MATH 482, Spring 2013 - Homework 3
Solutions

1. [5pts] Solve (in mixed strategies) for both players the game with the payoff matrix

$$\begin{pmatrix} -2 & 3 & 0 & -6 & 3 \\ 0 & -4 & 9 & 2 & 1 \\ 6 & -2 & 7 & 4 & 5 \\ 7 & -3 & 8 & 3 & 2 \end{pmatrix}.$$

Solution: There are multiple ways to arrive at the optimal strategies, but we only need two strategies with value that is feasible in the primal and the dual. However, the solutions MUST be presented as fractions, otherwise they are not exact!

$$P_1 = (0, 0, 2/11, 0, 9/11), \quad P_2 = (8/11, 3/11, 0, 0), \quad v_1 = v_2 = 27/11$$

Example Sage Worksheet is online.
One method to find this solution is to reduce the number of strategies using *domination*. If column i is coordinate-wise smaller than column j, then Player 1 would never choose strategy i over strategy j, since all payoff options are worse for Player 1 in column i. If row a is coordinate-wise larger than row b, then Player 2 would never choose strategy a over strategy b, since all payoff options are worse for Player 2 in row a.
Thus, for this matrix we can delete column 4 (dominated by column 3) and column 2 (dominated by column 4). We can then delete row 3 (dominates row 2) and row 4 (dominates row 1). Finally, we can now delete column 1 (dominated by column 5). This leaves the 2-by-2 matrix $\begin{bmatrix} 0 & 3 \\ 9 & 1 \end{bmatrix}$ corresponding to the submatrix with rows 1 & 2, and columns 3 & 5. Solving this smaller game is equivalent to the larger game.

2. Add slack variables y_1 & y_2 to be in standard form.
Use y_1 & y_2 as initial basis w/ $B = I_2$, $c_B = 0$, $x_B = b$.

CARRY$^{(1)}$:

0	0	0	
5	1	0	y_1
8	0	1	y_2
	y_1	y_2	

-7	
2	
3	
x_1	

CARRY$^{(2)}$

$35/2$	$7/2$	0	
$5/2$	$1/2$	0	x_1
$1/2$	$-3/2$	1	y_2
	y_1	y_2	

$-1/2$	
$3/2$	
$1/2$	
x_2	

$$\bar{c}_2 = -11 + \begin{pmatrix} 7/2 & 0 \end{pmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = -1/2$$

$$X_2 = B^{-1} A_2 = \begin{pmatrix} 1/2 & 0 \\ -3/2 & 1 \end{pmatrix} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix}$$

CARRY$^{(3)}$

18	2	1	
1	5	-3	x_1
1	-3	2	x_2
	y_1	y_2	

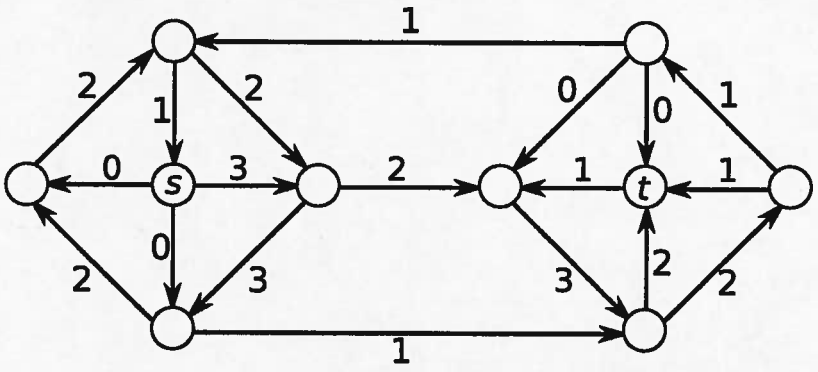
$$\bar{c}_3 = -4 + \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 0$$

$$\bar{c}_4 = -1 + \begin{pmatrix} 2 & 1 \end{pmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 4$$

All relative costs are nonnegative, so optimal!

Solution: $(1, 1, 0, 0)$ (x_1, x_2, x_3, y_4)
with value: -18.

3. [5pts] Let G be the network with the flow drawn below on the left (the numbers correspond to flow values). Write the flow as a linear combination of flows along cycles and paths from s to t.

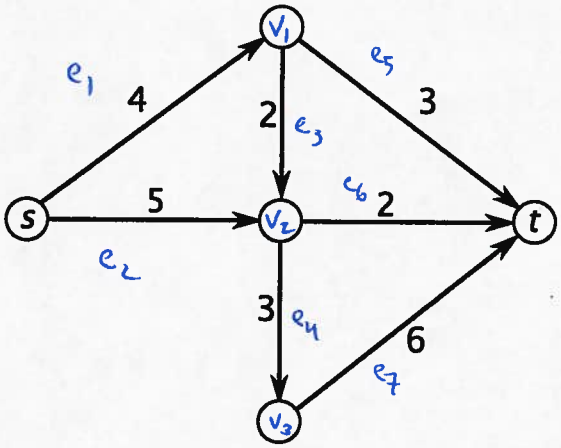


Label the vertices around s in clockwise order (starting at the top) as v_1, v_2, v_3, v_4. Label the vertices around t in clockwise order (starting at the top) as v_5, v_6, v_7, v_8. Then, the flow can be described as:

$$2\underbrace{sv_2v_8v_7t}_{\text{path}} + 2\underbrace{tv_8v_7v_6t}_{\text{cycle}} + 1\underbrace{sv_2v_3v_7v_6v_5v_1s}_{\text{cycle}} + 2\underbrace{v_1v_2v_3v_4}_{\text{cycle}}$$

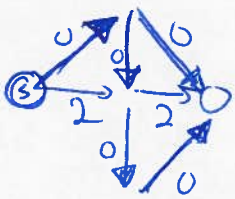
where the flow across an edge is the sum of the flows across all cycles and paths using that edge. (There are multiple ways to solve this problem; all involve at least one cycle and at least one path.)

