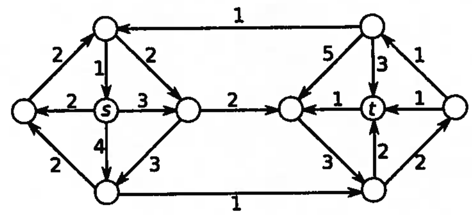
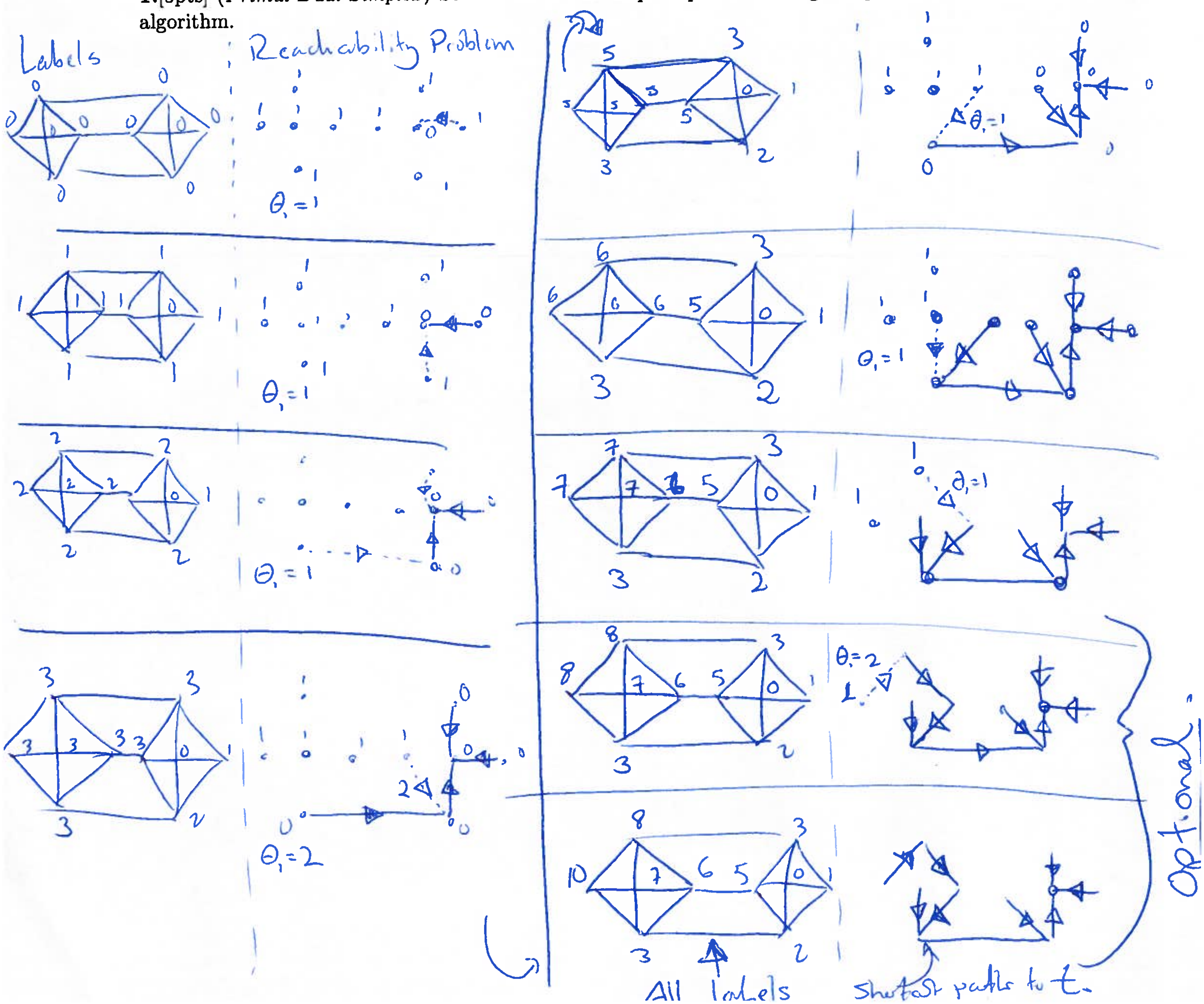
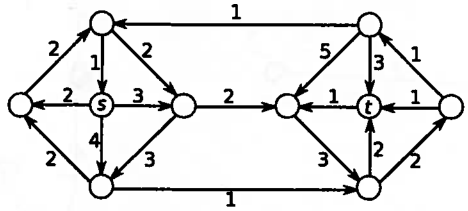
MATH 482, Spring 2013 - Homework 4
Solutions

