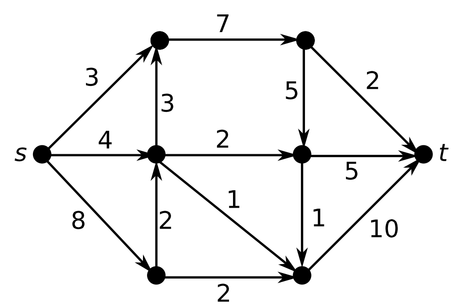
MATH 482, Spring 2013 - Homework 2

Assigned Wednesday 09/18. Due Friday 09/20.

Problems Assigned: 1, 2, 3.a., 4.c.

1. (*Assigned! 5pts*) Consider the graph below.



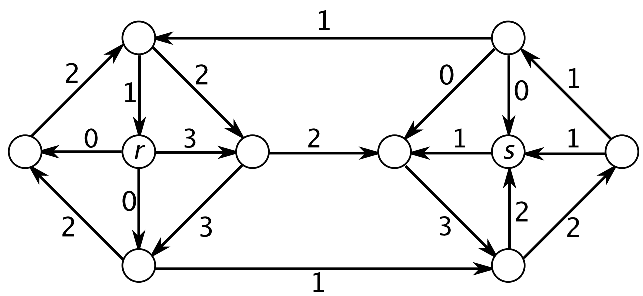
Find a shortest path and prove optimality using duality.

Label the vertices in the left column from top to bottom as a, b, c, and the vertices in the right column from top to bottom as i, j, k. The shortest path is s, b, j, t of weight 11. One dual solution (which assigns real values to all vertices) is to let

$$y_s = 0, \quad y_a = 3, \quad y_b = 4, \quad y_c = 8, \quad y_i = 10, \quad y_j = 6, \quad y_k = 5, \quad y_t = 11.$$

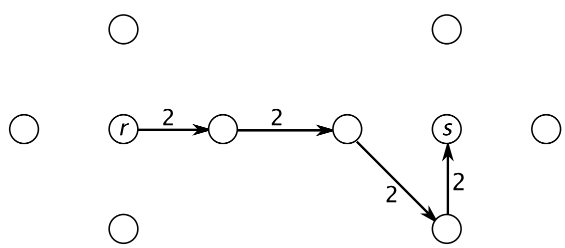
Observe that for every edge uv, we have $y_v \leq y_u + w(u, v)$, as required by the dual solution. Since $y_1 = 11$, all paths have weight at least 11.

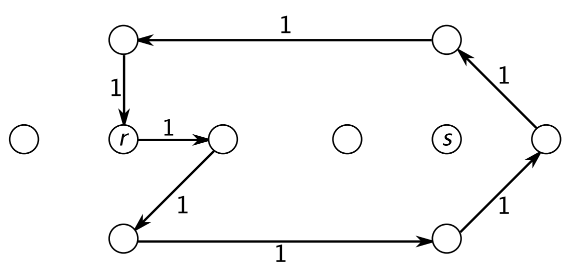
2. (*Assigned! 5pts*) Consider the network below with given edge values, forming an integer feasible flow.

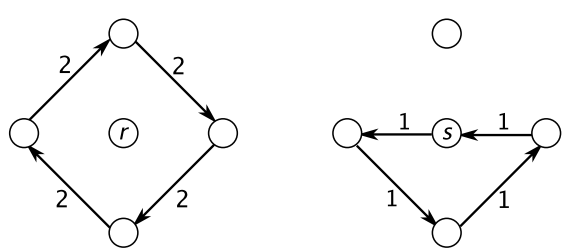


Demonstrate a list of path and cycle flows whose sum is this flow.

Below is a path flow, a cycle flow, and two cycle flows. The sum of the flows across these flows matches the flow above.



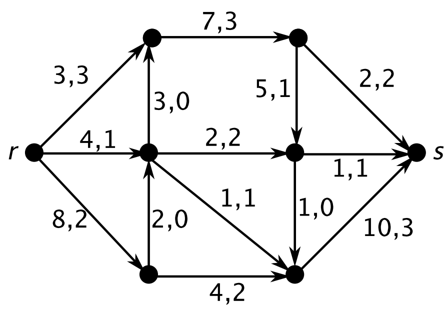




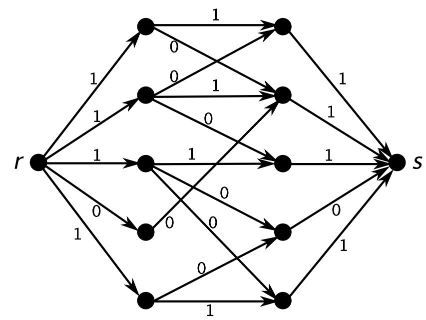
3.

a. (*Assigned! 5pts*) Consider the network below and on the left, with given capacity and flow values. (Recall that an edge label c, x has capacity c and flow-value x.) Find augmenting paths and augment the flow until it has value 8.

b. Consider the network below and on the right, where every edge has capacity 1 and the numbers below are the flow values. Find augmenting paths and augment the flow until it has value 5.



