Fall 2013 Math 566 Implementation Assignment 1

Linear and Integer Programs

Problem: Linear and Integer Programs

We have discussed linear programs at length now, but have not discussed actual algorithms to *solve* them! In this assignment, you will use a computer to solve given linear programs and find a dual solution. You will also use the computer to set variables to be integer-valued and solve the resulting integer program.

Questions To Answer.

Q1. Describe how you use the given primal problem to produce dual solution.

Q2. Describe what you do differently to make the problem have integer-valued variables instead of real-valued variables.

Q3. When the variables are set to be integer-valued, the optimal value can change from (and be worse than!) the linear solution. Prove that the relative gap between the optimal linear solution and optimal integer solutions can be arbitrarily large by demonstrating for every $n \geq 1$ a linear program P_n whose optimal linear solution $\mathbf{x}$ and optimal integer solution $\mathbf{x}^{(i)}$ has $\mathbf{c}^\top \cdot \mathbf{x} \leq \frac{1}{n}\mathbf{c}^\top \mathbf{x}^{(i)}$.

Q4. What language, libraries, and environments did you use?

Q5. What challenges did you encounter during your implementation?

Q6. What online/library resources did you use?

Problem Instances.

I1 & I2. Solve the following programs, once with real variables and once with integer variables. Produce a dual solution to the linear program.

$$
\begin{array}{rrcrcrcl}
\min & 3x_1 & - & x_2 & + & 2x_3 & & \\
\text{subject to} & x_1 & + & x_2 & & & = & 10 \\
 & 3x_1 & & & - & x_3 & \geq & 6 \\
 & & & 5x_2 & + & 3x_3 & \geq & 4 \\
 & x_1, & & x_2, & & x_3, & \geq & 0
\end{array}
$$

$$
\begin{array}{rrcrcrcrcrcrcrcrcrcrcl}
\max & x_1 & + & x_2 & + & x_3 & + & x_4 & + & x_5 & + & x_6 & + & x_7 & + & x_8 & + & x_9 & + & x_{10} & & \\
\text{s.t.} & x_1 & + & x_2 & & & & & & & & & & & & & & & + & x_{10} & \leq & 2 \\
 & & & x_2 & + & x_3 & & & & & & & + & x_7 & & & & & & & \leq & 2 \\
 & & & & & x_3 & + & x_4 & & & & & & & & & + & x_9 & & & \leq & 2 \\
 & & & & & & & x_4 & + & x_5 & + & x_6 & & & & & & & & & \leq & 2 \\
 & x_1 & + & & & & & & & x_5 & & & & & + & x_8 & & & & & \leq & 2 \\
 & x_1 & & & & & & & & & + & x_6 & + & x_7 & & & & & & & \leq & 2 \\
 & & & x_2 & & & & & & & & & & & + & x_8 & + & x_9 & & & \leq & 2 \\
 & & & & & x_3 & & & & & + & x_6 & & & & & & & + & x_{10} & \leq & 2 \\
 & & & & & & & x_4 & & & & & + & x_7 & + & x_8 & & & & & \leq & 2 \\
 & & & & & & & & & x_5 & & & & & & & + & x_9 & + & x_{10} & \leq & 2 \\
 & x_1, & & x_2, & & x_3, & & x_4, & & x_5, & & x_6, & & x_7, & & x_8, & & x_9, & & x_{10} & \geq & 0
\end{array}
$$

I3 & I4. See the files `lp3.txt` and `lp4.txt` on the course web page for a matrix A, and vectors c and b. Solve the canonical form linear program given by these parameters and produce dual solutions. (*Integer solutions to these problems are not required.*)

Solutions

Instance I1. Here is the solution to the linear program. It has integer value, so it is also the integer optimum.

```
Optimal Value: -2.0
x_{1} = 2.000000
x_{2} = 8.000000
x_{3} = 0.000000
```