For problems 1–3, consider the following graph.



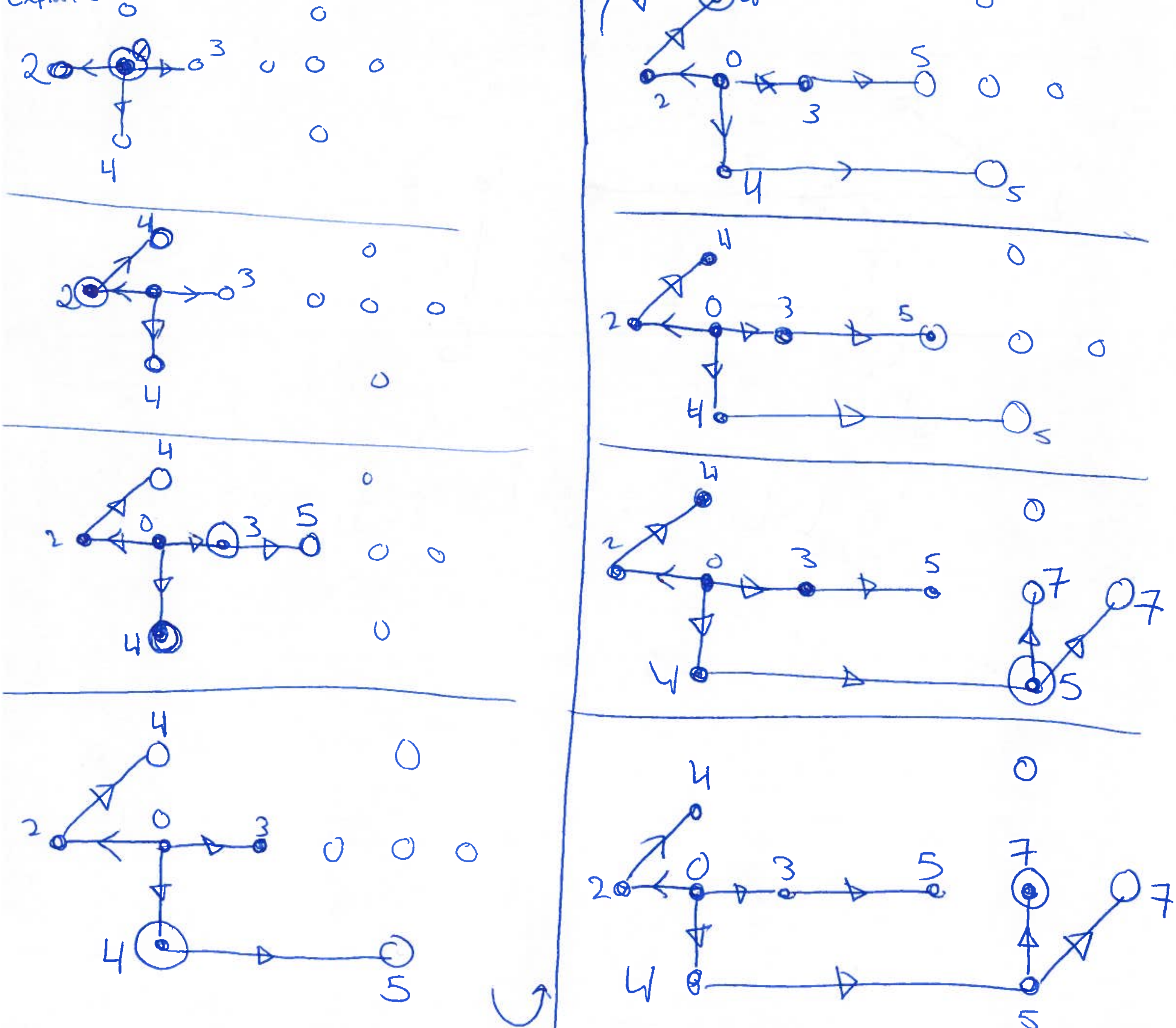
1.[5pts] (*Primal-Dual Simplex*) Solve the shortest *st*-path problem using the primal-dual simplex algorithm.





