MATH 412, FALL 2012 - HOMEWORK 9

WARMUP PROBLEMS: Section 4.2 #30. Section 4.3 #1, 2, 3, 4. Do not write up!

OTHER INTERESTING PROBLEMS: Section 4.2 #31, 32. Section 4.3 #5, 6, 7, 10, 11, 14. Do not write these up!

WRITTEN PROBLEMS: Write up five of the following six. Due Wed., October 31.

1. Let X and Y be disjoint sets of vertices in a k-connected graph G. Let $u(x)$ for $x \in X$ and $w(y)$ for $y \in Y$ be positive integers such that $\sum_{x \in X} u(x) = \sum_{y \in Y} w(y) = k$. Prove that G has k pairwise internally disjoint X, Y-paths so that $u(x)$ of them start at x and $w(y)$ of them end at y, for $x \in X$ and $y \in Y$.

2. A k-connected graph G is *minimally k-connected* if for every $e \in E(G)$, the graph $G - e$ is not k-connected.

a) Use ear decomposition to prove that $\delta(G) = 2$ if G is minimally 2-connected. (Comment: Halin [1969] proved that $\delta(G) = k$ when G is minimally k-connected.)

b) Prove that a minimally 2-connected graph G with at least 4 vertices has at most $2n(G) - 4$ edges, with equality only for $K_{2,n-2}$.

3. Let $[S, \overline{S}]$ and $[T, \overline{T}]$ be source/sink cuts in a network N.

a) Prove that $\text{cap}\,(S \cup T, \overline{S \cup T}) + \text{cap}\,(S \cap T, \overline{S \cap T}) \le \text{cap}\,([S, \overline{S}]) + \text{cap}\,(T, \overline{T})$. (Hint: Draw a picture and consider contributions from various types of edges.)

b) Suppose that $[S, \overline{S}]$ and $[T, \overline{T}]$ are minimum cuts. Conclude from part (a) that $[S \cup T, \overline{S \cup T}]$ and $[S \cap T, \overline{S \cap T}]$ are also minimum cuts. Conclude also that no edge between $S - T$ and $T - S$ has positive capacity.

4. Several companies send representatives to a conference; the ith company sends m_i people. The conference has simultaneous networking groups; the jth group holds up to n_j people. The organizers want to schedule all the people into groups, but people from the same company must be in different groups. The groups need not all be filled.

a) Show how to use network flows to test whether the constraints can be satisfied.

b) Let there be p companies and q groups, indexed so that $m_1 \ge \cdots \ge m_p$ and $n_1 \le \cdots \le n_q$. Prove that all the people can be scheduled into groups if and only if, for all $0 \le k \le p$ and $0 \le l \le q$, the inequality $k(q - l) + \sum_{j=1}^{l} n_j \ge \sum_{i=1}^{k} m_i$ holds.

5. Let G be a weighted graph. Let the *value* of a spanning tree be the minimum weight of its edges. Let the *cap* from an edge cut $[S, \overline{S}]$ be the maximum weight of its edges. Prove that in G the maximum value of a spanning tree equals the minimum cap of an edge cut.

6. Let x be a vertex of maximum outdegree in a tournament T. Prove that T has a spanning directed tree rooted at x such that every vertex has distance at most 2 from x and every vertex other than x has outdegree at most 2. (Hint: Create a network to model the desired paths to the non-successors of x, and show that every cut has enough capacity. Comment: This strengthens Proposition 1.4.30 about kings in tournaments.)