MATH 482, Spring 2013 - Homework 1
Solutions

Solve 4 of the first 5 problems below, and also solve problem 6. Students registered for 4 credits must solve all problems.

1. [5pts] A paper recycling machine can produce toilet paper, writing pads, and paper towels, which sell for 18, 29 and 25 cents and consume 0.5, 0.22 and 0.75 kilograms of newspaper and 0.2, 0.4, and 0.22 minutes. Each day 10 hours and 1500 kilograms of newspaper are available, and at least 1000 rolls of toilet paper, 200 writing pads and 400 rolls of paper towels are required. Formulate an appropriate LP to maximize revenue.

Solution: Let x_1 be the number of rolls of toilet paper, x_2 be the number of writing pads, and x_3 be the number of rolls of paper towels.

$$
\begin{array}{rrrrcr}
\text{max} & 18x_1 & +29x_2 & +25x_3 & & \\
\text{subject to} & 0.5x_1 & +0.22x_2 & +0.75x_3 & \leq & 1500 \\
 & x_1 & & & \geq & 1000 \\
 & & x_2 & & \geq & 200 \\
 & & & x_3 & \geq & 400 \\
 & x_1, & x_2, & x_3 & \geq & 0
\end{array}
$$

2. [5pts] You have 100 quarters and 90 dimes and no other money. You have to pay a given amount C. No change is given to you. You want to minimize your overpay. State this problem in mathematical terms. Is this problem linear? Solve it for $C = 15$ cents, for $C = \$1.02$, and for $C = \$50$.

Let q be the number of quarters to pay, and d be the number of dimes to pay. Since I can't split change (or steal change), q and d are nonnegative integers. Consider the integer program

$$
\begin{array}{rrrcr}
\text{min} & q & +d & & \\
\text{subject to} & 0.25q & +0.10d & \geq & C \\
 & q & & \leq & 100 \\
 & & d & \leq & 90 \\
 & q, & d & \in & \mathbb{Z}_{\geq 0}
\end{array}
$$

The constraints are as follows: (1) we must pay at least C dollars. (2) We can use at most 100 quarters, (3) We can use at most 90 dimes (4) Quarters and dimes are non-negative integers. Since we MUST pay at least C, we minimize overpay simply by minimizing our total pay.
The problem is NOT linear, since variables are integer. (The CONSTRAINTS are linear.)
When $C = 0.15$, the unique solution is $(q, d) = (0, 2)$. We cannot do better, since paying only a single quarter is too much, but paying fewer dimes is not enough.
When $C = 1.02$, the possible solutions are $(q, d) \in \{(3, 3), (1, 8)\}$. Since the gcd of 25 and 10 is 5, we can only pay amounts that are multiples of 5. The multiple of five with least overpay from \$1.02 is \$1.05, which is attainable by the above solutions.
When $C = 50$, observe that since $q \leq 100$ and $d \leq 90$, we have $0.25q + 0.10d \leq 25 + 9 = 34$. Thus, the problem is infeasible. We don't have enough money!

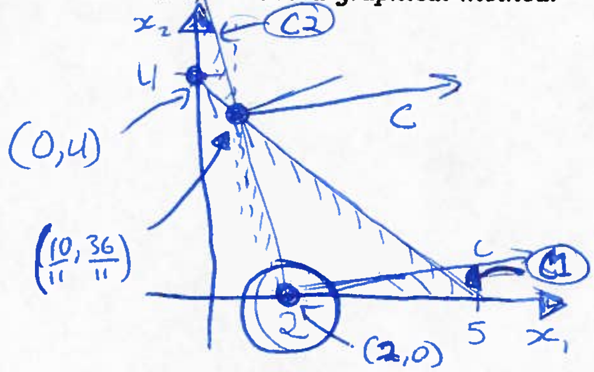
3. [5pts] Solve the problem:

$$\max 6x_1 + x_2$$

with respect to

$$\begin{array}{rrcl} 4x_1 & +5x_2 & \le & 20, \\ 3x_1 & +x_2 & \le & 6, \\ & x_1, x_2 & \ge & 0. \end{array}$$

Solution 1: *Use a graphical method.*



$-(4x_1 + 5x_2 = 20)$
$+(15x_1 + 5x_2 = 30)$

$11x_1 + 0x_2 = 10$

$x_1 = \frac{10}{11}, \quad x_2 = 6 - 3\frac{10}{11} = \frac{36}{11}$

Optimum: $(x_1, x_2) = (2, 0)$ w/ value 12.

