

MATH 482, Spring 2013 - Homework 2 Solutions

Solve 4 of the first 5 problems below, and also solve problem 6. Students registered for 4 credits must solve all problems.

- i. [5pts] Check whether the vector $[3, -1, 0, 2]$ is an optimal solution to the problem

$$\max 6x_1 + x_2 - x_3 - x_4$$

subject to

$$\begin{cases} x_1 + 2x_2 + x_3 + x_4 \leq 5, \\ 3x_1 + x_2 - x_3 \leq 8, \\ x_2 + x_3 + x_4 = 1, \\ x_3, x_4 \geq 0. \end{cases}$$

This problem should not require the use of the simplex algorithm. Any use thereof will get a maximum of 2 points.

If this entry is optimal, there exists a dual feasible point (y_1, y_2, y_3) with the same cost. We find this by complementary slackness.

Since x_1 , x_2 , and x_3 are nonzero, the complementary slackness equations imply the first, second, and fourth dual constraints are tight:

$$y_1 + 3y_2 = 6, \quad 2y_1 + y_2 + y_3 = 1, \quad y_1 + y_3 = -1.$$

These equations uniquely solve to $(y_1 = 0, y_2 = 2, y_3 = -1)$. While the cost of this dual point is $5(0) + 8(2) + 1(-1) = 15$ and matches the cost of the primal point, this dual point does not satisfy the third constraint

$$y_1 - y_2 + y_3 \geq -1, \quad (0 - 2 + (-1)) = -3 < -1.$$

Therefore, the primal point is *not* optimal!

2. [5pts] Solve with two-phase simplex algorithm the LP represented by the tableau below.

	x_1	x_2	x_3	x_4	x_5
0	-3	-4	0	0	0
6	2	1	1	0	0
2	1	-2	0	1	0
1	-3	0	9	9	1

The second row (starting 0) corresponds to the cost vector.

The variable x_5 is a good column for a third basis column, so we add variables y_1 & y_2 as first & second basis columns, where the costs are rewritten as $y_1 + y_2$ cost 1. So we start with tableau:

0	0	0	0	0	0	1	1
6	2	1	1	0	0	1	0
2	1	-2	0	1	0	0	1
1	3	0	9	9	1	0	0

Need to clear this out.
 $R_0 \leftarrow R_0 - R_1 - R_2$

y_1

y_2

x_5

-8	-3	1	-1	-1	0	0	0
6	2	1	1	0	0	1	0
2	1	-2	0	1	0	0	1
1	-3	0	9	9	1	0	0

y_1

y_2

x_5

-2	0	-5	-1	2	0	0	3
2	0	5	1	-2	0	1	-2
2	1	-2	0	1	0	0	1
4	0	0	3	9	12	1	0

pivot

clear out

y_2

x_1

$\cancel{x_5}$

0 cost, since y 's are out!

pivot

discard!

0	0	0	0	0			
$\frac{2}{5}$	0	1	$\frac{1}{5}$	$-\frac{2}{5}$	0		
$\frac{14}{5}$	1	0	$\frac{3}{5}$	$\frac{1}{5}$	0		
$\frac{32}{5}$	0	0	$\frac{51}{5}$	$\frac{48}{5}$	1		

x_2

x_1

x_5

0	-3	-4	0	0	0
$\frac{2}{5}$	0	1	$\frac{1}{5}$	$-\frac{2}{5}$	0
$\frac{14}{5}$	1	0	$\frac{3}{5}$	$\frac{1}{5}$	0
$\frac{32}{5}$	0	0	$\frac{51}{5}$	$\frac{48}{5}$	1

x_2

x_1

x_5

Diagonalize (Clear \bar{x}_1, \bar{x}_2 & \bar{x}_5 to zero)
 and continue by simplex...
 (Next page)

2. cont'd.

