MATH 412, SPRING 2013 - HOMEWORK 9

WARMUP PROBLEMS: Section 4.3 #1, 2, 3, 4. Section 5.1 #2, 3, 5. Do not write up!

EXTRA PROBLEMS: Section 4.3 #5, 6, 7, 10, 11, 13. Section 5.1 #22, 23, 29, 33. Do not write these up!

WRITTEN PROBLEMS: Write up five of the following six. Due Wed., March 27.

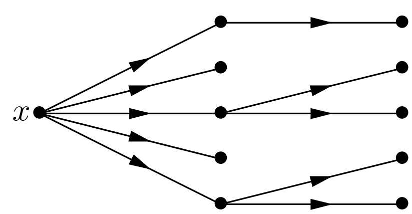
1. Use network flows to prove Menger's Theorem for internally-disjoint paths in digraphs: $\kappa(x, y) = \lambda(x, y)$ when xy is not an edge. (Hint: Use the first transformation suggested in Remark 4.3.15.)

2. A large university with $3k$ academic departments must appoint an important committee. One professor will be chosen from each department. Some professors have joint appointments in two or more departments, but each must be the designated representative of at most one department. We must use k assistant professors, k associate professors, and k full professors (each professor has only one rank). How can the committee be found? (Hint: Build a network where units of flow correspond to professors chosen for the committee and capacities enforce the various constraints. Explain how to use the network to test whether such a committee exists and to find it if it does.)

3. Let G be a weighted graph. Let the *value* of a spanning tree be the minimum weight of its edges. Let the *cap* from an edge cut $[S, \overline{S}]$ be the maximum weight of its edges. Prove that in G the maximum value of a spanning tree equals the minimum cap of an edge cut.

4. Use network flows to prove Hall's Theorem.

5. Let x be a vertex of maximum outdegree in a tournament T. Prove that T has a spanning directed tree rooted at x such that every vertex has distance at most 2 from x and every vertex other than x has outdegree at most 2. (Hint: Create a network to model the desired paths to the non-successors of x, and show that every cut has enough capacity. Comment: This strengthens Proposition 1.4.30 about kings in tournaments.)



6. Let G be a graph whose odd cycles are pairwise intersecting, meaning that every two odd cycles in G have a common vertex. Prove that $\chi(G) \leq 5$. Construct a graph to show that the bound cannot be improved.