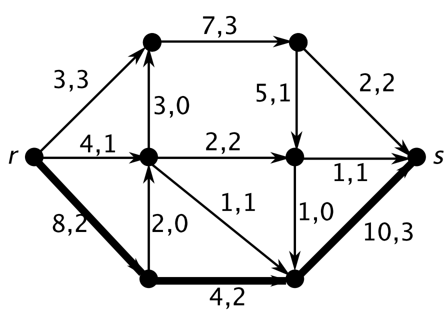
Problem 3.a.

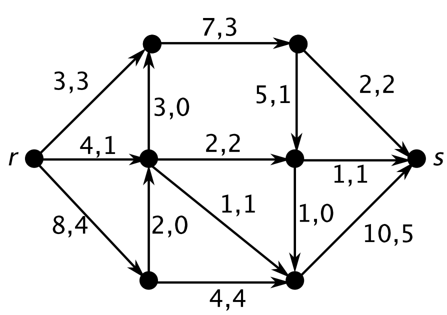


Problem 3.b.

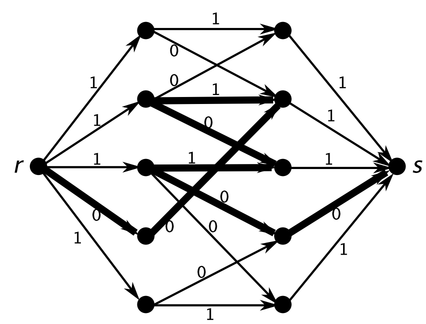
The two augmenting paths below complete this problem.

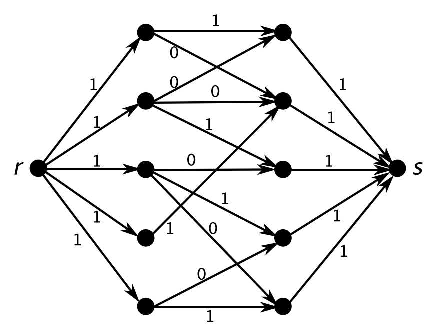
Problem 3.a.





Problem 3.b.



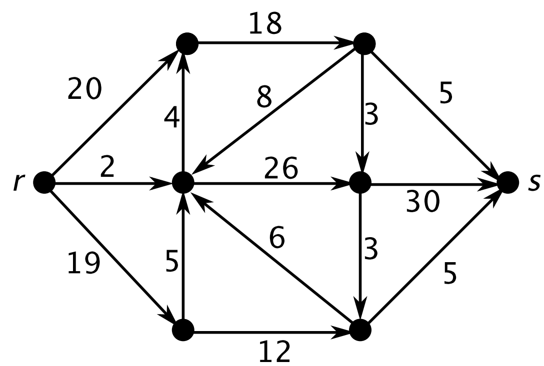


4.

a. Construct a network such that there is a unique maximum flow, but multiple distinct minimum cuts.

b. Construct a network such that there is a unique minimum cut, but multiple distinct maximum flows.

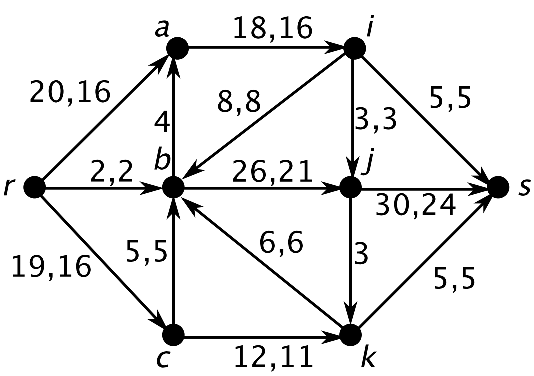
c. (*Assigned!* 5pts) Prove that the network below has a unique maximum flow and a unique minimum cut. (*Hint:* Find a max flow and a min cut. Use the flow to prove there is a unique cut, and use the cut to prove there is a unique flow.)



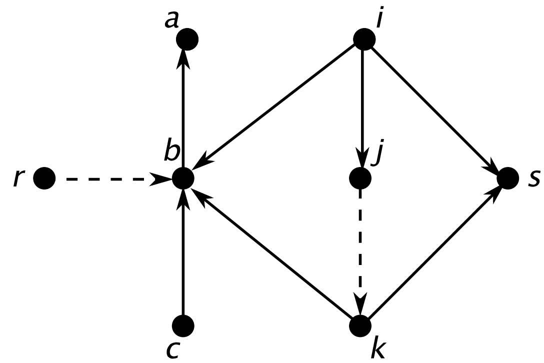
Note: I did not get to this result in class, so here is the result you should use for 4.c.