0	-3	-4	0	0	0
$\frac{2}{5}$	0	1	$\frac{1}{5}$	$-\frac{2}{3}$	0
$\frac{14}{5}$	1	0	$\frac{2}{5}$	$\frac{1}{5}$	0
$\frac{32}{5}$	0	0	$\frac{5}{15}$	$\frac{48}{15}$	1

x_2
 x_1
 x_5

10	0	0	2	-1	0
$\frac{2}{5}$	0	1	$\frac{1}{5}$	$-\frac{2}{5}$	0
$\frac{14}{5}$	1	0	$\frac{2}{5}$	$\frac{1}{5}$	0
$\frac{32}{5}$	0	0	$\frac{5}{15}$	$\frac{48}{15}$	1

x_1 x_2 x_3 x_4 x_5

$\frac{32}{3}$	0	0	$\frac{49}{16}$	$0 \frac{5}{48}$	← Non-negative, so stop!
$\frac{2}{3}$	0	1	$\frac{5}{8}$	$0 \frac{1}{24}$	x_2
$\frac{8}{3}$	1	0	$\frac{3}{16}$	$0 -\frac{1}{48}$	x_1
$\frac{2}{3}$	0	0	$\frac{13}{16}$	1 $\frac{5}{48}$	x_4

x_1 x_2 x_3 x_4 x_5

Solution: $(\frac{8}{3}, \frac{2}{3}, 0, \frac{2}{3}, 0)$

w/ cost: $-\frac{32}{3}$.

3. [5pts] Prove the theorem due to P. Gordan (1873) that the system $\mathbf{Ax} < \mathbf{0}$ is unsolvable if and only if the system $\mathbf{y}^T \mathbf{A} = \mathbf{0}, \mathbf{y} \geq \mathbf{0}, \mathbf{y} \neq \mathbf{0}$ is solvable. (Hint: In order to apply duality theorems, replace the system $\mathbf{Ax} < \mathbf{0}$ of strict inequalities by a system of nonstrict inequalities that is solvable if and only if $\mathbf{Ax} < \mathbf{0}$ is solvable).

Pf: Consider the linear program

$$\max \quad \varepsilon$$

s.t.

$$[\mathbf{A} | \underline{1}] \begin{bmatrix} \mathbf{x} \\ \varepsilon \end{bmatrix} \leq \underline{0}$$

\mathbf{x}, ε free,

where $\underline{1}$ is the vector in \mathbb{R}^m of all 1's. $\mathbf{x} = \underline{0}, \varepsilon = 0$ is a feasible solution

Observe the system $\mathbf{Ax} < \underline{0}$ is solvable if and only if this LP has a feasible solution \mathbf{x}, ε with $\varepsilon > 0$.

Also, in this case the LP is unbounded.

Consider its dual:

$$\min \quad \underline{0}^T \mathbf{y}$$

s.t.

$$\begin{bmatrix} \mathbf{A}^T \\ \underline{1}^T \end{bmatrix} \begin{bmatrix} \mathbf{y} \\ \varepsilon \end{bmatrix} = \begin{bmatrix} \underline{0} \\ 1 \end{bmatrix}$$

$$\mathbf{y} \geq \underline{0}.$$

If this dual is feasible, its optimum is zero.

Observe that a feasible point satisfies $\mathbf{A}^T \mathbf{y} = \underline{0}$ and $\underline{1}^T \mathbf{y} = 1$. For any $\mathbf{y}' \neq \mathbf{0}$ satisfying $\mathbf{A}^T \mathbf{y}' = \underline{0}$, some scalar multiple

(3. cont'd)

... Satisfies $A^T y \geq 0$ and $1^T y = 1$.

Hence, if the dual is feasible, then the primal is not unbounded.

In Review:

$Ax \leq 0$ is unsolvable



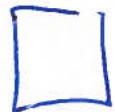
Primal LP ~~is~~ not unbounded (i.e. feasible, since $x = 0$ is a feasible pt.)



