MATH 482, Spring 2013 - Homework 5
Assigned Monday 11/04. Due Monday 11/11.

For this homework, solve four of the following five problems, but definitely complete problems 4 and 5. Each is worth 5 points. Problem 1 has a point breakdown for the parts, should you choose to do that problem.
Complete AT MOST four problems. If you complete all five, then your top score will be dropped!
1. Determine which of the matrices below are (i) unimodular, (ii) totally unimodular, or (iii) neither. Be sure to explain your answer.

$$\begin{bmatrix} 1 & -1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

a. (*1.5pts*) **b.** (*1.5pts*) **c.** (*2pts*)

a. It is totally unimodular since every column has exactly one 1 and exactly one -1 and the rest are zeroes.

b. It is not totally unimodular, since the 3×3 submatrix with rows and columns in $\{2, 3, 4\}$ has determinant 2. It is also not unimodular, since the full matrix has determinant 2.

c. It is not totally unimodular, since the 3×3 submatrix with rows and columns in $\{3, 4, 5\}$ has determinant 2. It is also not unimodular, since the full matrix has determinant -2.

2. An (n, k, λ, μ) *strongly regular graph* is an undirected graph with vertex set $\{v_1, \dots, v_n\}$ where every vertex is incident to k edges and for every pair v_i, v_j:

- If $v_i v_j$ is an edge, then v_i and v_j have exactly λ common neighbors.
- If $v_i v_j$ is not an edge, then v_i and v_j have exactly μ common neighbors.

For arbitrary n, k, λ, μ, construct an integer program encoding the constraints of an (n, k, λ, μ) strongly regular graph. Prove that the feasible integer solutions are in bijection with the (n, k, λ, μ) strongly regular graphs on vertex set $v_1, \dots, v_n$. (For this last part, we consider the graphs to be *labeled* and so do not worry about isomorphism.)

For each pair v_i, v_j, let $x_{i,j}$ be the indicator variable that $v_i v_j$ is an edge in the graph G. For each pair v_i, v_j and also a vertex v_k, let $y_{i,j}^k$ be the indicator variable that $v_i v_k$ and $v_k v_j$ are edges (and thus v_k is a common neighbor of v_i and v_j). If these variables do correspond to these notions, then the following constraints encode a strongly regular graph with parameters (n, k, λ, μ):

$$\sum_{i \neq j} x_{i,j} = k (\text{for all } v_j)$$

$$\sum_{k \notin \{i,j\}} y_{i,j}^k = \mu + (\lambda - \mu) x_{i,j} (\text{for all } i \neq j)$$

The first equation above implies that every vertex v_j has exactly k neighbors. Observe that the second equation above gives that v_i and v_j have exactly μ common neighbors when $x_{i,j} = 0$ and exactly λ common neighbors when $x_{i,j} = 1$, thus satisfying the strongly-regular constraints.
Now, to encode the property that $y_{i,j}^k$ is the indicator that $x_{i,k} = x_{k,j} = 1$, we use the following constraints:

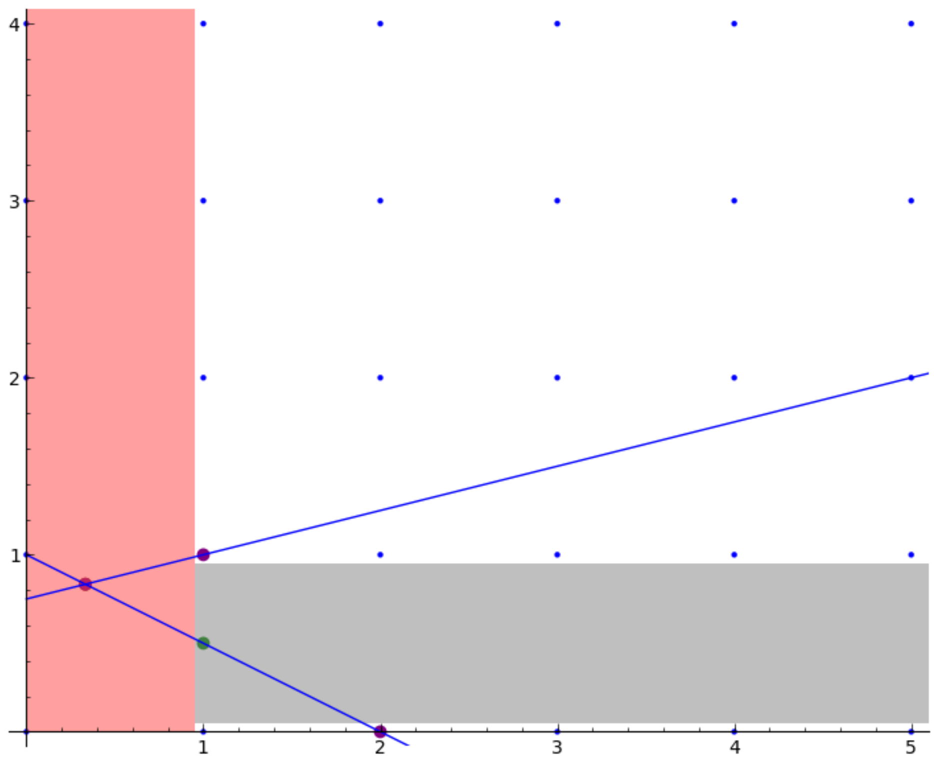
$$y_{i,j}^k \leq x_{i,k} \qquad y_{i,j}^k \leq x_{k,j}, \qquad y_{i,j}^k \geq x_{i,k} + x_{k,j} - 1.$$

