MATH 412, FALL 2012 - HOMEWORK 14

WARMUP PROBLEMS: Section 7.1 #29. Section 7.2 #1, 2, 3, 4 Do not write these up! Use them to check your understanding.

EXTRA PROBLEMS: Section 7.1: #31, 33, 35 Section 7.2: #6, 7, 8, 9, 11, 13, 15, 17, 25, 26, 29. Do not write these up!

WRITTEN HOMEWORK: Do five of the following six. Due Wednesday, December 12.

1. Let G be a bipartite graph. Prove that G has a k-edge-coloring in which at each vertex v, each color appears $\lceil d(v)/k \rceil$ or $\lfloor d(v)/k \rfloor$ times. (Hint: Use a graph transformation.)

2. Use Petersen's Theorem (every $2k$-regular graph has a 2-factor—Theorem 3.3.9) to prove that $\chi'(G) \leq 3\lceil \Delta(G)/2 \rceil$ when G is a loopless graph.

3. *Hamiltonian vs. Eulerian.*

a) Find a 2-connected non–Eulerian graph whose line graph is Hamiltonian.

b) For a graph G, prove that $L(G)$ is Hamiltonian if and only if G has a closed trail that contains at least one endpoint of each edge.

4. A graph G is *uniquely* k-edge-colorable if all proper k-edge-colorings of G induce the same partition of the edges.

a) Prove that every uniquely 3-edge-colorable 3-regular graph is Hamiltonian.

b) Give an example of a Hamiltonian 3-regular graph that is not uniquely 3-edge-colorable.

5. The kth *power* of a simple graph G is the simple graph G^k with vertex set $V(G)$ and edge set $\{uv\colon d_G(u,v) \leq k\}$.

a) Suppose that $G - x$ has at least three nontrivial components in each of which x has exactly one neighbor. Prove that G^2 is not Hamiltonian. (Hint: Consider the second graph in Example 7.2.5.)

b) Prove that the cube of each connected graph (with at least three vertices) is Hamiltonian. (Hint: Reduce this to the special case of trees, and prove it for trees by proving the stronger result that if xy is an edge of the tree T, then T^3 has a Hamiltonian cycle using the edge xy.)

6. *Sufficient condition for Hamiltonian-connected.*

a) Prove that a simple graph G is Hamiltonian-connected if $x \not\leftrightarrow y$ implies $d(x) + d(y) > n(G)$. (Hint: Prove that appropriate graphs related to G are Hamiltonian by considering their closures.)

b) Prove that part (a) is sharp by constructing, for each even n greater than 2, a simple n-vertex graph with minimum degree $n/2$ that is not Hamiltonian-connected.