MATH 412, SPRING 2013 - HOMEWORK 3

WARMUP PROBLEMS: Section 1.3 #8, 46. Section 1.4 #1, 3, 4, 5, 8, 10. Do not write these up! Use these to clarify your understanding.

EXTRA PROBLEMS: Section 1.3 #18, 25, 26, 31, 40, 41, 45, 47, 49, 52, 53, 57, 59, 61, 64. Section 1.4 #11, 14, 20, 21, 23, 25, 27, 28, 29, 32, 35, 36, 37, 40. Do not write these up! Think about some if you have time.

WRITTEN PROBLEMS: Solve and write up five of the following six (students registered for four credits or honors do all six problems). Due Wednesday, February 6.

COMMENT: When using induction to prove an implication, keep the template of Remark 1.3.25 in mind. Induction is useful when the current instance of the problem can be solved by using a solution to a smaller instance.

1. *Counting subgraphs.*

a) Count the n-cycles in K_n.

a) Count the $2n$-cycles in $K_{n,n}$.

a) Count the 6-cycles in $K_{m,n}$.

2. Let G be an n-vertex simple graph, where $n \geq 2$. Determine the maximum possible number of edges in G under each of the following conditions.

a) G has an independent set of size a.

b) G has exactly k components.

c) G is disconnected.

3. *Stronger version of Mantel's Theorem.*

a) For a vertex v in a graph G, let $f(v)$ be the maximum size of an independent set contained in $N(v)$. Prove that $\sum_{v \in V(G)} f(v) \leq \lfloor n(G)^2/2 \rfloor$, and determine for which graphs equality holds. (Hint: Consider a maximum independent set in G.)

b) Use part (a) to obtain Theorem 1.3.23 (Mantel's Theorem).

4. Let $d_1, \ldots, d_n$ be the vertex degrees of a simple graph G, indexed so that $d_1 \leq \cdots \leq d_n$. Prove that G is connected if $d_j \geq j$ when $j \leq n - 1 - d_n$. (Hint: Consider a component that omits some vertex of maximum degree.)

5. Given an alphabet of size k, prove that there exists a cyclic arrangement of $2k^{l-1}$ letters from the alphabet such that all the strings of l consecutive letters are distinct and all possible strings of $l-1$ consecutive letters appear.

6. Let T be an n-vertex tournament.

a) Prove that the number of 3-cycles in T is $\binom{n}{3} - \sum_{v \in V(T)} \binom{d^+(v)}{2}$.

b) Given that n is odd, determine the maximum possible number of 3-cycles in T.