Observe that for each of the four possible assigments to $x_{i,k}$ and $x_{k,j}$, there is exactly one satisfying value for $y_{i,j}^k$ and it is the correct one.

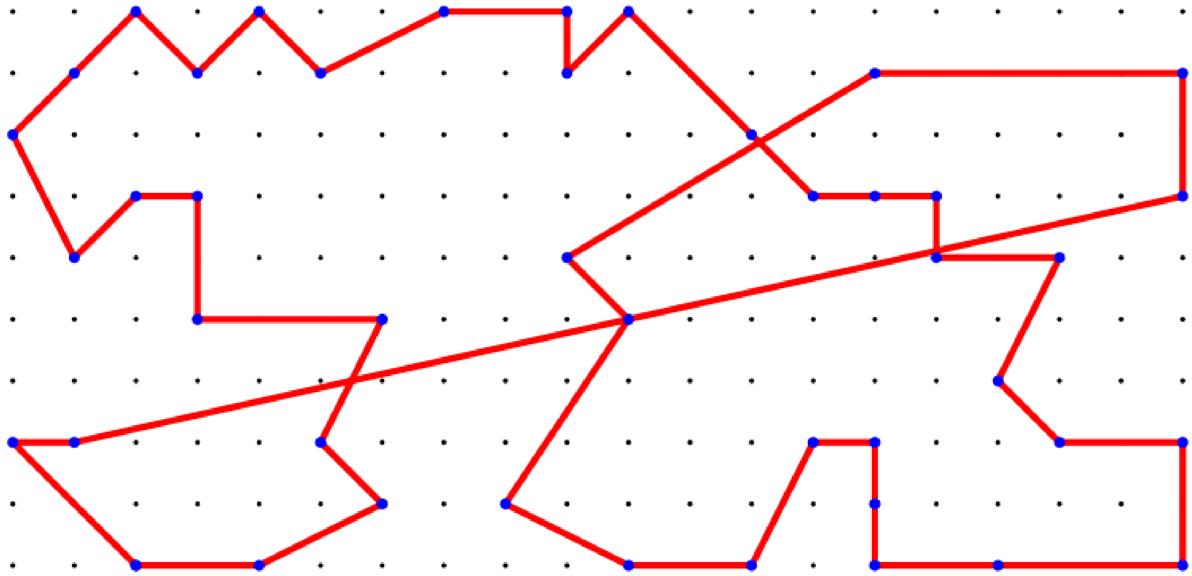
3. Solve the following integer linear program using branch-and-bound. Use the graphical method to solve each linear program relaxation. Plot your branch-and-bound cuts in the x_1, x_2-plane.

