MATH 482, Spring 2013 - Homework 1
(Will be) Assigned Wednesday, 09/04.
Due Friday, 09/06.

1. (*Assigned!*, 5pts) Convert the following (general form) linear program into standard and canonical form:

$$\begin{array}{rrcrcrcrcl} \max & 3x_1 & - & 2x_2 & & & + & x_4 & & \\ \text{subject to} & x_1 & + & x_2 & - & x_3 & & & \geq & 1 \\ & & & 2x_2 & & & + & x_4 & = & 0 \\ & & & & & x_3 & - & 3x_4 & \leq & 6 \\ & x_1, & & x_2 & & & & & \geq & 0 \\ & & & & & x_3 & & & \leq & 0 \\ & & & & & & & x_4 & & \text{free} \end{array}$$

2. Consider the linear programming formulation of the shortest paths problem. (*The statements below are adjusted versions of the questions asked in class.*)

a. Produce a digraph whose shortest paths linear program has a feasible point with $x_{i,j} > 1$ for some edge ij.

b. (*Assigned!*, 5pts) Prove that for every digraph, every *optimal* solution to the shortest paths linear program has $x_{i,j} \leq 1$ for all edges ij.

c. (*Assigned!*, 5pts) Produce a digraph whose shortest paths linear program has an optimal point where at least one variable $x_{i,j}$ has $0 < x_{i,j} < 1$. (*Hint: Consider what happens when there are multiple optimal solutions to the shortest paths problem.*)

d. Prove that for every digraph there exists an optimal solution to the shortest paths linear program where every variable $x_{i,j}$ has value in $\{0, 1\}$. (*Hint: Use duality, and look in the example implementation report 0.*)

3. Consider the following linear program:

$$\begin{array}{rrcrcl} \min & 2x_1 & + & 5x_2 & & \\ \text{subject to} & 3x_1 & + & x_2 & \geq & 6 \\ & x_1 & & & \geq & \frac{1}{2} \\ & x_1 & + & x_2 & \geq & 1 \\ & 2x_1 & + & 5x_2 & \geq & 10 \\ & -x_1 & + & x_2 & \geq & -3 \\ & -x_1 & - & x_2 & \geq & -100 \\ & x_1, & & x_2 & \geq & 0 \end{array}$$

a. Plot the constraints and solve the problem graphically.

b. List all of the basic feasible solutions.

c. Characterize the optimal solutions in terms of optimal basic feasible solutions.

4. ((*Assigned!*, 5pts) Consider a standard-form linear program P

$$\begin{aligned} \min \mathbf{c}^\top \mathbf{x} \\ \text{subject to } A\mathbf{x} = \mathbf{b} \\ \mathbf{x} \geq \mathbf{0} \end{aligned}$$

Let D be the dual of P. Prove that the dual of the dual of P is equivalent to P by forming the dual D, converting D into a standard form problem D', then forming the dual of D' (call it P') and converting P' into standard form. Here is an outline: