MATH 412, SPRING 2013 - HOMEWORK 7

WARMUP PROBLEMS: Section 3.2 #9, 10. Section 3.3 #1, 2, 4, 6. Section 4.1 #1, 2, 4, 5. Do not write these up!

EXTRA PROBLEMS: Section 3.2 #12, 13. Section 3.3 #7, 10, 13–16, 18–20, 23, 24, 26. Section 4.1 #8, 9, 10, 11, 12, 17, 20. Do not write these up!

WRITTEN PROBLEMS: Five of the following six. Due Wednesday, March 6.

1. Prove that if man x is paired with woman a in some stable matching, then a does not reject x when the Gale–Shapley Proposal Algorithm is run with men proposing. Conclude that among all stable matchings, *every* man is as happy in the matching produced by this algorithm as in any stable matching. (Hint: Consider the first occurrence of such a rejection.)

2. *Extension of König–Egerváry Theorem to general graphs.* Given a graph G, let $S_1, \ldots, S_k$ and T be subsets of $V(G)$ such that each S_i has odd size. These sets form a *generalized cover* of G if every edge of G has at least one endpoint in T or both endpoints in some S_i. The *weight* of a generalized cover is $|T| + \sum \lfloor |S_i| /2 \rfloor$. Let $\beta^*(G)$ be the minimum weight of a generalized cover. Prove that $\alpha'(G) = \beta^*(G)$. (Hint: Apply Corollary 3.3.7. Comment: Every vertex cover is a generalized cover, so $\beta^*(G) \leq \beta(G)$.)

3. Let G be a k-regular graph of even order that remains connected when any $k-2$ edges are deleted. Use Remark 3.3.5 to prove that every edge of G belongs to some 1-factor. (Comment: This strengthens Exercise 3.3.16.)

4. Prove that a 3-regular simple graph has a 1-factor if and only if it decomposes into copies of P_4.

5. Let M be a matching in a graph G, and let u be an M-unsaturated vertex. Prove that if G has no M-augmenting path that starts at u, then u is unsaturated in some maximum matching in G.

6. *Connectivity.* Let G be an n-vertex graph.

a) Prove that $\kappa(G) = \delta(G)$ if G is simple and $\delta(G) \geq n-2$. Prove that this is best possible for each $n \geq 4$ by constructing a simple n-vertex graph with minimum degree $n-3$ and connectivity less than $n-3$.

b) Prove that if $\delta(G) = n-3$ and $\overline{G}$ contains no 4-cycle, then $\kappa(G) = n-3$.