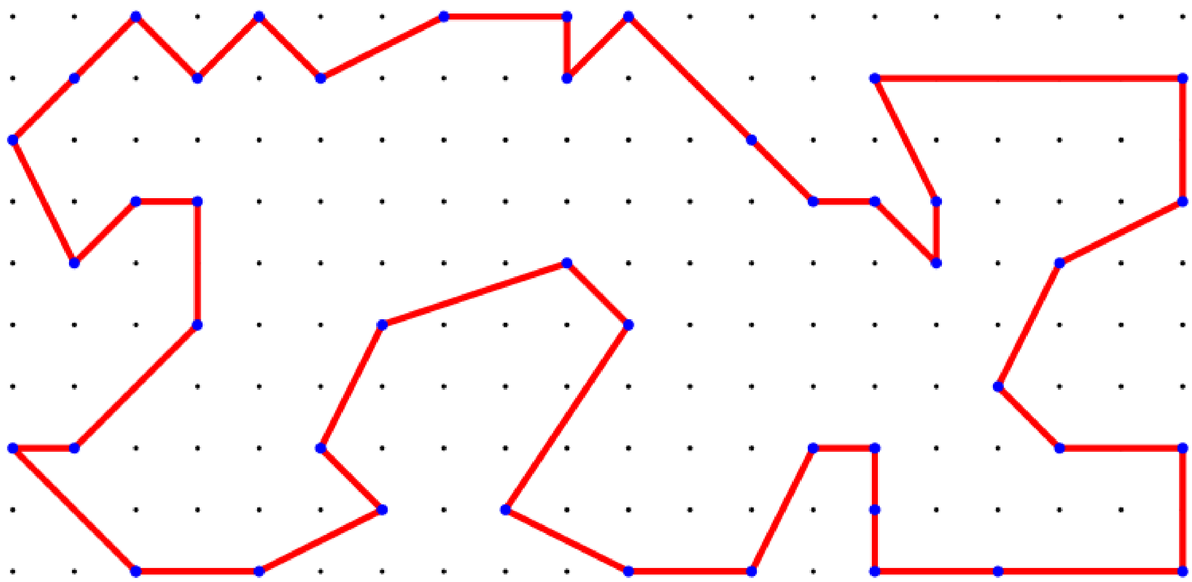
$$
\begin{array}{rrcrcl}
\min & x_1 & + & x_2 & = & z \\
\text{subject to} & x_1 & + & 2x_2 & \geq & 2 \\
& -x_1 & + & 4x_2 & \leq & 3 \\
& x_1, & & x_2 & \geq & 0, \text{ integer}
\end{array}
$$

We begin by solving the linear program to find linear optimal point $(1/3, 5/6)$.
We branch on $x_1 \leq 0$ and $x_1 \geq 1$ and find that $x_1 \leq 0$ is infeasible! For $x_1 \geq 1$ we have the linear optimal point $(1, 1/2)$. We then branch on $x_2 \leq 0$ and $x_2 \geq 1$. For each, we find the linear optimal points are $(2, 0)$ and $(1, 1)$, respectively. Each has cost 2, so they are both integer optimal points. See the plot below for the branching choices and the graphical plot.



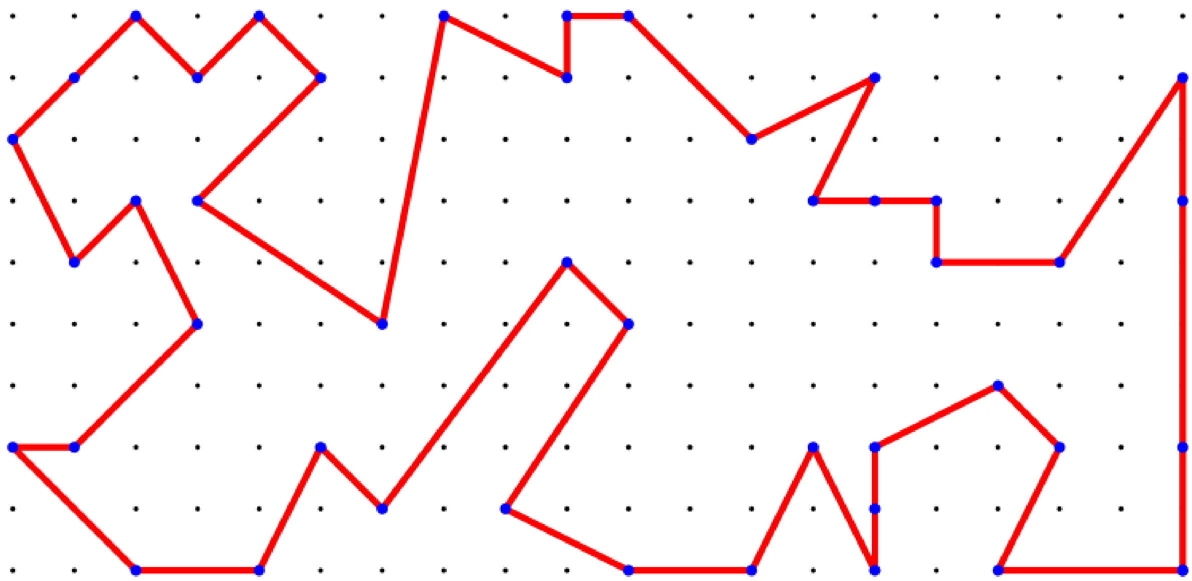
4. (*Assigned!*) Using the following set of points on the integer grid, compute a heuristic TSP using the Nearest Neighbor heuristic, then locally improve it using 2-switches until it is locally optimal.





Also see the course web page for an animation of these processes.

5. (*Assigned!*) Using the above set of points on the integer grid, compute a heuristic TSP using the Farthest Insertion heuristic, then locally improve it using 2-switches until it is locally optimal.



Also see the course web page for an animation of these processes.