MATH 412, FALL 2012 - HOMEWORK 11

WARMUP PROBLEMS: Section 5.2 #3, 4, 5.

EXTRA PROBLEMS: Section 5.2 #11, 13, 16, 17, 18, 20, 21, 26, 31, 37, 40

WRITTEN HOMEWORK: Do five of the following six. Due Wed., November 14.

1. *Turán's proof of Turán's Theorem*, including uniqueness.

a) Prove that a maximal simple graph having no $r+1$-clique has an r-clique.

b) Prove that $e(T_{n,r}) = \binom{r}{2} + (n-r)(r-1) + e(T_{n-r,r})$.

c) Use parts (a) and (b) to prove Turán's Theorem by induction on n, including the characterization of graphs achieving the bound.

2. *Partial analogue of Turán's Theorem for $K_{2,m}$.*

a) Let G be an n-vertex graph such that $\sum_{v\in V(G)} \binom{d(v)}{2} > (m-1)\binom{n}{2}$. Prove that G contains $K_{2,m}$. (Hint: View $K_{2,m}$ as two vertices with m common neighbors.)

b) Prove that $\sum_{v\in V(G)} \binom{d(v)}{2} \ge e(2e/n - 1)$, where G has e edges.

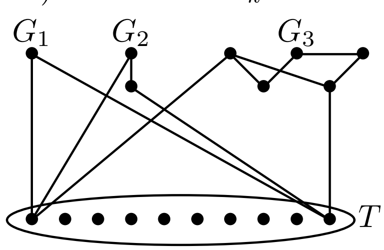
c) Use parts (a) and (b) to prove that a graph with more than $\frac{1}{2}(m-1)^{1/2}n^{3/2} + n/4$ edges contains $K_{2,m}$.

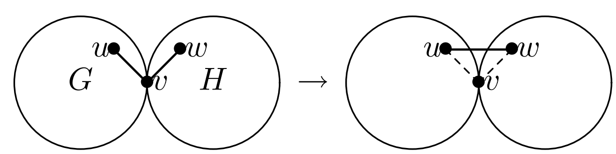
d) Application: Given n points in the plane, prove that the distance is exactly 1 for at most $\frac{1}{\sqrt{2}}n^{3/2} + n/4$ pairs.

3. Let $G_1 = K_1$. For $k > 1$, construct G_k as follows. To the disjoint union $G_1 + \cdots + G_{k-1}$, and add an independent set T of size $\prod_{i=1}^{k-1} n(G_i)$. For each choice of $(v_1, \ldots, v_{k-1})$ in $V(G_1) \times \cdots \times V(G_{k-1})$, let one vertex of T have neighborhood $\{v_1, \ldots, v_{k-1}\}$. (In the sketch of G_4 on the left below, neighbors are shown for only two elements of T.)

a) Prove that $\omega(G_k) = 2$ and $\chi(G_k) = k$.

b) Prove that G_k is k-critical.





4. *The Hajós construction.*

a) For $k \ge 3$, let G and H be k-critical graphs sharing only vertex v, with $vu \in E(G)$ and $vw \in E(H)$. Let F be the graph formed from $G \cup H$ by deleting the edges vu and vw and adding the edge uw (shown on the right above). Prove that F is k-critical.

b) For all $k \ge 3$, use part (a) to obtain a k-critical graph other than K_k.

c) For all $n \ge 4$ except $n = 5$, construct a 4-critical graph with n vertices. (Hint: Part (a) uses the properties of k-critical graphs for G and H. In part (c), one can give explicit examples with orders $4, 6, 8$ and then apply part (a).)

5. Let G be a claw-free graph (no induced $K_{1,3}$).

a) Prove that the subgraph induced by the union of any two color classes in a proper coloring of G consists of paths and even cycles.

b) Prove that if G has a proper coloring using exactly k colors, then G has a proper k-coloring where the color classes differ in size by at most one.

6. Let $m = k(k+1)/2$. Prove that $K_{m,m-1}$ has no K_{2k}-subdivision.