MATH 412, FALL 2012 - HOMEWORK 8

WARMUP PROBLEMS: Section 4.1 #6, 7, 30. Section 4.2 #1, 2, 4, 6. Do not write up!

OTHER INTERESTING PROBLEMS: Section 4.1 #31, 32. Section 4.2 #7, 8, 11, 12, 13, 14, 15, 19, 20, 21, 23. Do not write these up!

WRITTEN PROBLEMS: Solve and write five of the following six. Due Wednesday, October 24.

1. For a connected graph G with at least three vertices, prove that the following statements are equivalent (use of Menger's Theorem is permitted).

A) G is 2-edge-connected.

B) Every edge of G appears in a cycle.

C) G has a closed trail containing any specified pair of edges.

D) G has a closed trail containing any specified pair of vertices.

2. Let v be a vertex of a 2-connected graph G. Prove that v has a neighbor u such that $G - u - v$ is connected.

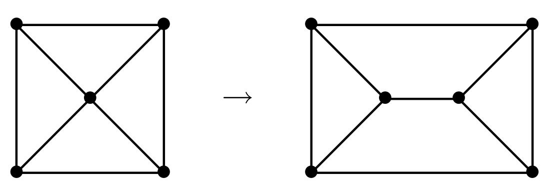
3. Let G be a graph without isolated vertices. Prove that if G has no even cycles, then every block of G is an edge or an odd cycle.

4. Suppose that $\kappa(G) = k$ and $\operatorname{diam} G = d$. Prove that $n(G) \geq k(d-1)+2$ and $\alpha(G) \geq \lceil (1+d)/2 \rceil$. For each $k \geq 1$ and $d \geq 2$, construct a graph with connectivity k and diameter d for which equality holds in both bounds.

5. A *vertex k-split* of a graph G is a graph H obtained from G by replacing one vertex $x \in V(G)$ by two adjacent vertices x_1, x_2 such that $d_H(x_i) \geq k$ and that $N_H(x_1) \cup N_H(x_2) = N_G(x) \cup \{x_1, x_2\}$.

a) Prove that every vertex k-split of a k-connected graph is k-connected.

b) Conclude that any graph obtained from a "wheel" $W_n = K_1 \vee C_{n-1}$ (Definition 3.3.6) by a sequence of edge additions and vertex 3-splits on vertices of degree at least 4 is 3-connected. (Comment: Tutte [1961b] proved also that every 3-connected graph arises in this way. The characterization does not extend easily for $k > 3$.)



6. Given a graph G, let D be the digraph obtained by replacing each edge with two oppositely-directed edges having the same endpoints (thus D is the symmetric digraph with underlying graph G). Assume that for all $x, y \in V(D)$ both $\kappa'_D(x,y) = \lambda'_D(x,y)$ and $\kappa_D(x,y) = \lambda_D(x,y)$ hold, the latter applying only when $x \not\to y$. Use this hypothesis to prove that also $\kappa'_G(x,y) = \lambda'_G(x,y)$ and $\kappa_G(x,y) = \lambda_G(x,y)$, the latter for $x \not\leftrightarrow y$.