MATH 412, SPRING 2013 - HOMEWORK 5

WARMUP PROBLEMS: Section 2.1 #33 Section 2.3 #2, 3, 4, 5, 22, 24, 28. Do not write these up!

EXTRA PROBLEMS: Section 2.2 #23, 24, 25, 26. Section 2.3 #6, 8, 9, 11, 12, 13, 17, 18, 19, 20, 22, 29. Do not write these up!

WRITTEN PROBLEMS: Five of the following six. Due Wednesday, February 20.

1. Use the method of Prufer codes to show that the number of rooted forests with vertex set $[n]$ whose set of roots is $[k]$ is kn^{n-k-1}. That is, let S be the set of rooted forests on $[n]$ with root set $[k]$. For $F \in S$, form a list by iteratively deleting the largest (nonroot) leaf and recording its neighbor, until only the roots remain. Prove that this establishes a bijection from S to $[n]^{n-k-1}[k]$.

2. An *up/down labeling* is a graceful labeling for which there exists a *critical value* α such that every edge joins vertices with labels above and below α. Prove that every caterpillar has an up/down labeling. Prove that the 7-vertex tree that is not a caterpillar has no up/down labeling.

3. *Prim's Algorithm* grows a spanning tree from a given vertex of a connected weighted graph G, iteratively adding the cheapest edge from a vertex already reached to a vertex not yet reached, finishing when all the vertices of G have been reached. (Ties are broken arbitrarily.) Prove that Prim's Algorithm produces a minimum-weight spanning tree of G.

4. Let G be a weighted graph.

a) Let T be a minimum-weight spanning tree. Prove that T omits some heaviest edge from every cycle in G.

b) Let C be a cycle in G, and let e be an edge of maximum weight on C. Prove that there is a minimum-weight spanning tree not containing e.

c) Use part (b) to prove that iteratively deleting a heaviest non-cut-edge until the remaining graph is acyclic produces a minimum-weight spanning tree.

5. *Breadth-First Search.*

a) Explain how to use BFS to compute the girth of a graph.

b) Explain how to use BFS to quickly test whether a graph is bipartite. The algorithm should not need to consider an edge more than twice.

6. Consider n messages occurring with probabilities $p_1, \ldots, p_n$, such that each p_i is a power of $1/2$ (each $p_i \geq 0$ and $\sum p_i = 1$).

a) Prove that the two least likely messages have equal probability.

b) Prove that the expected message length of the Huffman code for this distribution is $-\sum p_i \lg p_i$.