4. [5pts] Solve the flow problem above and on the right (*now below*) using revised simplex (the numbers correspond to capacity values).



We start by adding flow across any st-path.

$P_1 = s v_2 t$, value $= 2$.



CARRY$^{(1)}$

	e_1	e_2	e_3	e_4	e_5	e_6	e_7	f_{P_1}
0	0	0	0	0	0	0	0	-1
4	1							0
5		1				0		1
2			1					0
3				1				0
3					1			0
2			0			1		1
6							1	0

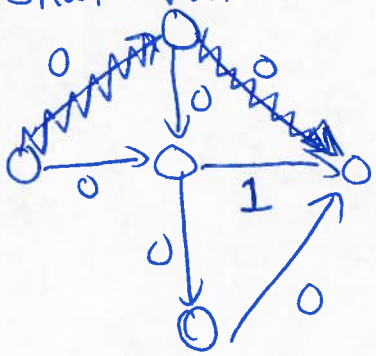
CARRY$^{(2)}$

								f_{P_2}
2	0	0	0	0	0	1	0	-1
4	1							1
3		1				-1		0
2			1					0
3				1				0
3					1			1
2						1		0
6							1	0

$f_{P_1} =$

$e_{P_2} = -1 + (0000010)\begin{pmatrix}1\\0\\0\\0\\0\\0\\0\end{pmatrix}$

Shortest Path!



$P_2 = s v_1 t$

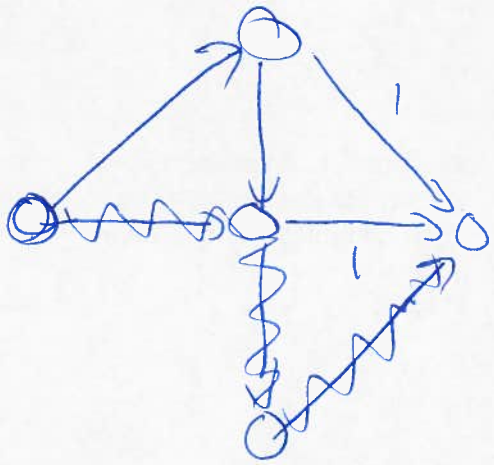
(4 Continued)

CARRY$^{(3)}$

	5	0	0	0	0	1	1	0	-1
	1	1				-1			0
	3		1				-1		1
	2			1					0
	3				1				1
$f_{P_2}=$	3					1			0
$f_{P_1}=$	2						1		0
	6							1	1

f_{P_3}

either works, choose top



$P_3 = s\, v_2 v_3\, t$

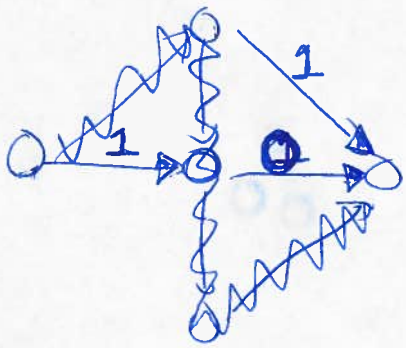
(Could choose $P_3' = s\, v_1 v_2 v_3\, t$,)

CARRY$^{(4)}$

	8	0	1	0	0	1	0	0	-1
	1	1				-1			1
$f_{P_3}=$	3		1				-1		0
	2			1					1
	0		-1		1		1		1
$f_{P_2}=$	3					1			0
$f_{P_1}=$	2						1		0
	3		-1				1	1	1

f_{P_4}

← Observe: Degeneracy!



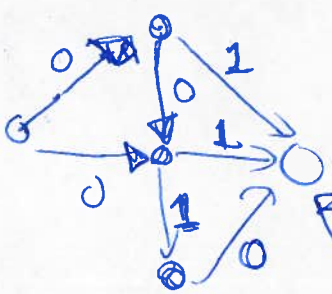
CARRY$^{(5)}$

	8	0	0	0	1	1	1	0
	1	1	+1		-1	-1	-1	
$f_{P_3}=$	3		1				-1	
	2		+1	1	-1		-1	
$f_{P_4}=$	0		-1		1		1	
$f_{P_2}=$	3					1		
$f_{P_1}=$	2						1	
	3		0		-1		0	1
		e_1	e_2	e_3	e_4	e_5	e_6	e_7

$P_4 = s\, v_1 v_2 v_3\, t$ (Could choose $P_4' = s\, v_1 v_2\, t$)

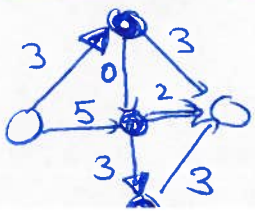
$\bar{c}_4 = -1 + 0 = -1$

$$X_{P_4} = B^{-1}\begin{pmatrix}1\\0\\1\\1\\0\\0\\1\end{pmatrix} = \begin{pmatrix}1\\0\\1\\1\\0\\0\\1\end{pmatrix}$$



No path of weight ~~length~~ < 1, so we are optimal!

Edge flows:



Value: 8 (as a flow)