An example of the revised 2-phase simplex method

Suppose we are given the problem

$$\text{Minimize } z = -19x_1 - 13x_2 - 12x_3 - 17x_4$$

subject to

$$\begin{cases} 3x_1 & +2x_2 & +x_3 & +2x_4 & = & 225, \\ x_1 & +x_2 & +x_3 & +x_4 & = & 117, \\ 4x_1 & +3x_2 & +3x_3 & +4x_4 & = & 420 \\ x_1, & x_2, & x_3, & x_4 & \geq & 0. \end{cases} \tag{1}$$

Add to each of the equations its own variable y_i and consider the auxiliary problem of the minimization of $\xi = y_1 + y_2 + y_3$. Subtracting each equation from Row 0 we get the following tableau.

		y_1	y_2	y_3	x_1	x_2	x_3	x_4
$y_0 = -\xi$	-762	0	0	0	-8	-6	-5	-7
y_1	225	1	0	0	3	2	1	2
y_2	117	0	1	0	1	1	1	1
y_3	420	0	0	1	4	3	3	4

The first four columns of this tableau form our matrix CARRY-0. Following Bland's Rule, the pivot column corresponds to x_1. The best ratio is in Row 1. Pivoting, we calculate only elements in the first four columns. Our CARRY-1 is

		y_1	y_2	y_3
$y_0 = -\xi$	-162	$8/3$	0	0
x_1	75		0	0
y_2	42	$-1/3$	1	0
y_3	120	$-4/3$	0	1

Now we calculate $\widetilde{d}_j$ using the formula

$$\widetilde{d}_j = d_j - \pi^T A_j, \tag{2}$$

where $-\pi^T$ is the vector in the last 3 entries of Row 0, and A_j is the jth column of the original matrix A. Since x_1 is in the basis, we first try $\widetilde{d}_2$:

$$\widetilde{d}_2 = -6 + (8/3,\ 0,\ 0) \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = -6 + 16/3 = -2/3 < 0.$$

So we will pivot on x_2. We calculate the column $\widetilde{A}_2$ using the formula

$$\widetilde{A}_j = B^{-1} A_j, \tag{3}$$

where B^{-1} is formed by the last 3 columns and 3 rows of the last tableau. We have $\widetilde{A}_2 = \begin{pmatrix} 1/3 & 0 & 0 \\ -1/3 & 1 & 0 \\ -4/3 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2/3 \\ 1/3 \\ 1/3 \end{pmatrix}$. Adding column $\begin{pmatrix} -2/3 \\ 2/3 \\ 1/3 \\ 1/3 \end{pmatrix}$ to the last tableau and pivoting on the first row we get CARRY-2:

		y_1	y_2	y_3
$y_0 = -\xi$	-87	3	0	0
x_2	$225/2$	$1/2$	0	0
y_2	$9/2$	$-1/2$	1	0
y_3	$165/2$	$-3/2$	0	1

Since x_2 is in the basis and x_1 just got out of it, we first calculate $\widetilde{d}_3$:

$$\widetilde{d}_3 = -5 + (3,\, 0,\, 0) \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = -2 < 0.$$

Then similarly to above $\widetilde{A}_3 = \begin{pmatrix} 1/2 & 0 & 0 \\ -1/2 & 1 & 0 \\ -3/2 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 3/2 \end{pmatrix}$. Adding column $\begin{pmatrix} -2 \\ 1/2 \\ 1/2 \\ 3/2 \end{pmatrix}$ to the last tableau and pivoting on the second row we get CARRY-3:

		y_1	y_2	y_3
$y_0 = -\xi$	-69	1	4	0
x_2	108	1	-1	0
x_3	9	-1	2	0
y_3	69	0	-3	1

Now we should check $\widetilde{d}_1$ again and it turns out to be negative:

$$\widetilde{d}_1 = -8 + (1,\, 4,\, 0) \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = -1 < 0.$$

Then $\widetilde{A}_1 = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & 0 \\ 0 & -3 & 1 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$. Adding column $\begin{pmatrix} -1 \\ 2 \\ -1 \\ 1 \end{pmatrix}$ to the last tableau and pivoting on the first row we get CARRY-4:

		y_1	y_2	y_3
$y_0 = -\xi$	-15	$3/2$	$7/2$	0
x_1	54	$1/2$	$-1/2$	0
x_3	63	$-1/2$	$3/2$	0
y_3	15	$-1/2$	$-5/2$	1

Note that x_1 first entered the basis, then exited it, and now entered again. Since x_1 and x_3 are in the basis and x_2 just got out of it, we calculate $\widetilde{d}_4$:

$$\widetilde{d}_4 = -7 + (3/2,\, 7/2,\, 0) \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = -1/2 < 0.$$

Then $\widetilde{A}_4 = \begin{pmatrix} 1/2 & -1/2 & 0 \\ -1/2 & 3/2 & 0 \\ -1/2 & -5/2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$. Adding column $\begin{pmatrix} -1/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}$ to the last tableau and pivoting on the last row we get CARRY-5:

		y_1	y_2	y_3
$y_0 = -\xi$	0	1	1	1
x_1	39	1	2	-1
x_3	48	0	4	-1
x_4	30	-1	-5	2

Thus we found a basic feasible solution of the original problem. Now we replace the row $-\mathbf{d}_B^T B^{-1}$ (the last 3 entries) by the row

$$-\mathbf{c}_B^T B^{-1} = -(-19,\, -12,\, -17) \begin{pmatrix} 1 & 2 & -1 \\ 0 & 4 & -1 \\ -1 & -5 & 2 \end{pmatrix} = (2,\, 1,\, 3).$$

Also, the current value of $-z$ is

$$-\mathbf{c}_B^T B^{-1}\mathbf{b} = (2,\, 1,\, 3) \begin{pmatrix} 225 \\ 117 \\ 420 \end{pmatrix} = 1827.$$

Hence, our CARRY-6 is

		y_1	y_2	y_3
$x_0 = -z$	1827	2	1	3
x_1	39	1	2	-1
x_3	48	0	4	-1
x_4	30	-1	-5	2

Now check $\widetilde{c}_1 = -19 + (2,\ 1,\ 3)\begin{pmatrix} 3 \\ 1 \\ 4 \end{pmatrix} = 0$, $\widetilde{c}_2 = -13 + (2,\ 1,\ 3)\begin{pmatrix} 2 \\ 1 \\ 3 \end{pmatrix} = 14 > 0$, $\widetilde{c}_3 = -12 + (2,\ 1,\ 3)\begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} = 0$, and $\widetilde{c}_4 = -17 + (2,\ 1,\ 3)\begin{pmatrix} 2 \\ 1 \\ 4 \end{pmatrix} = 0$. Since there are no negative $\widetilde{c}_j$, the optimal value is -1827 attained at $(39,\ 0,\ 48,\ 30)$.