MATH 412, FALL 2012 - HOMEWORK 6

WARMUP PROBLEMS: Section 3.1 #40. Section 3.2 #1, 2. Section 3.3 #1, 2, 4. Do not write these up!

EXTRA PROBLEMS: Section 3.1 #39, 43, 44. Section 3.2 #6, 7. Section 3.3 #7, 10, 13, 14, 15, 19, 22, 26. Do not write these up!

WRITTEN PROBLEMS: Solve and write five of the following six. Due Wednesday, October 10.

1. An algorithm to greedily build a large independent set S iteratively selects a vertex of minimum degree in the remaining graph, adds it to S, and deletes it and its neighbors from the graph. Prove that this algorithm produces an independent set of size at least $\sum_{v\in V(G)} \frac{1}{d_G(v)+1}$ in a simple graph G.

2. An edge e of a graph G is *α-critical* if $\alpha(G-e) > \alpha(G)$. Suppose that xy and xz are α-critical edges in G. Prove that G has an induced subgraph that is an odd cycle containing xy and xz. (Hint: Let Y, Z be maximum independent sets in $G - xy$ and $G - xz$, respectively. Let $H = G[Y \triangle Z]$. Prove that every component of H has the same number of vertices from Y and from Z. Use this to prove that y and z belong to the same component of H.)

3. Find a minimum-weight transversal in the matrix below, and use duality to prove that the solution is optimal. (Hint: Use a transformation of the problem.)

$$\begin{pmatrix} 4 & 5 & 8 & 10 & 11 \\ 7 & 6 & 5 & 7 & 4 \\ 8 & 5 & 12 & 9 & 6 \\ 6 & 6 & 13 & 10 & 7 \\ 4 & 5 & 7 & 9 & 8 \end{pmatrix}$$

4. *Extension of König–Egerváry Theorem to general graphs.* Given a graph G, let $S_1, \ldots, S_k$ and T be subsets of $V(G)$ such that each S_i has odd size. These sets form a *generalized cover* of G if every edge of G has at least one endpoint in T or both endpoints in some S_i. The *weight* of a generalized cover is $|T| + \sum \lfloor |S_i|/2 \rfloor$. Let $\beta^*(G)$ be the minimum weight of a generalized cover. Prove that $\alpha'(G) = \beta^*(G)$. (Hint: Apply Corollary 3.3.7. Comment: Every vertex cover is a generalized cover, so $\beta^*(G) \le \beta(G)$.)

5. Let G be a k-regular graph of even order that remains connected when any $k-2$ edges are deleted. Prove that G has a 1-factor.

6. A graph G is *factor-critical* if each subgraph $G - v$ obtained by deleting one vertex has a 1-factor. Prove that G is factor-critical if and only if $n(G)$ is odd and $o(G-S) \le |S|$ for all nonempty $S \subseteq V(G)$.