Here is the dual problem and its solution:

```
Maximization:
  10.0 y_0 + 6.0 y_1 + 4.0 y_2
Constraints:
  y_0 + 3.0 y_1 <= 3.0
  y_0 + 5.0 y_2 <= -1.0
  - y_1 + 3.0 y_2 <= 2.0
Variables:
  y_0 is a continuous variable (min=-oo, max=+oo)
  y_1 is a continuous variable (min=0.0, max=+oo)
  y_2 is a continuous variable (min=0.0, max=+oo)
Optimal Value: -2.0
y_{1} = -1.000000
y_{2} = 1.333333
y_{3} = 0.000000
```

Instance I2. Here is the optimal linear solution.

```
Optimal Value: 6.66666666667
x_{1} = 0.666667
x_{2} = 0.666667
x_{3} = 0.666667
x_{4} = 0.666667
x_{5} = 0.666667
x_{6} = 0.666667
x_{7} = 0.666667
x_{8} = 0.666667
x_{9} = 0.666667
x_{10} = 0.666667
```

Here is the optimal integer solution. *Not required: A careful observer may notice that this models the problem of maximizing the number of edges in a triangle-free graph of order 5 (there are 10 possible edges (making the variables), and 10 possible triangles (making the constraints)). So, the constraints say that every triple of vertices has at most 2 out of 3 edges, so the linear relaxation allows all edges to have value 2/3.*

```
Optimal Value: 5.0
x_{1} = 1.000000
x_{2} = 1.000000
x_{3} = 1.000000
x_{4} = 1.000000
x_{5} = 1.000000
x_{6} = 0.000000
x_{7} = 0.000000
x_{8} = 0.000000
x_{9} = 0.000000
x_{10} = 0.000000
```

Here is the linear dual and its solution.

```
Minimization:
  2.0 y_0 + 2.0 y_1 + 2.0 y_2 + 2.0 y_3 + 2.0 y_4 + 2.0 y_5 + 2.0 y_6 + 2.0 y_7 + 2.0 y_8 + 2.0 y_9
Constraints:
  - y_0 - y_4 - y_5 <= -1.0
  - y_0 - y_1 - y_6 <= -1.0
  - y_1 - y_2 - y_7 <= -1.0
  - y_2 - y_3 - y_8 <= -1.0
  - y_3 - y_4 - y_9 <= -1.0
  - y_3 - y_5 - y_7 <= -1.0
  - y_1 - y_5 - y_8 <= -1.0
  - y_4 - y_6 - y_8 <= -1.0
  - y_2 - y_6 - y_9 <= -1.0
  - y_0 - y_7 - y_9 <= -1.0
Variables:
  y_0 is a continuous variable (min=0.0, max=+oo)
  y_1 is a continuous variable (min=0.0, max=+oo)
  y_2 is a continuous variable (min=0.0, max=+oo)
  y_3 is a continuous variable (min=0.0, max=+oo)
  y_4 is a continuous variable (min=0.0, max=+oo)
  y_5 is a continuous variable (min=0.0, max=+oo)
  y_6 is a continuous variable (min=0.0, max=+oo)
  y_7 is a continuous variable (min=0.0, max=+oo)
  y_8 is a continuous variable (min=0.0, max=+oo)
  y_9 is a continuous variable (min=0.0, max=+oo)
Optimal Value: 6.66666666667
y_{1} = 0.333333
y_{2} = 0.333333
y_{3} = 0.333333
y_{4} = 0.333333
y_{5} = 0.333333
y_{6} = 0.333333
y_{7} = 0.333333
y_{8} = 0.333333
y_{9} = 0.333333
y_{10} = 0.333333
```

Instance I3.
The canonical form linear primal has optimal value 142.849296741 and is solved by the vector $\mathbf{x}$ with positive values (variables are 0-indexed, so $\mathbf{x} = (x_0, \ldots, x_{29})$.

$$x_0 = 1.768333, \quad x_5 = 3.933333, \quad x_7 = 7.900483, \quad x_8 = 3.562416, \quad x_{12} = 3.315287,$$

$$x_{15} = 4.100000, \quad x_{20} = 0.601932, \quad x_{24} = 3.537718, \quad x_{25} = 2.388889, \quad x_{29} = 2.788122.$$

