MATH 412, FALL 2012 - HOMEWORK 10

WARMUP PROBLEMS: Section 5.1 #2, 3, 5, 7, 12, 21. Section 5.2 #1, 2.

EXTRA PROBLEMS: Section 5.1 #22, 23, 29, 33, 38, 42, 48, 50, 54. Section 5.2 #6, 7, 9. Do not write these up!

WRITTEN HOMEWORK: Do five of the following six problems. Due Wednesday, November 7.

1. For all $k \in \mathbb{N}$, prove that a graph G is 2^k-colorable if and only if G is the union of k bipartite graphs. (Hint: This generalizes Theorem 1.2.23.)

2. Prove that every k-chromatic graph has at least $\binom{k}{2}$ edges. Use this to prove that if G is the union of m complete graphs of order m, then $\chi(G) \leq 1 + m\sqrt{m-1}$. (Comment: This bound is near tight, but the Erdős–Faber–Lovász Conjecture asserts that $\chi(G) = m$ when the complete graphs are pairwise edge-disjoint.)

3. Prove that $\chi(G) + \chi(\overline{G}) \leq n(G) + 1$. (Hint: Use induction on $n(G)$.)

4. *Improvement of Brooks' Theorem.*

a) Let $k_1, \ldots, k_t$ be positive integers with sum k. Prove that the vertex set of any graph G with $\Delta(G) < k$ can be partitioned into sets $V_1, \ldots, V_t$ such that $\Delta(G[V_i]) < k_i$ for $1 \leq i \leq t$. (Hint: Prove that the partition minimizing $\sum e(G_i)/k_i$ has the desired property.)

b) For $4 \leq r \leq \Delta(G) + 1$, use part (a) to prove that $\chi(G) \leq \left\lceil \frac{r-1}{r}(\Delta(G)+1) \right\rceil$ when G has no r-clique.

5. *Chromatic number and cycle lengths.*

a) Let v be a vertex in a graph G. Among all spanning trees of G, let T be one that maximizes $\sum_{u \in V(G)} d_T(u, v)$. Prove that every edge of G joins vertices belonging to a path in T starting at v.

b) Prove that if $\chi(G) > k \geq 2$, then G has a cycle whose length is one more than a multiple of k. (Hint: Use the tree T of part (a) to define a k-coloring of G.)

6. Prove that every triangle-free n-vertex graph has chromatic number at most $2\sqrt{n}$. (Comment: Thus every k-chromatic triangle-free graph has at least $k^2/4$ vertices.)