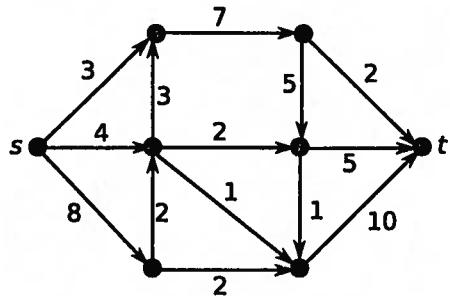
Dual is feasible ~~is~~



The system $A^T y = 0$, $y \neq 0$, $y \geq 0$
is solvable.



4. [5pts] Consider the network below.



Write the LP for the shortest-paths problem and the dual problem.

To describe these problems, we must label the vertices and edges of the graph. Let v_1, v_2, v_3 be the neighbors of s from top to bottom, and v_4, v_5, v_6 be the neighbors of t from top to bottom. Let $x_{i,j}$ denote the flow on the edge $v_i v_j$ (or sv_j when $i = s$, or $v_i t$ when $j = t$).

$$\text{Primal: } \min 3x_{s,1} + x_{s,2} + 8x_{s,3} + 7x_{1,4} + 3x_{2,1} + 2x_{2,5} + x_{2,6} + 2x_{3,2} + x_{3,6} + 5x_{4,5} + x_{4,6} + 2x_{4,t} + x_{5,t} + 10x_{6,t}$$

subject to:

$$\begin{array}{l}
 x_{s,1} + x_{s,2} + x_{s,3} \\
 -x_{s,1} + x_{1,4} - x_{2,1} \\
 -x_{s,2} + x_{2,1} + x_{2,5} + x_{2,6} - x_{3,2} \\
 -x_{s,3} + x_{3,2} + x_{3,6} \\
 -x_{1,4} + x_{4,5} + x_{4,t} \\
 -x_{2,5} - x_{4,5} + x_{5,6} + x_{5,t} \\
 -x_{2,6} - x_{3,6} - x_{5,6} + x_{6,t} \\
 -x_{4,t} - x_{5,t} - x_{6,t} = -1 \quad (t)
 \end{array}
 \begin{array}{l}
 = 1 \quad (s) \\
 = 0 \quad (v_1) \\
 = 0 \quad (v_2) \\
 = 0 \quad (v_3) \\
 = 0 \quad (v_4) \\
 = 0 \quad (v_5) \\
 = 0 \quad (v_6) \\
 = -1 \quad (t)
 \end{array}$$

$$x_{s,1}, \dots, x_{s,3}, \dots, x_{6,t} \geq 0$$

This last constraint may be removed, as it is not independent of previous rows.

4. Dual: The variable y_i is for v_i or s, t when $i \in \{s, t\}$.

$$\max Y_s - Y_t$$

$$\text{s.t. } y_s - y_t$$

$$\leq 3 \text{ (sv}_1\text{)}$$

$$y_s - y_t$$

$$\leq 4 \text{ (sv}_2\text{)}$$

$$y_s - y_t$$

$$\leq 8 \text{ (sv}_3\text{)}$$

$$y_1 - y_4$$

$$\leq 7 \text{ (v}_1 v_4\text{)}$$

$$-y_1 + y_2$$

$$\leq 3 \text{ (v}_2 v_1\text{)}$$

$$y_2 - y_5$$

$$\leq 2 \text{ (v}_2 v_5\text{)}$$

$$y_2 - y_6$$

$$\leq 1 \text{ (v}_2 v_6\text{)}$$

$$-y_2 + y_3$$

$$\leq 2 \text{ (v}_3 v_2\text{)}$$

$$y_3 - y_6$$

$$\leq 2 \text{ (v}_3 v_6\text{)}$$

$$y_4 - y_5$$

$$\leq 5 \text{ (v}_4 v_5\text{)}$$

$$y_5 - y_6$$

$$\leq 1 \text{ (v}_5 v_6\text{)}$$

$$y_4 - y_t$$

$$\leq 2 \text{ (v}_4 t\text{)}$$

$$y_s - y_t$$

$$\leq 5 \text{ (v}_s t\text{)}$$

$$y_6 - y_t$$

$$\leq 10 \text{ (v}_6 t\text{)}$$

$y_5, y_1, y_2, y_3, y_4, y_5, y_6, y_t$ free

5. [5pts] Demonstrate an optimal solution to the primal LP in Problem 4 and use duality to prove optimality.

