MATH 412, Spring 2012 - Homework 8

WARMUP PROBLEMS: Section 4.1: #6, 7, 30. Section 4.2: #1, 2, 4, 6. Do not write these up!
EXTRA PROBLEMS: Section 4.1: # 31, 32. Section 4.2: # 7, 8, 11, 12, 13, 14, 15, 19, 20, 21, 23, 27, 32, 35, 36, 37. Do not write these up!
WRITTEN PROBLEMS: Do five of the following six. Due Wednesday, March 13.

1. $\kappa'(G) = \delta(G)$ *for diameter 2.* Let G be a simple graph with diameter 2, and let $[S, \overline{S}]$ be a minimum edge cut with $|S| \le |\overline{S}|$.
a) Prove that every vertex of S has a neighbor in $\overline{S}$.
b) Use part (a) and Corollary 4.1.13 to prove that $\kappa'(G) = \delta(G)$.

2. (*The block-cutpoint graph*). Let H be the block-cutpoint graph of a graph G that has a cut-vertex.
a) Prove that H is a forest.
b) Prove that G has at least two blocks each of which contains exactly one cut-vertex of G.
c) Prove that a graph G with k components has exactly $k + \sum_{v \in V(G)} (b(v) - 1)$ blocks, where $b(v)$ is the number of blocks containing v.
d) Prove that every graph has fewer cut-vertices than blocks.

3. Let G be a 2-connected graph. Prove that if T_1 and T_2 are two spanning trees of G, then T_1 can be transformed into T_2 by a sequence of operations in which a leaf is removed and reattached using another edge of G.

4. Suppose that $\kappa(G) = k$ and $\operatorname{diam} G = d$. Prove that $n(G) \ge k(d-1) + 2$ and $\alpha(G) \ge \lceil (1+d)/2 \rceil$. For each $k \ge 1$ and $d \ge 2$, construct a graph for which equality holds in both bounds (simultaneously).

5. Let X and Y be disjoint sets of vertices in a k-connected graph G. Let $u(x)$ for $x \in X$ and $w(y)$ for $y \in Y$ be nonnegative integers such that $\sum_{x \in X} u(x) = \sum_{y \in Y} w(y) = k$. Prove that G has k pairwise internally disjoint X, Y-paths so that $u(x)$ of them start at $x \in X$ and $w(y)$ of them end at $y \in Y$.

6. Prove that applying the expansion operation of Example 1.3.26 to a 3-connected graph yields a 3-connected graph. Obtain the Petersen graph from K_4 by expansions. (Comment: Tutte prove that a 3-regular graph is 3-connected if and only if it arises from K_4 by a sequence of these operations.)