MATH 412, FALL 2012 - HOMEWORK 7

WARMUP PROBLEMS: Section 3.2 #9, 10. Section 3.3 #6 Section 4.1 #1, 2, 4, 5. Do not write these up!

EXTRA PROBLEMS: Section 3.2 #11, 12, 13. Section 3.3 #16, 18, 20. Section 4.1 #8, 9, 10, 11, 17. Do not write these up!

WRITTEN PROBLEMS: Solve and write five of the following six. Due Wednesday, October 17.

1. Prove that, in the Proposal Algorithm with men proposing, no man is ever rejected by all the women. (Hint: What happens after x is rejected by all but one woman?)

2. Let G be an X, Y-bigraph. Let H be the graph obtained from G by adding one vertex to Y if $n(G)$ is odd and then adding edges to make Y a clique.

a) Prove that G has a matching of size $|X|$ if and only if H has a 1-factor.

b) Prove that if G satisfies Hall's Condition ($|N(S)| \geq |S|$ for all $S \subseteq X$), then H satisfies Tutte's Condition ($o(H - T) \leq |T|$ for all $T \subseteq V(H)$).

c) Use parts (a) and (b) to obtain Hall's Theorem from Tutte's Theorem.

3. Let n, k be positive integers with n even, k odd, and $n > k > 1$. Let G be the k-regular simple graph formed by placing n vertices on a circle and making each vertex adjacent to the opposite vertex and to the $(k-1)/2$ nearest vertices in each direction. Prove that $\kappa(G) = k$.

4. Let G be a simple n-vertex graph with $n/2 - 1 \leq \delta(G) \leq n - 2$. Prove that G is k-connected for all k with $k \leq 2\delta(G) + 2 - n$. Prove that this is best possible for all $\delta \geq n/2 - 1$ by constructing a simple n-vertex graph with minimum degree δ that is not k-connected for $k = 2\delta + 3 - n$.

5. *Degree conditions for* $\kappa' = \delta$. Let G be a simple n-vertex graph, with $n \geq 4$. Use Lemma 4.1.   to prove the following statements.

a) If $\delta(G) \geq \lfloor n/2 \rfloor$, then $\kappa'(G) = \delta(G)$. Prove this best possible by constructing a simple n-vertex graph with $\delta(G) = \lfloor n/2 \rfloor - 1$ and $\kappa'(G) < \delta(G)$.

b) If $d(x) + d(y) \geq n - 1$ whenever $x \nleftrightarrow y$, then $\kappa'(G) = \delta(G)$. Prove that this is best possible by constructing for each $n \geq 4$ and $\delta(G) = m \leq n/2 - 1$ an n-vertex graph G with $\kappa'(G) < \delta(G) = m$ in which $d(x) + d(y) \geq n - 2$ whenever $x \nleftrightarrow y$.

6. *The block-cutpoint graph* (see Definition 4.1.   ). Let H be the block-cutpoint graph of a graph G that has a cut-vertex.

a) Prove that H is a forest.

b) Prove that G has at least two blocks each of which contains exactly one cut-vertex of G.

c) Prove that a graph G with k components has exactly $k + \sum_{v \in V(G)} (b(v) - 1)$ blocks, where $b(v)$ is the number of blocks containing v.

d) Prove that every graph has fewer cut-vertices than blocks.