Review for Applications of Definite Integrals

Sections 6.1–6.4

Math 166

Iowa State University
http://orion.math.iastate.edu/dstolee/teaching/15-166/

September 4, 2015

1. What type of problem:
   Volume? Arc Length? Surface Area?
2. What is the integration variable?
3. (Optional?) Sketch the 2D graph.
4. If volume, what method:
   Cross-Section, Disk/Washer, or Shell?
5. What are the bounds of integration?
6. Plug in the formula, clearly writing the definite integral.
7. Solve the integral and simplify result. (Skip this step today)

Example 1

Consider the solid of revolution given by taking the region where $y \leq -x^2 + 9x - 8$ and $y \geq x^2 - 3x + 2$ and rotating the region about the y-axis.

Compute the volume of this solid.

Solution to Example 1

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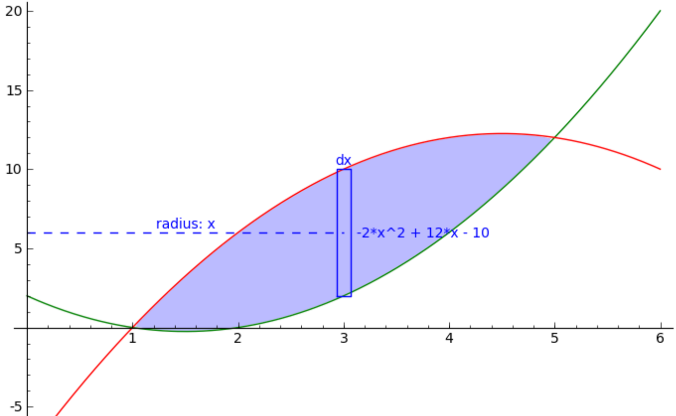

Solution to Example 1

Consider the solid of revolution given by taking the region where $y \leq -x^2 + 9x - 8$ and $y \geq x^2 - 3x + 2$ and rotating the region about the y-axis.

Compute the volume of this solid.

Integration Variable: x, Axis of rotation: y, $\longrightarrow$ Shell Method!

Bounds: Solution to $-x^2 + 9x - 8 = x^2 - 3x + 2$
$\longrightarrow 2x^2 - 12x + 10 = 0$
$\longrightarrow 2(x - 1)(x - 5) = 0$
$\longrightarrow x = 1$ or $x = 5$

Solution to Example 1

Consider the solid of revolution given by taking the region where $y \leq -x^2 + 9x - 8$ and $y \geq x^2 - 3x + 2$ and rotating the region about the y-axis.

Compute the volume of this solid.

$$V = \int_1^5 \underbrace{2\pi x}_{\text{circumference of shell}} \underbrace{\left[(-x^2 + 9x - 8) - (x^2 - 3x + 2)\right]}_{\text{height of shell}} dx$$

Example 2

Consider the surface of revolution given by taking the plot $x = e^{-y}$ from $y = -1$ to $y = 1$ and rotating about the y-axis.

Compute the area of this surface.

Solution to Example 2

Consider the surface of revolution given by taking the plot $x = e^{-y}$ from $y = -1$ to $y = 1$ and rotating about the y-axis.

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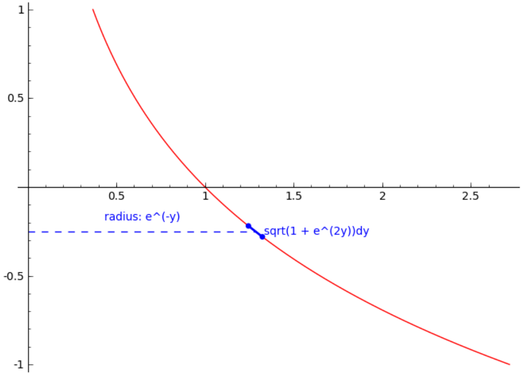

Solution to Example 2

Consider the surface of revolution given by taking the plot $x = e^{-y}$ from $y = -1$ to $y = 1$ and rotating about the y-axis.

Compute the area of this surface.

Integration Variable: y, Axis of rotation: y

Bounds: $y = -1$, $y = 1$.

Derivative: $\frac{dx}{dy} = -e^{-y}$

Solution to Example 2

Consider the surface of revolution given by taking the plot $x = e^{-y}$ from $y = -1$ to $y = 1$ and rotating about the y-axis.