The unique optimal solution applies the following weights to the edges (corresponding to the path directly through the middle of the graph:

$$(0, 4, 0, 0, 0, 0, 2, 0, 0, 0, 0, 5, 0, 0)$$

with cost 11. The dual solution can be found by the complementary slackness equations given by the three nonzero edges:

$$y_s - y_2 = 4, \quad y_2 - y_5 = 2, \quad y_5 - y_t = 5.$$

There is degeneracy, so one of these values can be selected arbitrarily. Thus, we select $y_t = 0$ and $y_5 = 5$, $y_2 = 7$, and $y_s = 11$. We must select values for the remaining variables, but there is "slack": some choice, but it must satisfy the dual constraints. Such points include

$$y_1 = 9, y_3 = 6, y_4 = 2, y_6 = 4.$$

(Here, the constraints $y_1 - y_4 \leq 7$, $y_3 - y_6 \leq 2$, and $y_4 - y_t \leq 2$ are tight.) The cost of this solution is $y_s - y_t = 11$, showing optimality.

[10pts] Use the dual simplex method to find an optimal solution to the problem

$$\text{Minimize } z = 7x_1 + x_2 + 3x_3 + x_4$$

subject to

$$\begin{cases} 2x_1 - 3x_2 - x_3 + x_4 \geq 8, \\ 6x_1 + x_2 + 2x_3 - 2x_4 \geq 12, \\ -x_1 + x_2 + x_3 + x_4 \geq 10, \\ x_1, x_2, x_3, x_4 \geq 0. \end{cases}$$

The additional constraints

$$\begin{aligned} x_1 + 5x_2 + x_3 + 7x_4 &\leq 50, \\ 3x_1 + 2x_2 - 2x_3 - x_4 &\leq 20 \end{aligned}$$

are added to those of Problem 6.a. Solve the new problem starting from the previous optimal tableau.

We start by adding slack variables y_1, y_2, y_3 , but since the constraints are \geq , they have coefficients -1 . So, to make y_1, y_2, y_3 as a basis, we negate the rows (other than last)

0	7	1	3	1	0	0	0
-8	-2	3	1	-1	1	0	0
-12	-6	-1	-2	2	0	1	0
-10	1	-1	-1	-1	0	0	1

since non-negative, we are dual-feasible?

$x_1 \ x_2 \ x_3 \ x_4 \ y_1 \ y_2 \ y_3$

Not primal feasible!

-8	5	4	4	0	1	0	0
8	2	-3	-1	1	-1	0	0
-28	-10	5	0	0	2	1	0
-2	3	-4	-2	0	-1	0	1

$x_1 \ x_2 \ x_3 \ x_4 \ y_1 \ y_2 \ y_3$



(next page)

-22	0	$\frac{13}{2}$	4	0	2	$\frac{1}{2}$	0
$\frac{12}{5}$	0	-4	-1	1	$-\frac{7}{5}$	$-\frac{1}{5}$	0
$\frac{14}{5}$	1	$\frac{1}{2}$	0	0	$\frac{1}{5}$	$\frac{1}{10}$	0
$\frac{-52}{5}$	0	$-\frac{5}{2}$	$\circled{-2}$	0	$-\frac{2}{5}$	$\frac{3}{10}$	1