Solution 2: *Use the simplex method.*

Requires standard form:

$$\begin{array}{lcl} 4x_1 + 5x_2 + 1x_3 & & = 20 \\ 3x_1 + x_2 & + x_4 & = 6 \\ x_1, x_2, x_3, x_4 & & \ge 0. \end{array}$$

Start w/ x_3, x_4 in basis,
pivot ~~so~~ x_1 enters basis. Optimal!
(Details omitted)

4. [5pts] State in the canonical form:

$$\min 2x_1 - 3x_2 + x_3$$

with respect to

$$\begin{array}{rrrrcl} 3x_1 & -2x_2 & +x_3 & & \le & -5, \\ -x_1 & -3x_2 & +4x_3 & & \le & -9, \\ x_1 & +x_2 & & +x_4 & = & 6, \\ & & x_1, & x_3 & \ge & 0. \end{array}$$

Solution:

$$\begin{array}{rrrrrrrcl} \min & 2x_1 & -3x_2^+ & +3x_2^- & +x_3 & & & & \\ \text{subject to} & -3x_1 & +2x_2^+ & -2x_2^- & -x_3 & & & \ge & 5 \\ & x_1 & 3x_2^+ & -3x_2^- & -4x_3 & & & \ge & 9 \\ & x_1 & +x_2^+ & -x_2^+ & & +x_4^+ & -x_4^- & \ge & 6 \\ & -x_1 & -x_2^+ & +x_2^+ & & -x_4^+ & +x_4^- & \ge & -6 \\ & x_1, & x_2^+, & x_2^+, & x_3, & x_4^+, & x_4^+ & \ge & 0 \end{array}$$

5. [5pts] Consider the constraints

$$\begin{array}{rrrrrcl} 4x_1 & +2x_2 & -3x_3 & & & = & 4 \\ & x_2 & +3x_3 & +x_4 & & = & 2 \\ -x_1 & & +x_3 & & +4x_5 & = & -3. \\ x_1, & x_2, & x_3, & x_4, & x_5 & \geq & 0 \end{array}$$

List all basic feasible solutions for these constraints.

Test all bases (5 choose $3 = 10$), and compute basic solutions.

(x_1, x_2, x_3): $\begin{bmatrix} 4 & 2 & -3 \\ 0 & 1 & 3 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ -3 \end{bmatrix}$

Solution: ______

Should Report $\boxed{NO}$ basic feasible solutions.

6. [10pts] (*You MUST do this problem!*) Solve the following LP using the simplex algorithm with tableaus.

$$\begin{array}{rrrrcl} \min & -60x_1 & -90x_2 & -300x_3 & & \\ \text{subject to} & x_1 & +x_2 & +x_3 & \leq & 600 \\ & x_1 & +3x_2 & & \leq & 600 \\ & 2x_1 & & +x_3 & \leq & 900 \\ & x_1, & x_2, & x_3 & \geq & 0 \end{array}$$

Add slack variables for standard form, w/ tableau:

0	-60	-90	300	0	0	0	
600	1	1	1	1	0	0	x_4
600	1	3	0	0	1	0	x_5
900	2	0	1	0	0	1	x_6
	x_1	x_2	x_3	x_4	x_5	x_6	

$\rightarrow$

180,000	240	210	0	300	0	0	← non-neg stop.
600	1	1	1	1	0	0	x_3
600	1	3	0	0	1	0	x_5
300	1	1	0	-1	0	1	x_6

~~Or, using Bland's Rule~~

(Longer w/ Bland's Rule.)

Solution:

$(0, 0, 600, 0, 600, 300)$

w/ cost: $-180,000$