MATH 412, SPRING 2013 - HOMEWORK 10

WARMUP PROBLEMS: Section 5.1 #7, 12, 21. Section 5.2 #1, 2.

EXTRA PROBLEMS: Section 5.1 #22, 23, 29, 32, 38, 39, 41, 47, 48, 51, 54. Section 5.2 #6, 7, 9, 14, 17, 21. Do not write these up!

WRITTEN HOMEWORK: Do five of the following six. Due Wednesday, April 3.

1. a) Prove that every graph G has a vertex ordering relative to which greedy coloring uses $\chi(G)$ colors.

b) For all $k \in \mathbb{N}$, construct a tree T_k with maximum degree k and an ordering σ of $V(T_k)$ such that greedy coloring relative to the ordering σ uses $k+1$ colors. (Hint: Use induction and construct the tree and ordering simultaneously. Comment: Thus the performance ratio of greedy coloring to optimal coloring can be as bad as $(\Delta(G)+1)/2$.)

2. The Kneser graph $K(n,k)$ is the disjointness graph of the k-element subsets of $[n]$. That is, the vertex set consists of all k-element subsets of $[n]$, and two vertices are adjacent if they are disjoint k-sets. For example, the Petersen graph is $K(5,2)$. Prove that $\chi(K(n,k))n-2k+2$ by covering the vertices with $n-2k+2$ independent sets. Prove that this is optimal when $n=2k+1$. (Comment: Lovász [1978] proved Kneser's conjecture that always $\chi(K(n,k)) = n-2k+2$.)

3. a) Prove that $\chi(G) + \chi(\overline{G}) \le n(G)+1$ for every graph G.

b) Let G be an n-vertex graph, and let $c = (n+1)/\alpha(G)$. Use part (a) to prove that $\chi(G)\cdot\chi(\overline{G}) \le (n+1)^2/4$, and use this to prove that $\chi(G) \le c(n+1)/4$. For each odd n, construct a graph such that $\chi(G) = c(n+1)/4$.

4. *Improvement of Brooks' Theorem.*

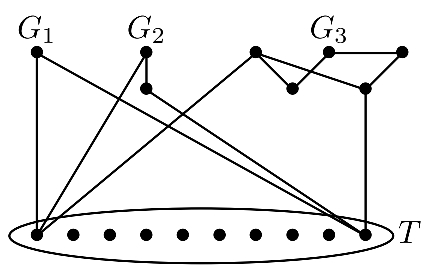
a) Let $k_1,\dots,k_t$ be positive integers with sum k. Prove that the vertex set of any graph G with $\Delta(G) < k$ can be partitioned into sets $V_1,\dots,V_t$ such that $\Delta(G[V_i]) < k_i$ for $1 \le i \le t$. (Hint: Prove that the partition minimizing $\sum e(G_i)/k_i$ works.)

b) For $4 \le r \le \Delta(G)+1$, use part (a) to prove that $\chi(G) \le \lceil \frac{r-1}{r}(\Delta(G)+1)\rceil$ when G has no r-clique.

5. Let $G_1 = K_1$. For $k>1$, construct G_k as follows. To the disjoint union $G_1+\cdots+G_{k-1}$, and add an independent set T of size $\prod_{i=1}^{k-1} n(G_i)$. For each choice of $(v_1,\dots,v_{k-1})$ in $V(G_1)\times\cdots\times V(G_{k-1})$, let one vertex of T have neighborhood $\{v_1,\dots,v_{k-1}\}$. (In the sketch of G_4 below, neighbors are shown for only two elements of T.)

a) Prove that $\omega(G_k) = 2$ and $\chi(G_k) = k$.

b) Prove that G_k is k-critical.



6. *Turán's proof of Turán's Theorem*, including uniqueness.

a) Prove that a maximal simple graph having no $r+1$-clique has an r-clique.

b) Prove that $e(T_{n,r}) = \binom{r}{2} + (n-r)(r-1) + e(T_{n-r,r})$.

c) Use parts (a) and (b) to prove Turán's Theorem by induction on n, including the characterization of graphs achieving the bound.