2.[5pts] (*Dijkstra's Algorithm*) Use Dijkstra's Algorithm to compute the distances from s to all other vertices in the graph above.

Explore s



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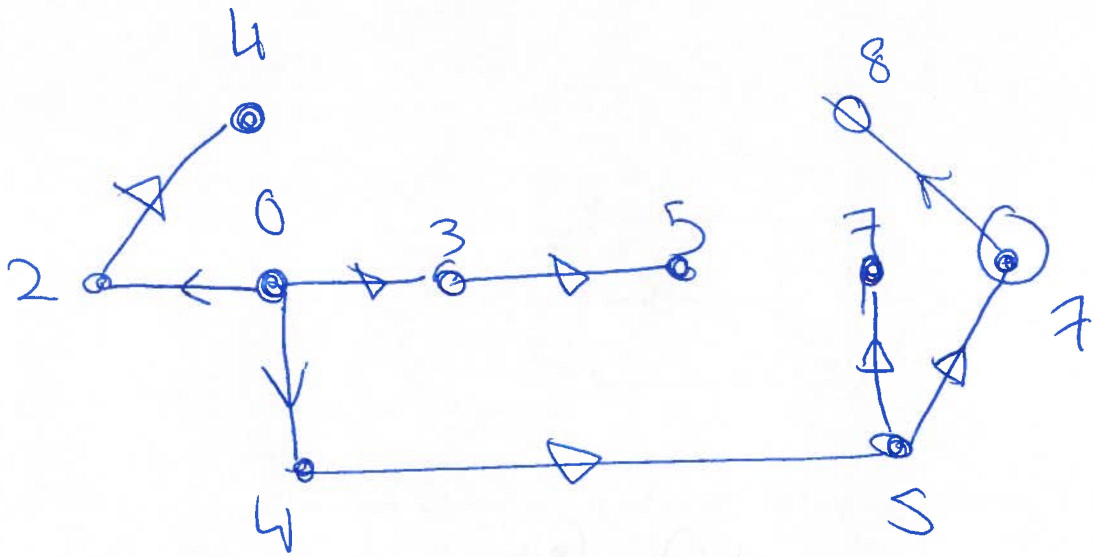
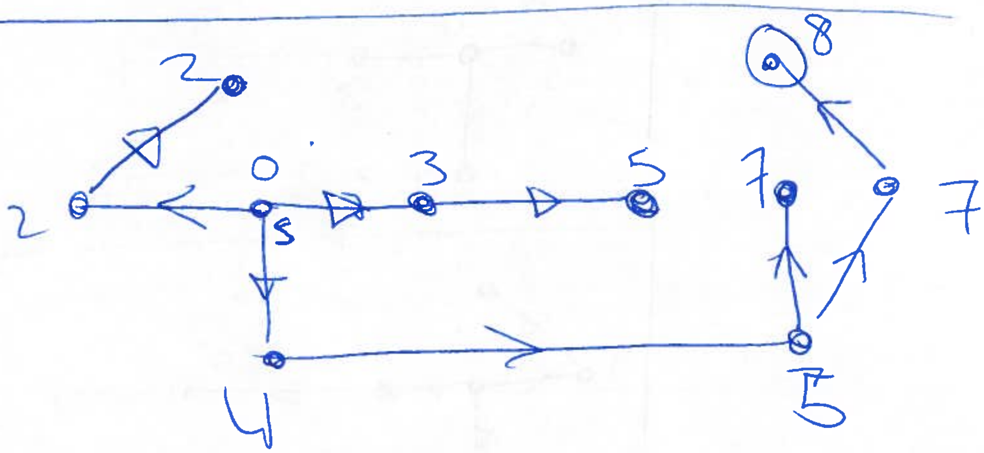
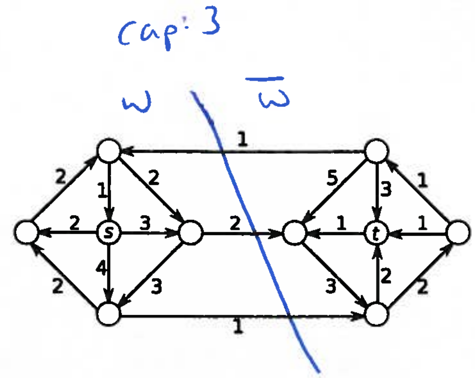




