MATH 412, SPRING 2013 - HOMEWORK 6

WARMUP PROBLEMS: Section 3.1 #1, 2, 4, 7, 40. Section 3.2 #1, 2. Do not write these up!

EXTRA PROBLEMS: Section 3.1 #8, 9, 11, 16, 18, 19, 21, 24, 25, 30, 31, 32, 39, 42, 43, 44, 45. Section 3.2 #6, 7, 8, 11, 12, 13. Do not write these up!

WRITTEN PROBLEMS: Five of the following six. Due Wednesday, February 20.

1. Let G be a k-regular bipartite graph, with $k \geq 2$. Prove that every edge of G appears in some perfect matching in G. Prove sharpness for $k \geq 3$ by constructing a k-regular bipartite graph having two independent edges that do not appear together in any perfect matching in G.

2. *Stronger versions of Hall's Theorem.* Let G be an nontrivial X, Y-bigraph satisfying Hall's Condition for X. Without assuming Hall's Theorem, prove the following.

(a) There is a vertex x in X such that every edge incident with x belongs to some matching that covers X. (Hint: Use induction on $|X|$.)

(b) If $d(x) \geq k$ for all x in X, and $|X| \geq k-1$, then there are at least $k!$ matchings that cover X.

3. Use the König–Egerváry Theorem to prove that every bipartite graph G has a matching of size at least $e(G)/\Delta(G)$. Use this to conclude that every subgraph of $K_{n,n}$ with more than $(k-1)n$ edges has a matching of size at least k.

4. An algorithm to greedily build a large independent set S iteratively selects a vertex of minimum degree in the remaining graph, adds it to S, and deletes it and its neighbors from the graph. Prove that this algorithm produces an independent set of size at least $\sum_{v \in V(G)} \frac{1}{d_G(v)+1}$ in a simple graph G.

5. Find a minimum-weight transversal in the matrix below, and use duality to prove that the solution is optimal. (Hint: Use a transformation of the problem.)

$$\begin{pmatrix} 5 & 3 & 2 & 8 & 1 \\ 6 & 4 & 7 & 1 & 2 \\ 4 & 3 & 5 & 8 & 3 \\ 2 & 0 & 1 & 6 & 3 \\ 6 & 4 & 3 & 7 & 3 \end{pmatrix}$$

6. Let the entries in matrix A have the form $w_{i,j} = a_i b_j$, where $a_1, \ldots, a_n$ are numbers associated with the rows and $b_1, \ldots, b_n$ are numbers associated with the columns. Determine the maximum weight of a transversal of A. What happens when $w_{i,j} = a_i + b_j$? (Hint: In each case, guess the general pattern by examining the solution when $n = 2$.)