MATH 412, SPRING 2013 - HOMEWORK 4

WARMUP PROBLEMS: Section 2.1 #3, 4, 8, 10, 11, 13. Section 2.2 #1, 2, 4, 5. Do not write these up!

EXTRA PROBLEMS: Section 2.1 #17, 19, 21, 23, 25, 37, 39, 47, 55, 62, 63, 66. Section 2.2 #6, 7, 11, 12, 15, 20, 38. Do not write these up!

WRITTEN PROBLEMS: Solve and write up five of the following six. Due Wednesday, February 13.

1. Let $d_1, \ldots, d_n$ be positive integers, with $n \geq 2$. Prove that there is a tree with vertex degrees $d_1, \ldots, d_n$ if and only if $\sum d_i = 2n - 2$.

2. *Diameter and radius.*

a) Prove that the distance function $d(u, v)$ on pairs of vertices of a graph satisfies the triangle inequality: $d(u, v) + d(v, w) \geq d(u, w)$.

b) Use part (a) to prove that $\operatorname{diam} G \leq 2\operatorname{rad} G$ for every graph G.

c) For all positive integers r and d that satisfy $r \leq d \leq 2r$, construct a simple graph with radius r and diameter d. (Hint: Build a suitable graph with one cycle.)

d) Let G be a simple graph. Prove that $\operatorname{rad} G \geq 3$ implies $\operatorname{rad} \overline{G} \leq 2$.

3. Prove that the isomorphism class of a tree is determined by the distances among its leaves. That is, up to isomorphism, a tree S with leaves $\{x_1, \ldots, x_k\}$ is determined by the values of $d_S(x_i, x_j)$ for all i and j.

4. *Distance and hypercubes.*

a) Determine the average distance between points in the k-dimensional hypercube Q_k (averaged over all pairs of points).

b) Determine the minimum radius among all spanning trees of Q_k.

5. Compute $\tau(K_{2,m})$. Also compute the number of isomorphism classes of spanning trees of $K_{2,m}$.

6. Use the Matrix Tree Theorem to compute $\tau(K_{r,s})$.