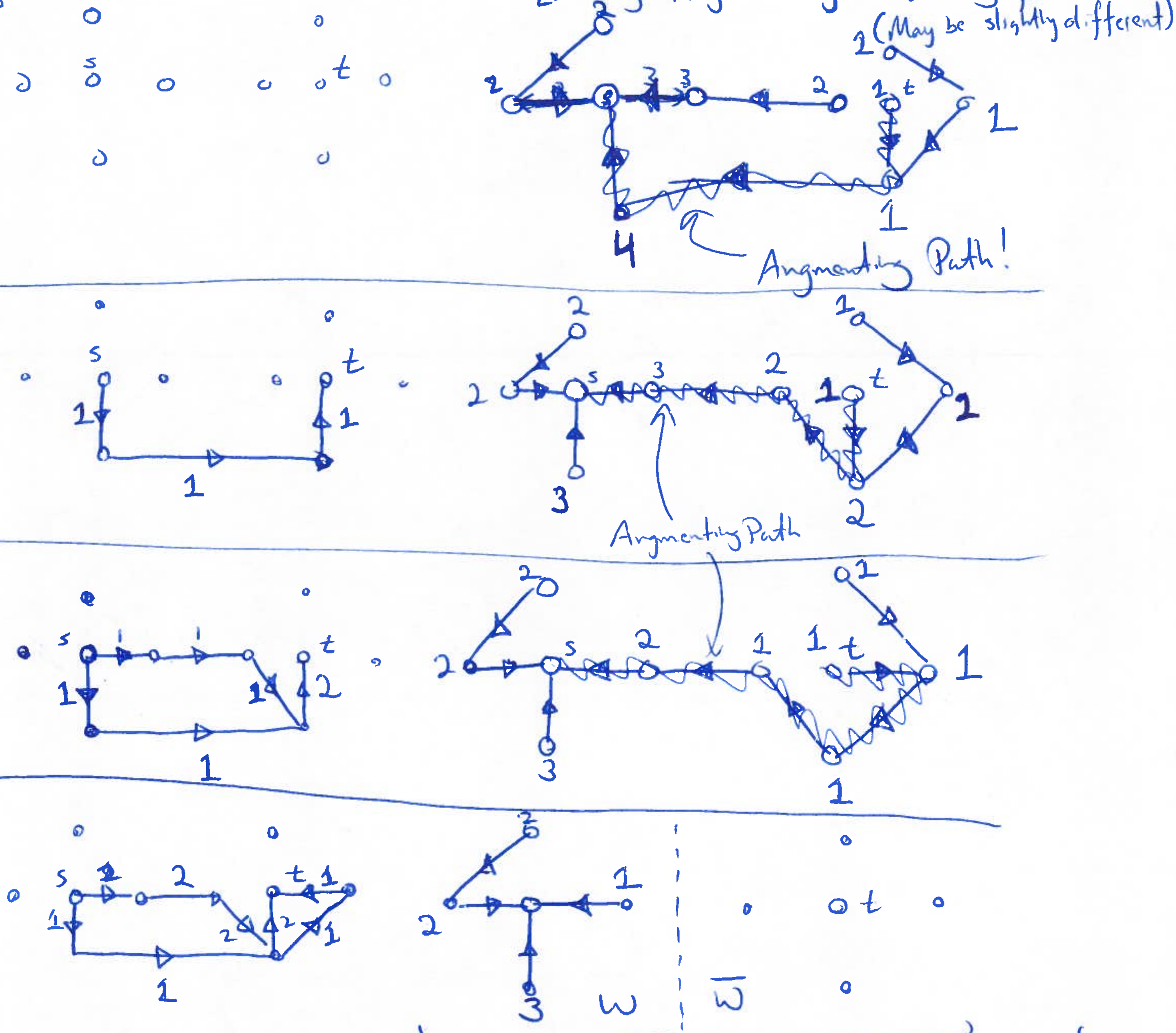
Labels are distances from s.



