Integral forms that you need to know

Let k be a real number, r a rational number and $a > 0$ be a positive real number.

1. $\displaystyle\int k\,du = ku + C$

2. $\displaystyle\int u^r\,du = \begin{cases} \dfrac{u^{r+1}}{r+1} + C, & r \neq -1 \\ \ln|u| + C, & r = -1 \end{cases}$

3. $\displaystyle\int e^u\,du = e^u + C$

4. $\displaystyle\int a^u\,du = \frac{a^u}{\ln a} + C, \quad a \neq 1$

5. $\displaystyle\int \sin u\,du = -\cos u + C$

6. $\displaystyle\int \cos u\,du = \sin u + C$

7. $\displaystyle\int \sec^2 u\,du = \tan u + C$

8. $\displaystyle\int \csc^2 u\,du = -\cot u + C$

9. $\displaystyle\int \sec u \tan u\,du = \sec u + C$

10. $\displaystyle\int \csc u \cot u\,du = -\csc u + C$

11. $\displaystyle\int \tan u\,du = -\ln|\cos u| + C$

12. $\displaystyle\int \cot u\,du = \ln|\sin u| + C$

13. $\displaystyle\int \frac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\frac{u}{a}\right) + C$

14. $\displaystyle\int \frac{du}{a^2 + u^2} = \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$

15. $\displaystyle\int \frac{du}{u\sqrt{u^2 - a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{|u|}{a}\right) + C$

16. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u| + C$

17. $\displaystyle\int \csc u\,du = \ln|\csc u - \cot u| + C$

Trigonometric Integrals

$$\begin{aligned} \sin^2 x &= (1-\cos 2x)/2 & \sin^2 x + \cos^2 x &= 1 & \sin mx \cos nx &= \tfrac{1}{2}[\sin(m+n)x + \sin(m-n)x] \\ \cos^2 x &= (1+\cos 2x)/2 & \tan^2 x + 1 &= \sec^2 x & \sin mx \sin nx &= -\tfrac{1}{2}[\cos(m+n)x - \cos(m-n)x] \\ \sin 2x &= 2\sin x \cos x & 1 + \cot^2 x &= \csc^2 x & \cos mx \cos nx &= \tfrac{1}{2}[\cos(m+n)x + \cos(m-n)x] \end{aligned}$$

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1. $\int \sin^n x\,dx, \int \cos^n x\,dx.$
    - If $\sin^n x$ and n is odd, take out a $\sin x$ term and express $\sin^{n-1} x$ in terms of $\cos x$. Do substitution. Similar for $\cos^n x$ and n odd.
    - If n is even, use the double-angle formulas for $\sin^2 x$ and $\cos^2 x$ above.
2. $\int \sin^m x \cos^n x\,dx.$
    - If m is odd (resp. n is odd), take out a $\sin x$ term (resp. $\cos x$ term) and express the rest in terms of $\cos x$ (resp. $\sin x$).
    - If m and n are even, use the the double-angle formulas for $\sin^2 x$ and $\cos^2 x$ above.
3. $\int \sin mx \cos nx\,dx, \int \sin mx \sin nx\,dx, \int \cos mx \cos nx\,dx.$
    - Use the angle sum formulas above for $\sin\cos$, $\sin\sin$ and $\cos\cos$, resp.
4. $\int \tan^n x\,dx, \int \cot^n x\,dx.$
    - If $\tan^n x$ (resp. $\cot^n x$), use the Pythagorean Theorem to factor out a $\tan^2 x = \sec^2 x - 1$ (resp. $\cot^2 x = \csc^2 x - 1$) and use $u = \tan x$, $du = \sec^2 x\,dx$ (resp. $u = \cot x$, $du = -\csc^2 x\,dx$).
5. $\int \tan^m x \sec^n x\,dx, \int \cot^m x \csc^n x\,dx.$
    - If n even, take out a $\sec^2 x$ term (resp $\csc^2 x$ term) and express the rest in terms of $\tan x$ (resp. $\cot x$). Do substitution.
    - If m odd, take out a $\tan x \sec x$ term (resp $\cot x \csc x$ term) and express the rest in terms of $\sec x$ (resp. $\csc x$). Do substitution.

Integration by parts

$$\int u\,dv = uv - \int v\,du \qquad\qquad \int_a^b u\,dv = [uv]_a^b - \int_a^b v\,du.$$

Look for a product of functions such that the derivative of one and the integral of another is a basic form.

Trigonometric substitution

$\sqrt{a^2 - x^2}$	$x = a\sin t$	$-\pi/2 \le t \le \pi/2$
$\sqrt{a^2 + x^2}$	$x = a\tan t$	$-\pi/2 < t < \pi/2$
$\sqrt{x^2 - a^2}$	$x = a\sec t$	$0 \le t \le \pi, t \ne \pi/2$

Can drop absolute value bars.

Use $\sin^{-1}$, $\tan^{-1}$, $\sec^{-1}$ or triangles to back-substitute.

$$\int \sec x\,dx = \ln|\sec x + \tan x| + C$$

$$\int \csc x\,dx = \ln|\csc x - \cot x| + C$$

Rational functions

Let $p(x), q(x)$ be polynomials.

$$\int \frac{p(x)}{q(x)}\,dx$$

0. Do long division to ensure that the degree of $p(x)$ is less than the degree of $q(x)$.
1. Factor $q(x)$ into linear and (unfactorable) quadratic terms.
2. Each term of the factorization of $q(x)$ gives a cluster of terms.
   - The term $(x-a)^n$ gives rise to the cluster of terms:
   $$\frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \cdots + \frac{A_n}{(x-a)^n}.$$
   - The term $(x^2+bx+c)^m$ gives rise to the cluster of terms:
   $$\frac{B_1x+C_1}{x^2+bx+c} + \frac{B_2x+C_2}{(x^2+bx+c)^2} + \cdots + \frac{B_mx+C_m}{(x^2+bx+c)^m}.$$
3. Solve for the unknown coefficients to get a Partial Fraction Decomposition.
4. Integrate using

$$\begin{aligned}
\int \frac{dx}{x-a} &= \ln|x-a| + C \\
\int \frac{2x+b}{x^2+bx+c}\,dx &= \ln|x^2+bx+c| + C \\
\int \frac{dx}{(x+d)^2+a^2} &= \frac{1}{a}\tan^{-1}\left(\frac{x+d}{a}\right) + C.
\end{aligned}$$