Cor: Let $\mathbf{x}$ be a feasible rs-flow and $\delta(R)$ be an rs-cut. Then, $f_{\mathbf{x}}(s) = c(\delta(R))$ if and only if $x_e = c(e)$ for all $e \in \delta(R)$ and $x_e = 0$ for all $e \in \delta(\overline{R})$.

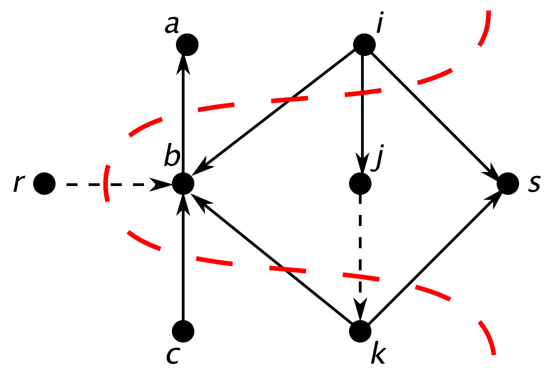
Proof. (*There is a unique minimum cut.*) Consider the following flow.



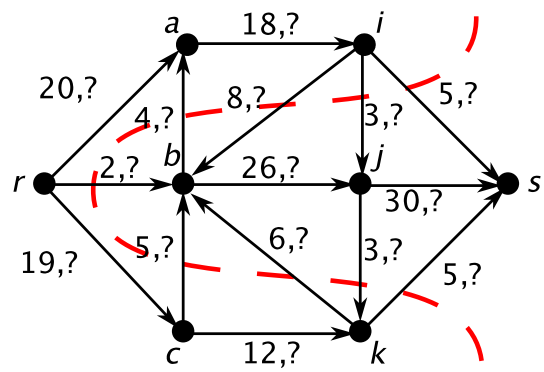
If this flow is maximum, then the only edges that can be used in a minimum cut are those that are at capacity or have zero flow. These edges are below, including the zero-flow edges being dotted.



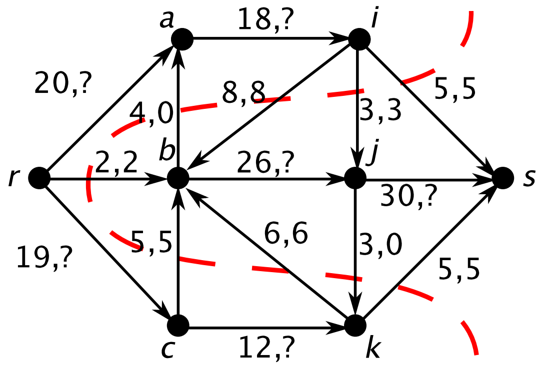
Any cut must use only edges in this set of edges, and must use them in their given direction. Since a, i, c, and k are reachable from r using edges not in this set, they must be in R for any minimum rs-cut $\delta(R)$. Since s is reachable from b and j using edges not in this set, they must not be in R for any minimum rs-cut $\delta(R)$. Thus, these restrictions uniquely define R, and $\delta(R)$ uses exactly these edges, as shown below.



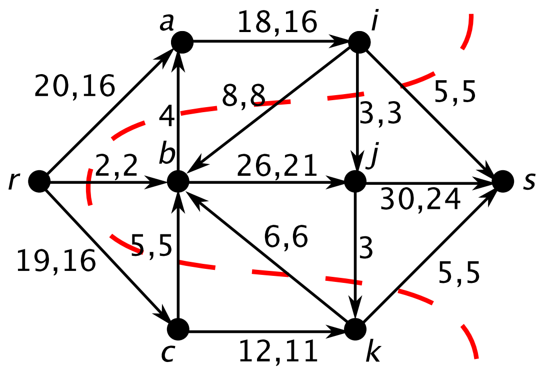
This cut matches our flow, so they are both optimal, and this cut is unique.
(*There is a unique maximum flow.* Consider our minimum rs-cut $\delta(R)$ with $R = \{r, a, c, i, k\}$. Consider unknown flow values in our flow network.



If we have a maximum flow, then by the corollary we must have full capacity for edges in $\delta(R)$ and no flow for edges not in $\delta(\overline{R})$.



For these values of flow, the vertex i requires an incoming flow of value 16, which determines the flow across ai and across ra. The vertex k requires an incoming flow of value 11, which determines the flow across ck. Then, the vertex c requires an incoming flow of value 16, which determines the flow across rc. The vertex b requires an outgoing flow of value 21, which determines the flow across bj. Finally, the vertex j requires an outgoing flow of value 24, which determines the flow across js.



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