Start w/ Zero Flow

3.[5pts] (*Max-Flow/Min-Cut*) Solve the max-*st*-flow problem using the Ford-Fulkerson Algorithm. Prove optimality using duality.

Labeling Algorithm gives following paths: (May be slightly different)



By this cut of capacity 2+1=3, our flow at value 3 is optimal! $\square$

4.[5pts] (*Floyd-Warshall Algorithm*) Use the Floyd-Warshall Algorithm to compute shortest distances among all pairs of vertices in the graph given by the following adjacency matrix. (The entry $a_{i,j}$ stores the length of the arc (i, j).)

$$\begin{bmatrix} \infty & 1 & 2 & \infty & \infty \\ \infty & \infty & \infty & 4 & \infty \\ 6 & 1 & \infty & \infty & 3 \\ 5 & 3 & \infty & \infty & \infty \\ \infty & \infty & \infty & 3 & \infty \end{bmatrix}$$

We perform the triangle updates on the five vertices in order (top-to-bottom/left-to-right). New values are in bold.

1. v_1

$$\begin{bmatrix} \infty & 1 & 2 & \infty & \infty \\ \infty & \infty & \infty & 4 & \infty \\ 6 & 1 & \mathbf{8} & \infty & 3 \\ 5 & 3 & \mathbf{7} & \infty & \infty \\ \infty & \infty & \infty & 3 & \infty \end{bmatrix}$$

2. v_2

$$\begin{bmatrix} \infty & 1 & 2 & \mathbf{5} & \infty \\ \infty & \infty & \infty & 4 & \infty \\ 6 & 1 & 8 & \mathbf{5} & 3 \\ 5 & 3 & 7 & \mathbf{8} & \infty \\ \infty & \infty & \infty & 3 & \infty \end{bmatrix}$$

3. v_3

$$\begin{bmatrix} \mathbf{8} & 1 & 2 & 5 & \mathbf{5} \\ \infty & \infty & \infty & 4 & \infty \\ 6 & 1 & 8 & 5 & 3 \\ 5 & 3 & 7 & 8 & \mathbf{10} \\ \infty & \infty & \infty & 3 & \infty \end{bmatrix}$$

4. v_4

$$\begin{bmatrix} 8 & 1 & 2 & 5 & 5 \\ \mathbf{9} & \mathbf{7} & \mathbf{11} & 4 & \mathbf{14} \\ 6 & 1 & 8 & 5 & 3 \\ 5 & 3 & 7 & 8 & 10 \\ \mathbf{8} & \mathbf{6} & \mathbf{10} & 3 & \mathbf{13} \end{bmatrix}$$

5. v_5

$$\begin{bmatrix} 8 & 1 & 2 & 5 & 5 \\ 9 & 7 & 11 & 4 & 14 \\ 6 & 1 & 8 & 5 & 3 \\ 5 & 3 & 7 & 8 & 10 \\ 8 & 6 & 10 & 3 & 13 \end{bmatrix}$$

5.[5pts] Consider the following linear program.