Compute the area of this surface.

$$\boxed{S = \int_{-1}^{1} \underbrace{2\pi e^{-y}}_{\text{circumference of frustum}} \underbrace{\sqrt{1 + (-e^{-y})^2}}_{\text{arc length contribution}} \, dy}$$

Example 3

You watch an ant walk on your counter (gross!). Considering your counter as an xy-plane with cm-units, you find out that the ant is traveling along the curve $y = \sin(\pi(1 + x^2))$ starting at $x = 1$. You also observe that the ant is traveling at 0.1 cm/sec.

You watch the ant until it hits the edge of your counter at $x = \sqrt{13}$. How long did you watch the ant?

(Describe how you would solve the problem, but do not solve any integrals.)

Solution to Example 3

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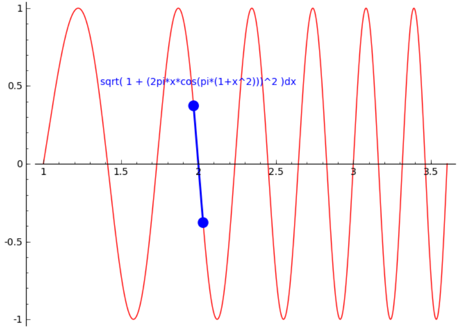

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You watch the ant until it hits the edge of your counter at $x = \sqrt{13}$. How long did you watch the ant?

Integration variable: x

Bounds: $x = 1$, $x = \sqrt{13}$

Derivative: $\frac{dy}{dx} = (2\pi x)\cos(\pi(1 + x^2))$

Solution to Example 3

You watch an ant walk on your counter (gross!). Considering your counter as an xy-plane with cm-units, you find out that the ant is traveling along the curve $y = x\sin(\pi(1+x^2))$ starting at $x = 1$. You also observe that the ant is traveling at 0.1 cm/sec.

You watch the ant until it hits the edge of your counter at $x = \sqrt{13}$. How long did you watch the ant?

$$L = \int_1^{\sqrt{13}} \underbrace{\sqrt{1 + \left(2\pi x \cos\left(\pi(1+x^2)\right)\right)^2}}_{\text{arc length contribution}}\, dx$$

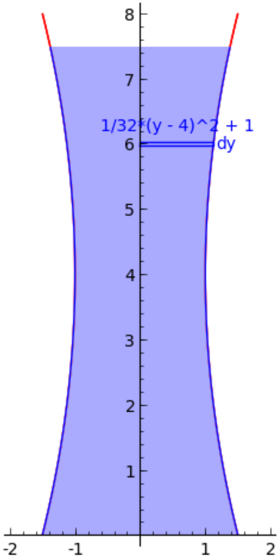
$$\boxed{\text{Amount of time: } \frac{L(\text{cm})}{0.1(\frac{\text{cm}}{\text{sec}})} = 10L(\text{sec}).}$$

Example 4

I have a vase at home that is 8 inches tall whose interior is given by rotating the region where $0 \leq x \leq 2\left(\frac{y-4}{8}\right)^2 + 1$ and $0 \leq y \leq 8$.

If I pour $10\pi \text{in}^2$ of water into the vase, then how high will the water level be?

Solution to Example 4



Solution to Example 4

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If I pour $10\pi \text{in}^2$ of water into the vase, then how high will the water level be?

Integration Variable: y, Axis of rotation: y, (Disk Method!)

Bounds: $y = 0$, $y = h$ (where h is an unknown value!)

Solution to Example 4

I have a vase at home that is 8 inches tall whose interior is given by rotating the region where $0 \leq x \leq 2\left(\frac{y-4}{8}\right)^2 + 1$ and $0 \leq y \leq 8$.

If I pour $10\pi \text{in}^2$ of water into the vase, then how high will the water level be?

$$10\pi = V = \int_0^h \pi \left(\frac{(y-4)^2}{32} + 1\right)^2 dy$$

Find the antiderivative formula for the above, then solve for h under equation $V = 10\pi$.

