MATH 412, FALL 2012 - HOMEWORK 12

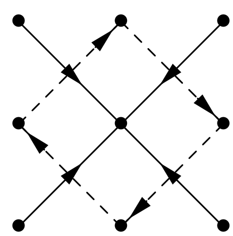
WARMUP PROBLEMS: Section 6.1 #1–5, 7, 8, 11. Section 6.2 #1–4. Do not write these up! Use them to check your understanding.

EXTRA PROBLEMS: Section 6.1: #13, 14, 15, 17, 20, 21, 24, 25, 27, 30. Section 6.2: #5, 6, 7. Do not write these up!

WRITTEN HOMEWORK: Do five of the following six problems. Due Wednesday, November 28.

1. Prove that a set of edges in a connected plane graph G forms a spanning tree of G if and only if the duals of the remaining edges form a spanning tree of G^*. Use this statement to prove Euler's Formula.

2. *Directed plane graphs.* Let G be a plane graph, and let D be an orientation of G. The *dual digraph* D^* is an orientation of the dual graph G^* such that when an edge of D is traversed from tail to head, the dual edge in D^* crosses it from right to left. For example, below we show part of D in solid edges: four edges entering a central vertex. If there are four faces incident to this vertex, then the corresponding edges in D^* form the cycle shown in dashed edges.



Prove that if D is strongly connected, then D^* has no cycle, and $\delta^-(D^*) = \delta^+(D^*) = 0$. Use this to prove that if D is strongly connected, then D has a face on which the edges form a clockwise cycle and another face on which the edges form a counterclockwise cycle.

3. For $n \geq 2$, determine the maximum number of edges in a simple outerplane graph with n vertices, giving three proofs.

a) By induction on n.

b) By using Euler's Formula.

c) By adding a vertex in the unbounded face and using Theorem 6.1.23.

4. Use Euler's Formula to count the bounded regions formed by the diagonals of a convex n-gon, assuming that no three diagonals meet at a point. (Hint: Begin by finding a simple formula in terms of n for the number of intersections of diagonals.)

5. Use Kuratowski's Theorem to prove that G is outerplanar if and only if it has no subgraph that is a subdivision of K_4 or $K_{2,3}$. (Hint: To apply Kuratowski's Theorem, consider an appropriate modification of G. This is much easier than trying to mimic a proof of Kuratowski's Theorem.)

6. Wagner [1937] proved that a graph G is planar if and only if neither K_5 nor $K_{3,3}$ can be obtained from G by performing deletions and/or edge-contractions.

a) Show that deletion and contraction of edges (and deletion of vertices) preserve planarity. Conclude from this that Wagner's condition is necessary.

b) Use Kuratowski's Theorem to prove that Wagner's condition is sufficient.