$$\begin{array}{rrcrcr} \max & x_1 & + & x_2 & & \\ \text{subject to} & -\frac{8}{3}x_1 & + & x_2 & \geq & -\frac{8}{3} \\ & x_1 & - & x_2 & \geq & -\frac{1}{2} \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

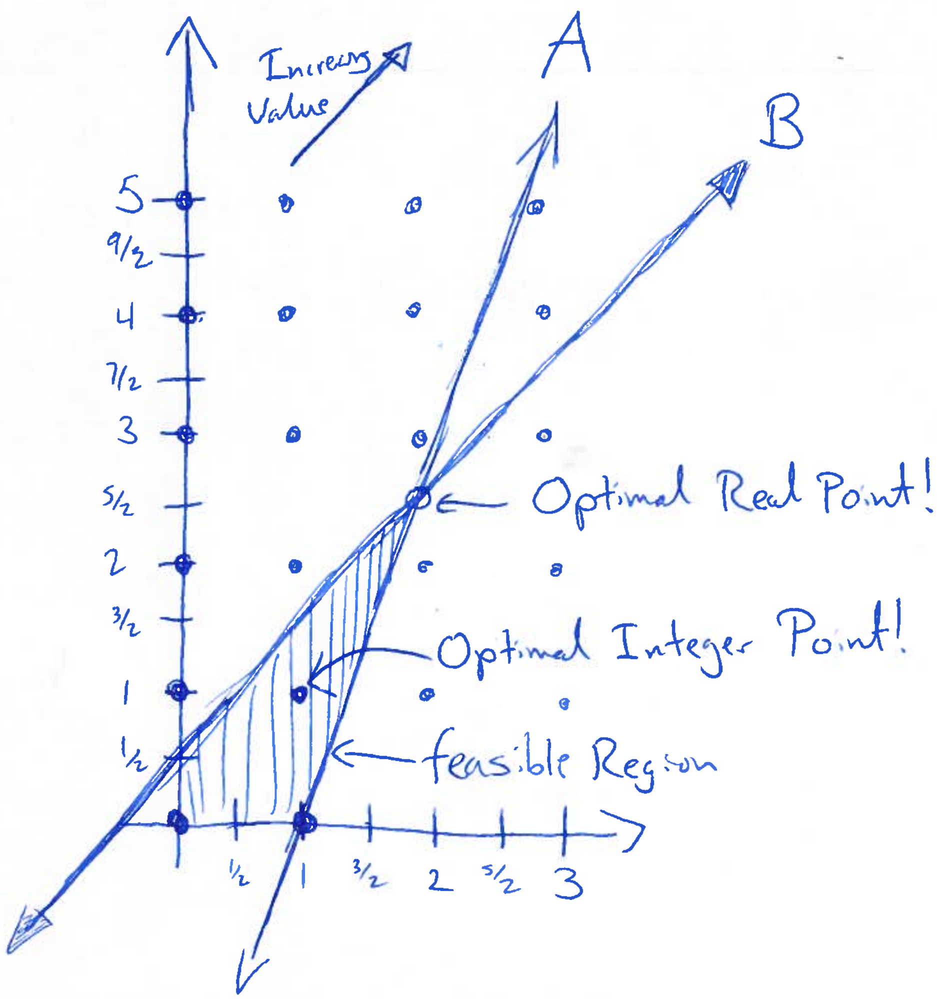
Solve the linear problem graphically, then also solve the problem graphically when x_1 and x_2 are constrainted to be integers, demonstrating a gap between the real and integer solutions.

The constraints are bounds at the following lines:

$$x_2 = \frac{8}{3}x_1 - \frac{8}{3} \qquad \text{(line A)}$$

$$x_2 = x_1 + \frac{1}{2} \qquad \text{(line B)}$$

Plot:



Optimal Real Point:

$x_1 = \frac{19}{10}$, $x_2 = \frac{24}{10}$

with value $\frac{43}{10}$.