Solution to Example 4

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If I pour $10\pi\text{in}^2$ of water into the vase, then how high will the water level be?

$$
\begin{aligned}
V &= \int_0^h \pi\left(\frac{(y-4)^2}{32} + 1\right)^2 dy \\
&= \pi \int_0^h \left(\frac{1}{32}y^2 - \frac{1}{4}y + \frac{1}{2} + 1\right)^2 dy \\
&= \pi \int_0^h \left[\frac{1}{1024}y^4 - \frac{1}{64}y^3 + \frac{5}{32}y^2 - \frac{3}{4}y + \frac{9}{4}\right] dy \\
&= \pi \left[\frac{1}{5\cdot 1024}y^5 - \frac{1}{4\cdot 64}y^4 + \frac{5}{3\cdot 32}y^3 - \frac{3}{2\cdot 4}y^2 + \frac{9}{4}y\right]_0^h \\
&= \pi \left[\frac{1}{5\cdot 1024}h^5 - \frac{1}{4\cdot 64}h^4 + \frac{5}{3\cdot 32}h^3 - \frac{3}{2\cdot 4}h^2 + \frac{9}{4}h\right] = 10\pi
\end{aligned}
$$

Solution to Example 4

I have a vase at home that is 8 inches tall whose interior is given by rotating the region where $0 \le x \le 2\left(\frac{y-4}{8}\right)^2 + 1$ and $0 \le y \le 8$.

If I pour $10\pi \text{in}^2$ of water into the vase, then how high will the water level be?

Solving the equation

$$10 = \frac{1}{5 \cdot 1024}h^5 - \frac{1}{4 \cdot 64}h^4 + \frac{5}{3 \cdot 32}h^3 - \frac{3}{2 \cdot 4}h^2 + \frac{9}{4}h$$

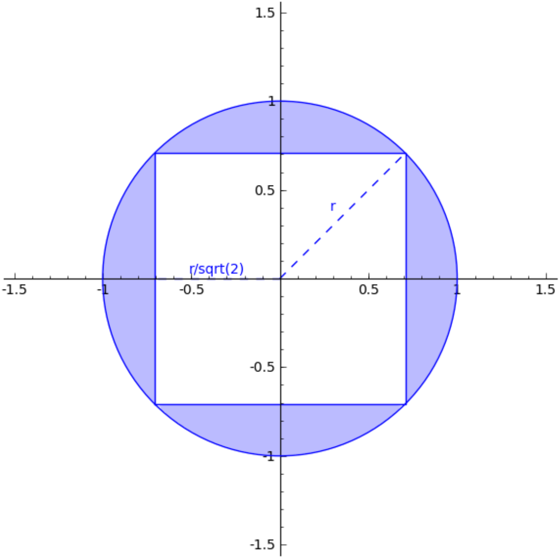
with a computer algebra system results in $\boxed{h \approx 7.48472}$

Example 5

I have a 3D-model of a solid I want to create with a 3D-printer. To print the model, the machine prints an *xy*-plane layer is for each z value from 0 to $\pi/2$. I want each plane to be the region inside a circle of radius $\cos(z)$ but outside the square with corners on the circle.

What volume of 3D printing material will this model use?

Solution to Example 5



Solution to Example 5

I have a 3D-model of a solid I want to create with a 3D-printer. To print the model, the machine prints an xy-plane layer is for each z value from 0 to $\pi/2$. I want each plane to be the region inside a circle of radius $\cos(z)$ but outside the square with corners on the circle.

What volume of 3D printing material will this model use?

Integration Variable: z, Rotation Axis: NONE!

Bounds: $z = 0$, $z = \pi/2$

Area Function:
$A(z) = \pi(\cos(z))^2 - (\sqrt{2}\cos(z))^2 = (\pi - 2)\cos^2(z)$.

Solution to Example 5

I have a 3D-model of a solid I want to create with a 3D-printer. To print the model, the machine prints an xy-plane layer is for each z value from 0 to $\pi/2$. I want each plane to be the region inside a circle of radius $\cos(z)$ but outside the square with corners on the circle.

What volume of 3D printing material will this model use?

$$\boxed{V = \int_0^{\pi/2} (\pi - 2) \cos^2 z dz}$$

Solution to Example 5

$$
\begin{aligned}
V &= \int_0^{\pi/2} (\pi - 2)\cos^2 z dz \\
&= (\pi - 2)\int_0^{\pi/2} \frac{1}{2}(1 + \cos(2z))dz \quad \text{(Half Angle Formula)} \\
&= \frac{\pi - 2}{2}\left[z + \frac{1}{2}\sin(2z)\right]_0^{\pi/2} \\
&= \frac{\pi - 2}{2}\left[\left(\frac{\pi}{2} + \frac{1}{2}\sin(\pi)\right) - \left(0 + \frac{1}{2}\sin(0)\right)\right] \\
&= \frac{\pi - 2}{2}\left(\frac{\pi}{2}\right) \\
&= \frac{\pi^2 - 2\pi}{4}
\end{aligned}
$$