and only if G has a matching of the desired size, and prove that G' satisfies Hall's Condition.)