Optimal Integer Point:

$x_1 = 1$, $x_2 = 1$

with value 2.

$$2 < \frac{43}{10}.$$

6.[10pts] (*Matchings and Vertex Covers*) Let G be a graph with edges spanning two sets of vertices, X and Y. A *matching* is a set M of edges, where $M = \{x_i y_i : 1 \leq i \leq k\}$ for some k, $x_i \in X$, and $y_i \in Y$, with $x_i y_i \in E(G)$. A *vertex cover* is a set $Q \subset V(G)$ such that all edges have at least one endpoint in Q. Use Max-Flow/Min-Cut to prove that the maximum size of a matching in a bipartite graph G is equal to the minimum size of a vertex cover. (*Hint:* Add vertices s and t to G, direct the edges, and show that the max st-flow and min st-cut problems are equivalent to the max matching and min vertex cover problems.)

Proof. Given a bipartite graph G with bipartition $X \cup Y$, we will build a network N whose flows correspond to matchings of G and whose minimum cuts correspond to minimum vertex covers in G.
Let N have vertex set $V(N) = \{s, t\} \cup X \cup Y$. For each $x \in X$, let sx be an edge of capacity 1. For each $y \in Y$, let yt be an edge of capacity 1. For each edge $xy \in E(G)$, let xy be an edge of N with capacity $|X| + |Y|$. Since the capacities are integers, the Ford-Fulkerson algorithm guarantees that maximum flows will have integer values on the edges.
Given a feasible integer flow f in N, let $M_f = \{xy : x \in X, y \in Y, f(xy) = 1\}$. Since each $x \in X$ has a maximum incoming flow of 1, there is at most one edge $xy \in M_f$. Since each $y \in Y$ has a maximum outgoing flow of 1, there is at most one edge $xy \in M_f$. Thus, M_f is a matching and observe that $|M_f|$ is equal to the value of f.
Given a matching M in G, let f be a flow defined as

1. $f(sx) = 1$ if and only if x is saturated by M,
2. $f(yt) = 1$ if and only if y is saturated by M, and
3. $f(xy) = 1$ if and only if $xy \in M$,

where $x \in X$ and $y \in Y$. Observe that since M is a matching, f is a feasible flow in N and f has value equal to $|M|$.
Let $[W, \overline{W}]$ be a minimum st-cut in N. Since assigning $W = \{s\}$ or $\overline{W} = \{t\}$ presents a cut of capacity $|X|$ or $|Y|$, a minimum st-cut $[W, \overline{W}]$ never contains an edge from X to Y, since their capacities are strictly larger. Thus, let $Q = (\overline{W} \cap X) \cup (W \cap Y)$. We claim that Q is a vertex cover of size equal to the capacity of $[W, \overline{W}]$. Observe that no edges span $W \cap X$ and $\overline{W} \cap Y$ or else the capacity of the cut is too large (by earlier argument). Thus, every edge $xy \in E(G)$ has at least one endpoint in Q, so Q is a vertex cover. Also, since $sx \in [W, \overline{W}]$ for all $x \in Q$ and $yt \in [W, \overline{W}]$ for all $y \in Q$, the capacity of W is equal to the size of Q.
For any vertex cut Q, let $W = \{s\} \cup (X \setminus Q) \cup (Y \cap Q)$. We claim that the capacity of W is equal to the size of Q: if $x \in Q \cap X$, then $sx \in [W, \overline{W}]$; if $y \in Q \cap Y$, then $yt \in [W, \overline{W}]$. Since Q is a vertex cover, no edges from X to Y are in $[W, \overline{W}]$, so hence the capacity of this st-cut is equal to $|Q|$.
Since the size of a maximum matching equals the value of a maximum flow, the value of a maximum flow equals the capacity of a minimum cut, and the capacity of a minimum cut is the size of some vertex cover, we have the maximum matching is bounded below by the minimum size of a vertex cover. Since a minimum vertex cover has size equal to the capacity of an st-cut, the capacity of an st-cut is at least the value of a maximum flow, and the value of a maximum flow is the size of a maximum mathing, we have the minimum vertex cover is bounded below by the maximum matching. Thus, the size of a maximum matching is equal to the size of a minimum vertex cover. $\square$