x_4
 x_1
 y_3

$\frac{-214}{5}$	0	$\frac{3}{2}$	0	0	$\frac{6}{5}$	$\frac{1}{10}$	2
$\frac{38}{5}$	0	$-3/4$	0	1	$-\frac{2}{5}$	$\frac{1}{20}$	$-\frac{1}{2}$
$\frac{14}{5}$	1	$-\frac{1}{2}$	0	0	$-\frac{1}{5}$	$-\frac{1}{10}$	0
$\frac{26}{5}$	0	$\frac{5}{4}$	1	0	$\frac{1}{5}$	$-\frac{3}{20}$	$-\frac{1}{2}$

x_4
 x_1
 x_3

primal feasible, so optimal!

Optimum Solution
 $(\frac{14}{5}, 0, \frac{26}{5}, \frac{38}{5}, 0, 0, 0)$
w/ value $\frac{214}{5}$.

► Add Constraints by adding rows & columns -

Since they are " \leq "
the slack variables
have +1 coefficient.

$\frac{-214}{5}$	0	$\frac{3}{2}$	0	0	$\frac{6}{5}$	$\frac{1}{10}$	2	0	0
$\frac{38}{5}$	0	$-3/4$	0	1	$-\frac{2}{5}$	$\frac{1}{20}$	$-\frac{1}{2}$	0	0
$\frac{14}{5}$	1	$-\frac{1}{2}$	0	0	$-\frac{1}{5}$	$-\frac{1}{10}$	0	0	0
$\frac{26}{5}$	0	$\frac{5}{4}$	1	0	$\frac{1}{5}$	$-\frac{3}{20}$	$-\frac{1}{2}$	0	0
$+50$	$+1$	$+5$	$+1$	$+7$	0	0	0	1	0
$+20$	$+3$	2	-2	-1	0	0	0	0	1

x_1 x_2 x_3 x_4 y_1 y_2 y_3 y_4 y_5

↑ ↑ ↑ ↑
Need to diagonalize!

(next page)

$-2\frac{14}{5}$	0	$\frac{3}{2}$	0	0	$\frac{6}{5}$	$\frac{11}{10}$	2	0	0
$3\frac{8}{5}$	0	$-\frac{3}{4}$	0	1	$-\frac{4}{5}$	$\frac{1}{20}$	$-\frac{1}{2}$	0	0
$1\frac{4}{5}$	1	$-\frac{1}{2}$	0	0	$-\frac{4}{5}$	$-\frac{1}{10}$	0	0	6
$2\frac{6}{5}$	0	$\frac{5}{4}$	1	0	$\frac{1}{5}$	$-\frac{3}{20}$	$-\frac{1}{2}$	0	0
$-5\frac{6}{5}$	0	$\frac{19}{2}$	0	0	$\frac{14}{5}$	$-\frac{1}{10}$	4	1	0
$14\frac{8}{5}$	0	$2\frac{1}{4}$	0	0	$\frac{3}{5}$	$\frac{1}{20}$	$-\frac{3}{2}$	0	1

$x_1 \ x_2 \ x_3 \ x_4 \ y_1 \ y_2 \ y_3 \ y_4 \ y_5$

Not primal feasible!

-166	0	106	0	0	32	0	46	11	0
2	0	4	0	1	1	0	$\frac{3}{2}$	$\frac{1}{2}$	0
14	1	-10	0	0	-3	0	-4	-1	0
22	0	-13	1	0	-4	0	$\frac{-13}{2}$	$-\frac{3}{2}$	0
112	0	-95	0	0	-28	1	-40	-10	0
24	0	10	0	0	2	0	$\frac{1}{2}$	$\frac{1}{2}$	1

$x_1 \ x_2 \ x_3 \ x_4 \ y_1 \ y_2 \ y_3 \ y_4 \ y_5$

Primal & Dual Feasible, so optimal!

Solution: $(14, 0, 22, 2, 0, 112, 0, 0, 24)$

w/ Cost: 166.