(The other values are all equal to 0.) The dual of this canonical form primal has optimal value 142.849296741 and is solved by the vector $\mathbf{y} = (y_0, \ldots, y_{24})$ with positive values

$$y_1 = 0.178891, \quad y_6 = 0.162393, \quad y_{11} = 0.456887, \quad y_{13} = 0.444444, \quad y_{14} = 0.091387,$$

$$y_{15} = 0.250000, \quad y_{18} = 0.232873, \quad y_{19} = 0.286111, \quad y_{20} = 0.055556, \quad y_{22} = 0.172415.$$

Instance I4. The canonical form linear primal has optimal value 282.179260865 and the optimal vector $\mathbf{x} = (x_0, \ldots, x_{99})$ has nonzero entries

$$x_3 = 1.389332, \quad x_{10} = 1.590434, \quad x_{11} = 1.912015, \quad x_{15} = 2.600000, \quad x_{17} = 0.677519,$$

$$x_{18} = 0.182167, \quad x_{19} = 0.952895, \quad x_{20} = 1.261353, \quad x_{21} = 1.631156, \quad x_{23} = 1.911912,$$
$$x_{25} = 0.441051, \quad x_{29} = 1.529277, \quad x_{35} = 0.390272, \quad x_{36} = 9.615976, \quad x_{37} = 0.219902,$$
$$x_{40} = 1.737279, \quad x_{42} = 1.042787, \quad x_{43} = 0.455099, \quad x_{46} = 0.354613, \quad x_{49} = 0.471414,$$
$$x_{51} = 1.247912, \quad x_{58} = 1.084308, \quad x_{69} = 3.475900, \quad x_{71} = 0.855168, \quad x_{73} = 2.071256,$$
$$x_{75} = 0.873261, \quad x_{77} = 0.809284, \quad x_{78} = 5.564642, \quad x_{79} = 0.637189, \quad x_{81} = 2.340148,$$
$$x_{83} = 1.396435, \quad x_{85} = 0.847659, \quad x_{86} = 3.089552, \quad x_{87} = 0.129206, \quad x_{88} = 0.280616,$$
$$x_{89} = 0.354456, \quad x_{95} = 1.877156, \quad x_{96} = 2.011611, \quad x_{97} = 0.313615.$$

The dual of this program has the same optimal value, with optimal vector $\mathbf{y} = (y_0, \ldots, y_{249})$ with nonzero entries

$$y_0 = 0.015264, \quad y_6 = 0.096743, \quad y_7 = 0.014533, \quad y_{15} = 0.006191, \quad y_{20} = 0.043478, \quad y_{21} = 0.037216,$$
$$y_{32} = 0.012829, \quad y_{36} = 0.032258, \quad y_{49} = 0.010472, \quad y_{50} = 0.037133, \quad y_{52} = 0.064745, \quad y_{56} = 0.028085,$$
$$y_{57} = 0.058527, \quad y_{67} = 0.025011, \quad y_{71} = 0.057364, \quad y_{86} = 0.041726, \quad y_{88} = 0.059776, \quad y_{95} = 0.084844,$$
$$y_{103} = 0.016565, \quad y_{112} = 0.009418, \quad y_{118} = 0.000528, \quad y_{120} = 0.000750, \quad y_{155} = 0.036650, \quad y_{157} = 0.132666,$$
$$y_{159} = 0.089204, \quad y_{171} = 0.020550, \quad y_{176} = 0.002848, \quad y_{179} = 0.015446, \quad y_{180} = 0.060115, \quad y_{183} = 0.030579,$$
$$y_{185} = 0.049984, \quad y_{190} = 0.033400, \quad y_{194} = 0.015333, \quad y_{196} = 0.024498, \quad y_{200} = 0.001548, \quad y_{204} = 0.176471,$$
$$y_{216} = 0.014735, \quad y_{234} = 0.021960, \quad y_{246} = 0.019401.$$